

# Chain groups of homeomorphisms of the interval

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**ABSTRACT.** We introduce and study the notion of a chain group of homeomorphisms of a one-manifold, which is a certain generalization of Thompson's group  $F$ . The resulting class of groups exhibits a combination of uniformity and diversity. On the one hand, a chain group either has a simple commutator subgroup or the action of the group has a wandering interval. In the latter case, the chain group admits a canonical quotient which is also a chain group, and which has a simple commutator subgroup. On the other hand, every finitely generated subgroup of  $\text{Homeo}^+(I)$  can be realized as a subgroup of a chain group. As a corollary, we show that there are uncountably many isomorphism types of chain groups, as well as uncountably many isomorphism types of countable simple subgroups of  $\text{Homeo}^+(I)$ . We consider the restrictions on chain groups imposed by actions of various regularities, and show that there are uncountably many isomorphism types of chain groups which cannot be realized by  $C^2$  diffeomorphisms.

## 1. INTRODUCTION

In this paper, we introduce and study the notion of a chain group of homeomorphism of a connected one-manifold. A chain group can be viewed as a generalization of Thompson's group  $F$  which sits inside of the group of homeomorphisms of the manifold in a particularly nice way.

We denote by  $\text{Homeo}^+(\mathbb{R})$  the group of orientation preserving homeomorphisms on  $\mathbb{R}$ . The *support* of  $f \in \text{Homeo}^+(\mathbb{R})$  is the set of  $x \in \mathbb{R}$  such that  $f(x) \neq x$ . The support of a group  $G \leq \text{Homeo}^+(\mathbb{R})$  is defined as the union of the supports of all the elements in  $G$ . For an interval  $J \subseteq \mathbb{R}$ , let us denote the left- and the right-endpoints of  $J$  by  $\partial^- J$  and  $\partial^+ J$ , respectively.

Suppose  $\mathcal{J} = \{J_1, \dots, J_n\}$  is a collection of nonempty open subintervals of  $\mathbb{R}$ . We call  $\mathcal{J}$  a *chain of intervals* (or an *n-chain of intervals* if the cardinality of  $\mathcal{J}$  is important) if  $J_i \cap J_k = \emptyset$  if  $|i - k| > 1$ , and if  $J_i \cap J_{i+1}$  is a proper nonempty subinterval of  $J_i$  and  $J_{i+1}$  for  $1 \leq i \leq n - 1$ . See Figure 1.

**Setting 1.1.** We let  $n \geq 2$  and let  $\mathcal{J} = \{J_1, \dots, J_n\}$  be a chain of intervals such that  $\partial^- J_i < \partial^- J_{i+1}$  for each  $i < n$ . We consider a collection of homeomorphisms

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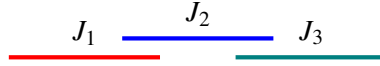


FIGURE 1. A chain of three intervals.

$\mathcal{F} = \{f_1, \dots, f_n\}$  such that  $\text{supp } f_i = J_i$  and such that  $f_i(t) \geq t$  for each  $t \in \mathbb{R}$ . We set  $G_{\mathcal{F}} = \langle \mathcal{F} \rangle \leq \text{Homeo}^+(\mathbb{R})$ .

We call the group  $G = G_{\mathcal{F}}$  a *prechain group*. We say that  $G$  is a *chain group* (sometimes *n-chain group*) if, moreover, the group  $\langle f_i, f_{i+1} \rangle$  is isomorphic to Thompson's group  $F$  for each  $i = 1, 2, \dots, n-1$ . Whereas this definition may seem rather unmotivated, part of the purpose of this article is to convince the reader that chain groups are natural objects.

In this paper, we will consider chain groups as both abstract groups and as groups with a distinguished finite generating set  $\mathcal{F}$  as above. Whenever we mention “the generators” of a chain group, we always mean the distinguished generating set  $\mathcal{F}$  which realizes the group as a chain group of homeomorphisms.

**1.1. Main results.** Elements of the class of chain groups enjoy many properties which are mostly independent of the choices of the homeomorphisms generating them, and at the same time can be very diverse. Moreover, chain groups are abundant in one-dimensional dynamics. Our first result establishes the naturality of chain groups:

**Theorem 1.1.** *If  $G$  is a prechain group as in Setting 1.1, then the group*

$$G_N := \langle f^N \mid f \in \mathcal{F} \rangle \leq \text{Homeo}^+(\mathbb{R})$$

*is a chain group for all  $N \gg 0$ .*

Choosing a two-chain group whose generators are  $C^\infty$  diffeomorphisms of  $\mathbb{R}$ , we obtain the following immediate corollary, which is a complement to (and an at least partial recovery of) a result of Ghys–Sergiescu [16]:

**Corollary 1.2.** *Thompson's group  $F$  can be realized as a subgroup of  $\text{Diff}^\infty(M)$ , the group of  $C^\infty$  orientation preserving diffeomorphisms of  $M$ .*

Let  $M$  be a one-manifold. We say  $G \leq \text{Homeo}^+(M)$  is *minimal* if every orbit is dense in  $\text{supp } G = \cup_{g \in G} \text{supp } g$ . General chain groups have remarkably uncomplicated normal subgroup structure, in relation to their dynamical features:

**Theorem 1.3.** *For an n-chain group  $G$ , exactly one of the following holds:*

- (i) *The action of  $G$  is minimal; in this case, every proper quotient of  $G$  is abelian and the commutator subgroup  $G' \leq G$  is simple;*
- (ii) *The closure of some  $G$ -orbit is perfect and totally disconnected; in this case,  $G$  canonically surjects onto an n-chain group that acts minimally.*

A general chain group may fail to have a simple commutator subgroup (Proposition 4.7).

Every finitely generated subgroup of  $\text{Homeo}^+(\mathbb{R})$  embeds into a minimal chain group, so that the subgroup structure of chain groups can be extremely complicated:

**Theorem 1.4.** *Let  $G = \langle f_1, f_2, \dots, f_n \rangle \leq \text{Homeo}^+(\mathbb{R})$  for some  $n \geq 2$ .*

- (1) *Then  $G$  embeds into an  $(n + 2)$ -chain group.*
- (2) *If  $\text{supp } f_1$  has finitely many components, then  $G$  embeds into an  $(n + 1)$ -chain group.*

There does not seem to be a suitable notion of the “rank” of a chain group. Indeed, the next proposition shows that a given  $n$ -chain group not only contains  $m$ -chain groups for all  $m$ , but is in fact isomorphic to an  $m$ -chain group for all  $m \geq n$ :

**Proposition 1.5.** *For  $m \geq n \geq 2$ , each  $n$ -chain group is isomorphic to some  $m$ -chain group.*

Throughout this paper, we will say that a group is *infinitely presented* if it is not finitely presentable. Theorem 1.4 has several consequences which we note here. Theorem 1.1 in the case  $n = 2$  implies that there is only one isomorphism type of 2-chain groups. This is in stark contrast to the case  $n \geq 3$ , where there exist uncountably many isomorphism types of  $n$ -chain groups. More precisely, we have the following:

**Theorem 1.6.** *Let  $n \geq 3$ . Then there exists a uncountable set  $\mathcal{G}$  of infinitely presented  $n$ -chain groups with simple commutator subgroups such that  $G' \not\cong H'$  for all distinct  $G, H \in \mathcal{G}$ .*

In the case  $n = 3$ , Theorem 1.6 answers a question posed to the authors by J. McCammond. We deduce the following.

**Corollary 1.7.** *For every one-manifold  $M$ , there exist uncountably many isomorphism types of countable simple subgroups of  $\text{Homeo}^+(M)$ . These simple groups can be realized as the commutator subgroups of 3-chain groups.*

The simple subgroups in Corollary 1.7 will necessarily be infinitely generated. We remark that it is not difficult to establish Corollary 1.7 for general  $n$ -manifolds, though we will not require such a statement here.

Theorem 1.6 and its corollaries above show that chain groups can be very diverse. However, chain groups exhibit a remarkable phenomenon called *stabilization* (see Section 3 for precise definitions and discussion). We have the following results, which show that chain groups acquire certain *stable isomorphism types*:

**Proposition 1.8.** *Let  $G = G_{\mathcal{F}}$  be an  $n$ -chain group for  $n \geq 2$ . Then for all  $N \gg 0$ , the group  $G_N = \langle f^N \mid f \in \mathcal{F} \rangle$  is isomorphic to the Higman–Thompson group  $F_n$ .*

The Higman–Thompson groups  $\{F_n\}_{n \geq 2}$  are defined and discussed in Subsection 2.1 below. We remark that Proposition 1.8 was observed independently (and in a different context) by Bleak–Brin–Kassabov–Moore–Zaremsky [5].

## 1.2. Notes and references.

1.2.1. *Motivations.* The original motivation for studying chain groups comes from the first and second authors’ joint work with Baik [1] which studied right-angled Artin subgroups of  $\text{Diff}^2(S^1)$ . If  $G = G_{\mathcal{F}} < \text{Homeo}^+(I)$  is a prechain group, then one can form a graph whose vertices are elements of  $\mathcal{F}$ , and where an edge is drawn between two vertices if the supports are disjoint. The graph one obtains this way is an *anti-path*, i.e. a graph whose complement is a path. In light of general stabilization results for subgroups generated by “sufficiently high powers” of homeomorphisms or of group elements or of mapping classes as occur in right-angled Artin group theory and mapping class group theory (see for instance [10, 23, 20, 21, 22]), one might guess that after replacing elements of  $\mathcal{F}$  by sufficiently high powers, one would obtain the right-angled Artin group on the corresponding anti-path. It is the somewhat surprising stabilization of a 2-chain group at the isomorphism type of Thompson’s group  $F$  instead of the free group on two generators that provides the key for systematically approaching the study of chain groups. Finally, Proposition 1.8 gives the correct analogous general stabilization of isomorphism type result for general chain groups.

1.2.2. *Uncountable families of countable simple groups.* Within any natural class of countable (and especially finitely generated) groups, it is typical to encounter a countable infinity of isomorphism types, but uncountable infinities of isomorphism types often either cannot be exhibited, or finding them is somewhat non-trivial. Within the class of finitely generated groups, the Neumann groups provide some of the first examples of an uncountable family of distinct isomorphism types of two-generated groups. The reader may find a description of the Neumann groups in Section 5, where we use these groups allow us to find uncountably many isomorphism classes of chain groups.

Uncountable families of simple groups, or more generally groups with a specified property, are also difficult to exhibit in general. Uncountably many distinct isomorphism types of finitely generated simple groups can be produced as a consequence of variations on [2]. For other related results on uncountable families of countable groups satisfying various prescribed properties, the reader is directed to [26, 29, 4, 25, 31, 6, 19, 24], for instance. To the authors’ knowledge, the present work produces the first examples of uncountable families of distinct isomorphism types of countable subgroups of  $\text{Homeo}^+(I)$ , as well as an uncountable families of distinct isomorphism types of countable simple subgroups of  $\text{Homeo}^+(I)$  (and more generally of  $\text{Homeo}^+(M)$  for an arbitrary manifold  $M$ ).

## 2. PRELIMINARIES

We gather some well-known facts here which will be useful to us in the sequel. Throughout this paper, we fix the notations

$$a(x) = x + 1 \quad \text{and} \quad b(x) = \begin{cases} x & \text{if } x \leq 0, \\ 2x & \text{if } 0 < x < 1, \\ x + 1 & \text{if } 1 \leq x \end{cases}$$

and let  $F = \langle a, b \rangle \leq \text{Homeo}^+(\mathbb{R})$  be the *standard* copy of Thompson's group  $F$ .

**2.1. Thompson's group and Higman–Thompson group.** The canonical references for this section are the classical Cannon–Floyd–Parry notes [9] and the contemporary treatment in Burillo's in-progress book [8].

The group  $F$  was originally defined as the group of piecewise linear orientation preserving homeomorphisms of the unit interval with dyadic breakpoints, and where all slopes are powers of two. With the maps  $A, B \in \text{PL}^+(I)$  illustrated in Figure 2, we can write a presentation

$$F = \langle A, B \mid [AB^{-1}, A^{-k}BA^k] \text{ for } k = 1, 2 \rangle.$$

Using  $h \in \text{PL}^+(\mathbb{R})$  in Figure 2 (c), the maps  $a, b$  introduced above satisfy

$$a = hA^{-1}h^{-1}, \quad b = hB^{-1}h^{-1},$$

and hence,

$$F = \langle a, b \mid [b^{-1}a, a^kba^{-k}] \text{ for } k = 1, 2 \rangle = \langle u, b \mid [u, (bu)^k b (bu)^{-k}] \text{ for } k = 1, 2 \rangle.$$

Thompson's group  $F$  has a very diverse array of subgroups. An example of a finitely generated subgroup of  $F$  which is not finitely presented is the lamplighter group  $\mathbb{Z} \wr \mathbb{Z}$ . We record this fact for use later in the paper:

**Lemma 2.1.** *Thompson's group  $F$  contains a copy of the lamplighter group  $\mathbb{Z} \wr \mathbb{Z}$ .*

*Proof.* Pick a compactly supported nontrivial element  $f \in F$ . Then for all  $N \gg 0$ , it is easy to check that  $\langle a^N, f \rangle \cong \mathbb{Z} \wr \mathbb{Z}$ .  $\square$

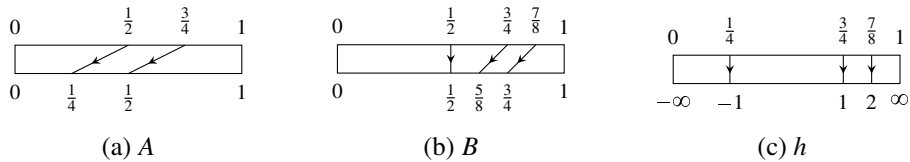


FIGURE 2. Elements  $A, B, h$  of Thompson's group  $F$ .

The Higman–Thompson groups  $\{F_n\}_{n \geq 2}$  are certain generalizations of Thompson’s group  $F$ , which first appear in [18] (see also [7], where the notation  $F_{n,1}$  is used for  $F_n$ ). Their relationship to Thompson’s group  $F$  is most evident from the following well-known presentation, which the reader may find in [8] for instance:

$$F \cong \langle \{g_i\}_{i \in \mathbb{Z}_{\geq 0}} \mid g_j^{g_i} = g_{j+1} \text{ if } 0 \leq i < j \rangle.$$

The group  $F_n$  can be defined by the following infinite presentation:

$$F_n = \langle \{g_i\}_{i \in \mathbb{Z}_{\geq 0}} \mid g_j^{g_i} = g_{j+n-1} \text{ if } 0 \leq i < j \rangle.$$

Observe that  $F_2$  is just Thompson’s group  $F$ . The groups  $\{F_n\}_{n \geq 2}$  have many remarkable algebraic features, the essential ones of which we record here:

**Lemma 2.2.** *For each  $n \geq 2$ , the group  $F_n$  has a trivial center and a simple commutator subgroup.*

It follows that  $F_n'' = F_n'$  and that every proper quotient of  $F_n$  is abelian.

**2.2. Higman’s Theorem.** The main technical tool we will require to prove Theorem 1.3 is Higman’s Theorem [17]. To properly state this result, consider a group  $H$  acting on a set  $X$ . For  $g \in H$ , we will retain standard notation and write

$$\text{supp } g = \{x \in X \mid g(x) \neq x\}.$$

**Theorem 2.3** (Higman). *Let  $H$  be a group acting faithfully on a set  $X$ , and suppose that for all triples  $r, s, t \in H \setminus \{1\}$  there is an element  $u \in H$  such that*

$$(\text{supp } r \cup \text{supp } s) \cap (t^u(\text{supp } r \cup \text{supp } s)) = \emptyset.$$

*Then the commutator subgroup  $H' = [H, H]$  is simple.*

The following definitions will be useful when Higman’s Theorem is applied.

**Definition 2.4.** Let  $X$  be a topological space, and let  $H \leq \text{Homeo}(X)$ .

- (1) We say  $H$  acts *CO-transitively* (or, *compact–open-transitively*) if for each proper compact subset  $A \subseteq X$  and for each nonempty open subset  $B \subseteq X$ , there is  $u \in H$  such that  $u(A) \subseteq B$ .
- (2) We say  $H$  is *compact–compressible* if for each proper compact subset  $A \subseteq X$  there is  $x \in X$  such that for each open neighborhood  $J$  of  $x$ , we have  $h(A) \subseteq J$  for some  $h \in H$ .

We shall use the following variation of Theorem 2.3; see also [15, 14, 3].

**Lemma 2.5.** *Let  $X$  be a non-compact Hausdorff space, and let  $\text{Homeo}_c(X)$  denote the group of compactly supported homeomorphisms. If  $H \leq \text{Homeo}_c(X)$  is CO-transitive, then  $H'$  is simple.*

*Proof.* In order to apply Higman's Theorem, we fix  $r, s, t \in H \setminus \{1\}$ . Since  $X$  is Hausdorff, we can choose a nonempty open set  $B \subseteq X$  such that  $B \cap tB = \emptyset$ . Choose a proper compact subset  $A$  containing  $\text{supp } r \cup \text{supp } s$ . By CO-transitivity, there is  $u \in H$  such that  $uA \subseteq B$ . From  $uA \cap tuA = \emptyset$ , we see

$$u(\text{supp } r \cup \text{supp } s) \cap tu(\text{supp } r \cup \text{supp } s) = \emptyset,$$

as desired in the condition of Higman's Theorem.  $\square$

Recall a group action on a topological space is *minimal* if every orbit is dense.

**Lemma 2.6.** *A minimal, compact-compressible group action is CO-transitive.*

*Proof.* Let  $G$  be the given group acting on a space  $X$ , let  $A \subseteq X$  be proper and compact, and let  $B \subseteq X$  be open. Suppose  $x$  is an accumulation point of the  $G$ -orbit of  $A$ , given by the hypothesis. By minimality, we can find an open neighborhood  $J$  of  $x$  and an element  $s \in G$  such that  $s(J) \subseteq B$ . Then there is  $r \in G$  such that  $r(A) \subseteq J$  and  $u = sr$  is a desired element.  $\square$

**2.3. Left-orderability and subgroups of  $\text{Homeo}^+(\mathbb{R})$ .** Recall that a group  $G$  is *left orderable* if there exists a total ordering  $\leq$  of  $G$  which is left invariant, i.e. for all  $g, h, k \in G$  we have  $h \leq k$  if and only if  $gh \leq gk$ . While orderability is an algebraic property of a group, it has a very useful dynamical interpretation. The reader may find the following fact as Theorem 2.2.19 of [28] (see also [13]):

**Lemma 2.7.** *Let  $G$  be a countable group. We have that  $G < \text{Homeo}^+(\mathbb{R})$  (equivalently  $G < \text{Homeo}^+(I)$ ) if and only if  $G$  is left orderable.*

In our proofs of Theorems 1.6 and its corollaries, we will need a recipe for assembling orderable groups from orderable pieces. We have the following well-known fact, whose proof we omit:

**Lemma 2.8.** *Let  $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$  be an extension of groups. Suppose that the groups  $Q$  and  $N$  are both left orderable. Then  $G$  is left orderable.*

### 3. DYNAMICAL ASPECTS

**3.1. Stabilization.** Let us note a dynamical condition that guarantees a prechain group to be a chain group.

**Lemma 3.1.** .

(1) *Suppose the supports of  $f, g \in \text{Homeo}^+(\mathbb{R})$  are given by*

$$\text{supp } f = (x, z), \quad \text{supp } g = (y, w)$$

*for some  $-\infty \leq x < y < z < w \leq \infty$ . If  $gf(y) \geq z$ , then  $\langle f, g \rangle \cong F$ .*

(2) If a prechain group  $G$  as in Setting 1.1 satisfies

$$f_n \cdots f_1(\text{supp } f_1 \setminus \text{supp } f_2) \cap (\text{supp } f_n \setminus \text{supp } f_{n-1}) \neq \emptyset,$$

then  $G$  is isomorphic to the Higman–Thompson group  $F_n$ .

*Proof.* (1) Note that  $f$  and  $g$  move points to the right. For  $k \geq 1$  we have that

$$\text{supp } f \cap \text{supp}(gf)^k g(gf)^{-k} = \text{supp } f \cap (gf)^k \text{supp } g = \emptyset.$$

So  $F$  surjects onto  $G = \langle f, g \rangle$ . Since every proper quotient of  $F$  is abelian and  $G$  is nonabelian, we see  $F \cong G$ .

(2) For  $0 \leq i \leq n-1$  and for  $q \geq 0$ , we define

$$h_i = f_{i+1}^{-1} \cdots f_{n-1}^{-1} f_n^{-1}, \quad h_{i+q(n-1)} = h_0^{-q} h_i h_0^q,$$

so that  $\langle h_0, \dots, h_{n-1} \rangle = G$ . By the same idea as in (1), we verify the relations of  $F_n$  in  $G$ :

$$h_j^{h_i} = h_{j+n-1} \text{ for } 0 \leq i < j.$$

Since every proper quotient of  $F_n$  is abelian, we have  $G \cong F_n$ .  $\square$

Let  $G = G_{\mathcal{F}}$  be a prechain group. We say that  $G$  stabilizes if for all  $N \gg 0$ , the groups  $\{G_N = \langle \{f^N \mid f \in \mathcal{F}\} \rangle\}$  form a single isomorphism class, called the *stable type* of  $G$ . The following asserts that an  $n$ -prechain group stabilizes for all  $n \geq 2$ .

**Theorem 3.2.** *For a prechain group  $G_{\mathcal{F}}$  as in Setting 1.1, the group  $G_N := \langle f^N \mid f \in \mathcal{F} \rangle$  is a chain group isomorphic to  $F_n$ , whenever  $N$  is sufficiently large.*

*Proof.* Let  $G = G_{\mathcal{F}}$  be a prechain group. By replacing the elements of  $\mathcal{F}$  by sufficiently high positive or negative powers, we may assume that the dynamical conditions of Lemma 3.1 hold. The resulting group generated by these powers of homeomorphisms will therefore be the desired chain group.  $\square$

For groups  $H \leq G$ , the normal closure of  $H$  in  $G$  is denoted as  $\langle\langle H \rangle\rangle_G$ .

**Lemma 3.3.** *For every chain group  $G$ , we have  $Z(G) = \{1\}$  and  $G'' = G'$ .*

*Proof.* Let  $G = G_{\mathcal{F}}$  be a chain group for  $\mathcal{F} = \{f_1, \dots, f_n\}$  as in Section 1. We may assume  $\text{supp } G = \mathbb{R}$ . If  $1 \neq g \in Z(G)$ , then  $\text{supp } g$  is  $G$ -invariant and accumulates at  $\pm\infty$ . Since the germ of  $G$  at  $-\infty$  is  $\langle f_1 \rangle$ , we see the restriction of  $g$  on  $(-\infty, t)$  for some  $t \in \mathbb{R}$  coincides with  $f_1^k$  for some  $k \neq 0$ . By the  $G$ -invariance, we see  $\text{supp } g = \mathbb{R}$ . As  $\text{supp } f_1$  is  $\langle g \rangle$ -invariant, we have a contradiction.

The second part of the lemma follows from

$$\begin{aligned} [f_i, f_{i+1}] &\in \langle f_i, f_{i+1} \rangle' = \langle f_i, f_{i+1} \rangle'' \leq G'' \trianglelefteq G, \\ G' &= \langle\langle \{[f_i, f_{i+1}] : 1 \leq i < n\} \rangle\rangle_G \leq G''. \end{aligned}$$

$\square$



**3.2. A dynamical dichotomy.** Let  $G$  be a group faithfully acting on  $\mathbb{R}$ . A closed nonempty  $G$ -invariant set  $\Lambda \subseteq \mathbb{R}$  is called a *minimal invariant set* of  $G$  if no proper nonempty closed subset of  $\Lambda$  is  $G$ -invariant. Such a set  $\Lambda$  exists in the case when  $G$  is finitely generated [27], but it may not be unique.

Assume a minimal invariant set  $\Lambda$  of  $G \leq \text{Homeo}^+(\mathbb{R})$  exists. We say  $G$  is *minimal* if  $\Lambda = \mathbb{R}$ . If  $\Lambda$  is perfect and totally disconnected, then  $\Lambda$  is called an *exceptional minimal invariant set*.

Let us first note two lemmas on general dichotomy of one-dimensional homeomorphism groups; the proofs are variations of [27, Section 2.1.2], which we omit.

**Lemma 3.4.** *Let  $G \leq \text{Homeo}^+(\mathbb{R})$  be a group such that  $\text{supp } G = \mathbb{R}$ .*

- (1) *If all the  $G$ -orbits share a common accumulation point  $x$ , then the closure  $\Lambda$  of  $Gx$  is the unique minimal invariant set; furthermore, either  $\Lambda = \mathbb{R}$  or  $\Lambda$  is exceptional.*
- (2) *If  $G$  admits an exceptional minimal invariant set  $\Lambda$ , then there exists a monotone continuous surjective map  $h: \mathbb{R} \rightarrow \mathbb{R}$  and a homomorphism  $\Phi: G \rightarrow \text{Homeo}^+(\mathbb{R})$  such that  $\Phi(G)$  is minimal and such that  $hg = \Phi(g)h$  for each  $g \in G$ .*

*Remark 3.5.* The group  $\Phi(G) \leq \text{Homeo}^+(\mathbb{R})$  above is called a *minimalization* of  $G$ . Note that  $h$  maps the closure of each component of  $\mathbb{R} \setminus \Lambda$  to a single point. Such a map  $h$  is called the *devil's staircase map*.

**Lemma 3.6.** *For a prechain group  $G$  as in Setting 1.1, all of the following hold.*

- (1) *There exists  $x \in \text{supp } G$  such that every orbit accumulates at  $x$ .*
- (2) *For each  $g \in G$  and for each compact set  $A \subseteq \text{supp } G$ , there exists  $u \in G'$  such that the actions of  $g$  and  $u$  agree as functions on  $A$ .*
- (3) *Every  $G$ -orbit is a  $G'$ -orbit.*
- (4)  *$G'$  is compact-compressible.*

*Proof.* We may assume  $\text{supp } G = \mathbb{R}$  as usual.

(1) Every orbit accumulates at  $x = \partial^+ f_1$ .

(2) Let us write  $g = s_k \cdots s_2 s_1$  for some  $s_i \in \mathcal{F} \cup \mathcal{F}^{-1}$ . There exist open intervals  $J \subseteq \text{supp } f_1$  and  $K \subseteq \text{supp } f_n$  such that

$$\partial^- J = \partial^- \text{supp } f_1, \quad \partial^+ K = \partial^+ \text{supp } f_n$$

and such that for each  $i \in \{1, 2, \dots, k\}$  we have  $s_i \cdots s_2 s_1(A) \cap (J \cup K) = \emptyset$ . There exists an  $h_i \in G$  such that  $h_i \text{supp } s_i \subseteq J \cup K$ . Then the following element has the desired properties:

$$u = \left( \prod_{i=1}^k h_i s_i^{-1} h_i^{-1} \right) g \in G'.$$

- (3) This is immediate by applying the part (2) for the case that  $A$  is a singleton.

(4) By the part (3), it suffices to show that  $G$  is compact-compressible. We let  $x = \partial^+ \text{supp } f_1$ . Let us fix a compact set  $A \subseteq \mathbb{R}$  and an open neighborhood  $J$  of  $x$ . For  $h = f_n \cdots f_1$ , we can find  $\ell, m \gg 0$  such that  $h^{-m}(A) \subseteq \text{supp } f_1$  and such that  $f_1^\ell h^{-m}(A) \subseteq J$ .  $\square$

We have the following dichotomy of chain group actions.

**Theorem 3.7.** *For a chain group  $G$ , exactly one of the following holds:*

- (i)  $G$  is minimal; in this case,  $G'$  is simple.
- (ii)  $G$  admits a unique exceptional minimal invariant set; in this case,  $G$  surjects onto a minimal chain group.

*Proof.* We may assume  $\text{supp } G = \mathbb{R}$ . From Lemmas 3.4 and 3.6 (1), we have that either  $G$  is minimal or exceptional.

Suppose  $G$  is minimal, and write  $H = G'$ . Lemma 3.6 implies that  $H$  is minimal and CO-transitive. Elements of  $H$  have the property that their supports are contained in compact intervals of  $\mathbb{R}$ , since the germs of  $G$  at  $-\infty$  and  $\infty$  are abelian quotients of  $G$ . Hence,  $H$  consists of compactly supported homeomorphisms of  $\mathbb{R}$ . Since  $H$  is perfect and nonabelian, Lemma 2.5 implies that  $H = H'$  is simple.

Suppose  $G$  is exceptional. We use the notation from Setting 1.1 and Lemma 3.4. The map  $h$  is two-to-one or one-to-one at each point in  $\Lambda$ . Since  $\Lambda$  accumulates at both boundary points of each interval  $\text{supp } f_i$ , the supports of the  $\Phi(f_i)$  are the non-degenerate open intervals  $h(\text{supp } f_i)$ , which together still form a chain of intervals for  $i = 1, 2, \dots, n$ . Since  $\langle f_i, f_{i+1} \rangle$  has only abelian proper quotients, we see  $\Phi(\langle f_i, f_{i+1} \rangle) \cong F$ . Hence  $\Phi(G)$  is a chain group which acts minimally on its support, namely  $\Phi(\Lambda)$ .  $\square$

It follows that every proper quotient of a minimal chain group is abelian.

#### 4. CHAIN GROUPS AND SUBGROUPS OF $\text{Homeo}^+(I)$

In this section, we prove Theorem 1.4, which is fundamental for establishing many of the remaining results claimed in the introduction.

**4.1. Groups of homeomorphisms.** For an interval  $J \subseteq \mathbb{R}$ , recall that  $\text{Homeo}^+(J)$  may be regarded as a subgroup of  $\text{Homeo}^+(\mathbb{R})$ , using the extension by the identity outside  $J$ . We continue to use the notations  $a, b \in \text{Homeo}^+(\mathbb{R})$  from Section 2. Let us note the following independent observation:

**Lemma 4.1.** *If  $G \leq \text{Homeo}^+(0, 1)$  is an  $n$ -generated group such that  $G/G' \cong \mathbb{Z}^n$  for some  $n \geq 1$ , then  $G$  embeds into  $\langle G, a \rangle' \leq \text{Homeo}^+(\mathbb{R})$ .*

*Proof.* Let us write

$$\begin{aligned} G &= \langle g_1, \dots, g_n \rangle, \quad G/G' \cong \bigoplus_{i=1}^n \mathbb{Z}g_i, \\ \bar{g}_i &= a^i g_i a^{-i} \in \text{Homeo}^+(i, i+1), \quad h_i = g_i \bar{g}_i^{-1} \in \text{Homeo}^+(\mathbb{R}), \\ H &= \langle h_1, \dots, h_n \rangle \leq \langle G, a \rangle'. \end{aligned}$$

It is straightforward to see that  $\phi(h_i) = g_i$  defines an isomorphism  $H \cong G$ .  $\square$

We define  $\mathcal{A}$  as the set of  $g \in \text{Homeo}^+(\mathbb{R})$  such that

$$g(x) \begin{cases} = x & \text{if } x \leq 0, \\ \in (x, x+1) & \text{if } 0 < x < 1, \\ = x+1 & \text{if } x \geq 1. \end{cases}$$

Note that  $b \in \mathcal{A}$ .

**Lemma 4.2.** *For each  $g_1, \dots, g_n \in \mathcal{A}$ , the group  $\langle g_1, \dots, g_n, a \rangle$  is abstractly isomorphic to an  $(n+1)$ -chain group.*

*Proof.* Write  $B = \{g_1, \dots, g_n\}$  and define

$$\begin{aligned} f_0 &= g_1^{-1} a, \quad f_n = a^{n-1} g_n a^{1-n}, \\ f_i &= (a^i g_{i+1}^{-1} a^{-i})(a^{i-1} g_i a^{1-i}) \text{ for } 1 \leq i \leq n-1, \\ \mathcal{F} &= \{f_0, \dots, f_n\}, \text{ and } G = \langle \mathcal{F} \rangle = \langle B, a \rangle. \end{aligned}$$

We make the following easy observations.

**Claim.** *We have the following:*

- (1)  $\text{supp } f_0 = (-\infty, 1)$ ,  $\text{supp } f_n = (n-1, \infty)$ .
- (2)  $\text{supp } f_i = (i-1, i+1)$  for  $1 \leq i \leq n-1$ .
- (3)  $f_{i+1} f_i(i) = i+1$  for  $1 \leq i \leq n-1$ .

Now Lemma 3.1 and the claim above imply that  $\mathcal{F}$  generates  $G$  as an  $(n+1)$ -chain group.  $\square$

**Corollary 4.3.** *Thompson's group  $F$  is isomorphic to an  $n$ -chain group for each  $n \geq 2$ .*

*Proof.* Apply Lemma 4.2 for  $g_1 = \dots = g_{n-1} = b \in \mathcal{A}$ .  $\square$

**Lemma 4.4.** *If  $g \in \text{Homeo}^+(1/4, 1/2) \leq \text{Homeo}^+(\mathbb{R})$ , and if  $c \in \mathcal{A}$  satisfies  $c(1/4, 1/2) = (1/2, 1)$ , then  $cg \in \mathcal{A}$ .*

*Proof.* If  $x \notin (1/4, 1/2)$ , then  $cg(x) = c(x)$ . If  $x \in (1/4, 1/2)$ , then

$$c^{-1}(x) < c^{-1}(1/2) = \frac{1}{4} < g(x) < \frac{1}{2} = c^{-1}(1) < c^{-1}(x+1). \quad \square$$

**Theorem 4.5.** *Let  $G = \langle h_1, h_2, \dots, h_n \rangle \leq \text{Homeo}^+(\mathbb{R})$  for some  $n \geq 1$ .*

- (1) *The group  $G$  embeds into an  $(n + 2)$ -chain group  $L$  such that  $L'$  is simple.*
- (2) *If, moreover,  $\text{supp } h_1$  has finitely many components then  $G$  embeds into an  $(n + 1)$ -chain group  $L$  such that  $L'$  is simple.*

*In both of the cases, if the additional hypothesis  $G/G' \cong \mathbb{Z}^n$  holds, then we can further require that  $G$  embeds into  $L'$ .*

*Proof.* (1) Write  $G = \langle B \rangle$  for some  $B = \{h_1, \dots, h_n\} \subseteq \text{Homeo}^+(\mathbb{R})$ . We may suppose  $\text{supp } G \subseteq (1/4, 1/2)$  after a suitable conjugation. Lemma 4.4 implies that  $bB, b \subseteq \mathcal{A}$ . From Lemma 4.2, we see that  $L = \langle B, b, a \rangle = \langle bB, b, a \rangle$  is an  $(n + 2)$ -chain group. Moreover,  $L$  acts minimally since  $F = \langle a, b \rangle$  does as well. Hence,  $L'$  is simple by Theorem 3.7.

If  $G/G' \cong \mathbb{Z}^n$ , then Lemma 4.1 gives us a desired embedding  $G \hookrightarrow \langle G, a \rangle' \hookrightarrow L'$ .

(2) We assume  $B = \{h_1, \dots, h_n\} \subseteq \text{Homeo}^+(1/4, 1/2)$ , and define  $L = \langle B, a, b \rangle$ . We may further suppose that  $h_1 \in F = \langle a, b \rangle$ , possibly after a conjugation. Then

$$L = \langle B \setminus \{h_1\}, b, a \rangle = \langle b(B \setminus \{h_1\}), b, a \rangle$$

is an  $(n + 1)$ -chain group. The rest of the proof is identical to the part (1).  $\square$

Theorem 1.4 is implied by the part (1).

**4.2. Isomorphisms between chain groups.** In this subsection, we prove Proposition 1.5, which follows fairly easily from the ideas in Subsection 4.1.

**Theorem 4.6.** *For  $m \geq n \geq 2$ , every  $n$ -chain group is isomorphic to an  $m$ -chain group.*

*Proof.* The case  $n = 2$  follows from Corollary 4.3. Assume  $n \geq 3$  and let  $\{f_1, f_2, \dots, f_n\}$  generate  $G$  as an  $n$ -chain group. We will show that  $G$  is isomorphic to an  $(n + 1)$ -chain group, which will establish the result by an easy induction. We consider the rightmost three intervals in the chain and we write  $p = f_{n-2}$ ,  $q = f_{n-1}$ , and  $r = f_n$ . We assume each generator moves points to the right in its support.

We let  $M$  be a sufficiently large integer which will be determined later. Define

$$d = q^{-M} p q^M, \quad f = (pr)^M q (pr)^{-M}, \quad e = f^{-M} q f^M.$$

Notice that  $d, e, f, r$  generate a 4-prechain group, as shown in Figure 3.

Lemma 3.1 implies that for  $M \gg 0$ , we have  $\langle e, f^M \rangle \cong \langle q, f^M \rangle \cong \langle f^M, r \rangle \cong F$ . Moreover, we have

$$\begin{aligned} \langle f_{n-3}, d \rangle &\cong q^M \langle f_{n-3}, d \rangle q^{-M} = \langle f_{n-3}, p \rangle \cong F, \\ \langle d, e \rangle &\cong f^M \langle d, e \rangle f^{-M} = \langle d, q \rangle = \langle p, q \rangle \cong F, \\ \langle d, e, f^M, r \rangle &= \langle d, q, f^M, r \rangle = \langle p, q, f^M, r \rangle = \langle p, q, r \rangle. \end{aligned}$$

Finally, notice that the left endpoints of  $\text{supp } e$  and  $\text{supp } q$  coincide. Therefore,  $G$  is an  $(n + 1)$ -chain group generated by

$$f_1, \dots, f_{n-3}, d, e, f^M, r. \quad \square$$

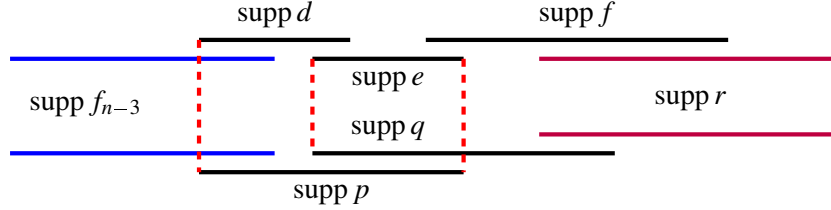


FIGURE 3. Converting an  $n$ -chain group to an  $(n + 1)$ -chain group.

#### 4.3. Chain group with non-simple commutator subgroup.

**Proposition 4.7.** *For each  $n \geq 3$ , there exists an  $n$ -chain group with a non-simple commutator subgroup.*

*Proof.* Let us describe an example of a 3-chain group with a non-simple commutator subgroup. We begin by finding a 3-chain group whose minimalization map is not injective. We will find such an example by blowing-up (sometimes called *Denjoy-ing* after [12]) an orbit. The case  $n \geq 4$  will then follow by Theorem 4.6.

By Lemma 4.2, the following generate  $F \leq \text{Homeo}^+(\mathbb{R})$  as a 3-chain group:

$$f_0 = b^{-1}a, \quad f_1 = ab^{-1}a^{-1}b, \quad f_2 = aba^{-1}.$$

Pick the orbit  $\mathcal{O} = F(0)$ , and let  $\{\ell_y : y \in \mathcal{O}\} \subseteq \mathbb{R}_{>0}$  satisfy  $\sum_{y \in \mathcal{O}} \ell_y = 1$ . Denote the closed interval  $[0, \ell_y]$  by  $I_y$ .

We now replace each point  $y \in \mathcal{O}$  by the interval  $I_y$ . Formally, we let

$$\mathbb{R}^\sim = (\mathbb{R} \setminus \mathcal{O}) \amalg \left( \coprod_{y \in \mathcal{O}} I_y \right)$$

Define  $q: \mathbb{R}^\sim \rightarrow \mathbb{R}$  by  $q(x) = x$  for  $x \in \mathbb{R} \setminus \mathcal{O}$  and  $q(I_y) = \{y\}$ , and topologize  $\mathbb{R}^\sim$  by the devil's staircase map  $q$ . Note  $\mathbb{R} \approx \mathbb{R}^\sim$ . We have a natural map

$$\Phi: \text{Homeo}_{\mathbb{R} \setminus \mathcal{O}}^+(\mathbb{R}^\sim) \rightarrow \text{Homeo}_{\mathcal{O}}^+(\mathbb{R})$$

from the set of  $\mathbb{R} \setminus \mathcal{O}$ -preserving homeomorphisms of  $\mathbb{R}^\sim$  to the set of  $\mathcal{O}$ -preserving homeomorphisms of  $\mathbb{R}$ . The map  $\Phi$  is determined by the condition

$$q \circ f = \Phi(f) \circ q \quad \text{for } f \in \text{Homeo}_{\mathbb{R} \setminus \mathcal{O}}^+(\mathbb{R}^\sim).$$

For each  $y, z \in \mathcal{O}$ , we will fix a homeomorphism  $h_z^y: I_y \rightarrow I_z$  which is *equivariant* in the following sense (for example, we can choose  $h_z^y$  to be linear):

$$h_y^y = \text{Id} \quad \text{and} \quad h_z^y \circ h_y^x = h_z^x \quad \text{for} \quad x, y, z \in \mathcal{O}.$$

Then for  $f \in \text{Homeo}_{\mathcal{O}}^+(\mathbb{R})$ , the equation

$$\phi(f) = \prod_{y \in \mathcal{O}} h_{f(y)}^y$$

defines a section  $\phi$  of  $\Phi$ .

Write  $g_i = \phi(f_i)$  for  $i = 0, 2$ . We now pick an arbitrary homeomorphism  $h$  such that  $\text{supp } h = I_1$ , and let  $g_1 = h \circ \phi(f_1)$ . Note that  $[h, g_0] = [h, g_2] = 1$ .

**Claim 1.** *The group  $G = \langle g_0, g_1, g_2 \rangle$  is a three-chain group.*

We note  $G$  is a prechain group as illustrated in Figure 4. The claim follows from the dynamical condition of Lemma 3.1.

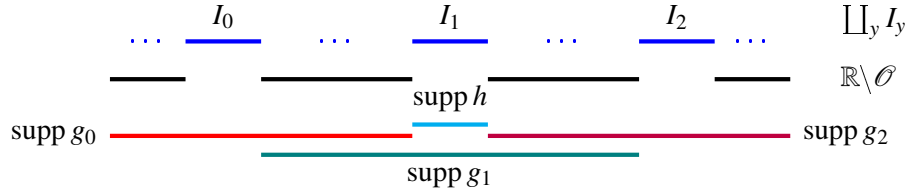


FIGURE 4. Blowing-up an orbit  $\mathcal{O}$ . Note that  $\mathbb{R} \setminus \mathcal{O}$  is totally disconnected.

**Claim 2.** *We have  $1 \neq h \in \ker \Phi \cap G$ .*

The part  $1 \neq h \in \ker \Phi$  follows from the definition. To show  $h \in G$ , we note  $g_1 g_0 g_2 g_1 (g_2 g_1 g_0)^{-1} = h^2 \phi((f_1 f_0)(f_2 f_1)(f_2 f_1 f_0)^{-1}) h^{-1} = h^2 \phi(ab^{-1} \cdot ba^{-1}) h^{-1} = h$ . So  $\Phi$  is not injective, as desired.  $\square$

It is straightforward to blow-up an orbit of a minimal chain group to obtain a non-minimal chain group which is algebraically isomorphic to the original one. The proof of Proposition 4.7 shows how to alter the isomorphism type by a blow-up.

## 5. UNCOUNTABILITY RESULTS

**5.1. Uncountable families of isomorphism types.** We first establish the existence of uncountably many isomorphism types of subgroups of  $\text{Homeo}^+(\mathbb{R})$ .

**Lemma 5.1.** *There exists a two-generated left orderable group  $\Gamma$  and a collection of normal subgroups  $\{N_i\}_{i \in I}$  of  $\Gamma$  with the following properties:*

- (1) *The collection  $\{N_i\}_{i \in I}$  is uncountable;*

- (2) For each  $i$ , the group  $N_i < \Gamma$  is central;  
(3) For each  $i \in I$ , the quotient  $\Gamma_i = \Gamma/N_i$  is left orderable.

The group  $\Gamma$  and its subgroups as in Lemma 5.1 appear in III.C.40 of de la Harpe's book [11]. Here, we merely observe that  $\Gamma$  and the quotients  $\Gamma_i$  are all left orderable.

*Proof of Lemma 5.1.* To establish the first two claims, we reproduce the argument given by de la Harpe nearly verbatim. Let  $S = \{s_i\}_{i \in \mathbb{Z}}$ , and let

$$R = \{[[s_i, s_j], s_k] = 1\}_{i,j,k \in \mathbb{Z}} \cup \{[s_i, s_j] = [s_{i+k}, s_{j+k}]\}_{i,j,k \in \mathbb{Z}}.$$

Define  $\Gamma_0 = \langle S \mid R \rangle$  and let  $\Gamma = \Gamma_0 \rtimes \mathbb{Z}$ , where the conjugation action of  $\mathbb{Z} = \langle t \rangle$  is given by  $t^{-1}s_it = s_{i+1}$ . For each  $i$ , we set  $u_i = [s_0, s_i]$ . Note the following easy observations:

- (i) The group  $[\Gamma_0, \Gamma_0]$  is central in  $\Gamma$ , is generated by  $\{u_i\}_{i \in \mathbb{Z}}$ , and is isomorphic to an infinite direct sum of copies of  $\mathbb{Z}$ ;
- (ii) The group  $\Gamma$  is generated by  $s_0$  and  $t$ ;
- (iii) The quotient group  $\Gamma/[\Gamma_0, \Gamma_0]$  is isomorphic to the lamplighter group  $\mathbb{Z} \wr \mathbb{Z}$ .

For each subset  $X \subset \mathbb{Z} \setminus \{0\}$ , we can consider the group  $N_X = \langle u_i \mid i \in X \rangle$ . Evidently these groups are distinct for distinct subsets of  $\mathbb{Z} \setminus \{0\}$ , and they are all normal because  $[\Gamma_0, \Gamma_0]$  is central in  $\Gamma$ . We thus establish the first two claims of the lemma.

For the third claim, note that the lamplighter group  $\mathbb{Z} \wr \mathbb{Z}$  is left orderable since it lies as a subgroup of  $F$  (see Lemma 2.1). For each  $X \subset \mathbb{Z} \setminus \{0\}$ , the groups  $[\Gamma_0, \Gamma_0]/N_X$  are all free abelian and therefore left orderable. By Lemma 2.8, it follows that the group  $\Gamma/N_X$  is left orderable.  $\square$

**Lemma 5.2.** *There exist uncountably many isomorphism types of two-generated subgroups of  $\text{Homeo}^+(\mathbb{R})$ .*

*Proof.* By Lemma 2.7, it suffices to prove that there are uncountably many isomorphism types of two-generated left orderable groups. To this end, suppose there exist only countably many isomorphism types of two-generated left orderable groups. Then the class of groups  $\mathcal{N} = \{\Gamma/N_X\}_{X \subset \mathbb{Z}}$  furnished by Lemma 5.1 consists of only countably many isomorphism types. It follows that there is an element  $N \in \mathcal{N}$  and uncountably many surjective homomorphisms  $\Gamma \rightarrow N$ . Since  $N$  and  $\Gamma$  are both finitely generated, this is a contradiction.  $\square$

Note that the case  $n \geq 4$  of Theorem 1.6 follows from Lemma 5.2.

**5.2. Uncountably many isomorphism types of 3-chain groups.** We now prove Theorem 1.6 in the case  $n = 3$ . We retain notation from Subsection 5.1 and write  $\mathcal{N} = \{\Gamma/N_X\}_{X \subset \mathbb{Z}}$ . As before, each  $N \in \mathcal{N}$  is generated by two elements,  $s (= s_0)$  and  $t$ .

**Lemma 5.3.** *Let  $N \in \mathcal{N}$  be generated by elements  $s, t \in N$ . There exists a faithful action of  $N$  on  $\mathbb{R}$  such that the element  $t \in N$  acts without fixed points.*

*Proof.* Let  $<_N$  be a left-invariant ordering on  $N$  as furnished by Lemma 2.8, using the surjection  $\phi: N \rightarrow N/([\Gamma_0, \Gamma_0]/N_X) \cong \mathbb{Z} \wr \mathbb{Z}$ . In this ordering, every positive element which is nontrivial under the surjection to the lamplighter group  $\mathbb{Z} \wr \mathbb{Z}$  is strictly larger than any element in  $[\Gamma_0, \Gamma_0]/N_X$ .

We first specify the ordering on  $\mathbb{Z} \wr \mathbb{Z}$  which we use. Write  $K$  for the kernel of the map  $\mathbb{Z} \wr \mathbb{Z} \rightarrow \mathbb{Z}$  given by  $s \mapsto 0$ . Observe now that we can realize  $\mathbb{Z} \wr \mathbb{Z}$  as acting on  $\mathbb{R}$ , where  $t$  acts by  $x \mapsto x + 1$  and where  $s$  is an arbitrary homeomorphism supported on the interval  $(0, 1/2)$ . This action of  $\mathbb{Z} \wr \mathbb{Z}$  on  $\mathbb{R}$  gives rise to an ordering  $<_{\mathbb{Z} \wr \mathbb{Z}}$  on  $\mathbb{Z} \wr \mathbb{Z}$  with  $k <_{\mathbb{Z} \wr \mathbb{Z}} t$  for any  $k \in K$ , and we use this ordering to build an ordering  $<_N$  on  $N$  via Lemma 2.8.

To obtain an action of  $N$  on  $\mathbb{R}$ , we follow the Ghys' dynamical realization (see [28]). We define a map  $\tau: N \rightarrow \mathbb{Q} \subset \mathbb{R}$  inductively. First, we set  $\tau(1) = 0$ , we set  $\tau(t^{-1}) = -1$ , and we set  $\tau(t) = 1$ . We now enumerate the remaining elements of  $N$  arbitrarily, subject to the following requirement: let  $C_n \subset N$  be the coset of  $\phi^{-1}(K) < N$  corresponding to  $t^n$ . If  $n \geq 0$  then we require  $t^n$  and  $t^{n+1}$  to appear on the list before any other element of  $C_n$  appears. Similarly, if  $n \leq 0$  then we require  $t^n$  and  $t^{n-1}$  to appear on the list before any other element of  $C_n$  appears. Moreover, if  $t^m$  appears before  $t^n$  on the list then we require  $|m| \leq |n|$ .

With this enumeration, we extend  $\tau$  to all of  $N$ . Having defined  $\{\tau(g_1), \dots, \tau(g_n)\}$ , we consider the next element  $g_{n+1}$  on the list. If  $g_{n+1} >_N \max\{g_1, \dots, g_n\}$  then we set  $\tau(g_{n+1}) = \max\{\tau(g_1), \dots, \tau(g_n)\} + 1$ . Similarly, if  $g_{n+1} <_N \min\{g_1, \dots, g_n\}$  then we set  $\tau(g_{n+1}) = \min\{\tau(g_1), \dots, \tau(g_n)\} - 1$ . Otherwise, we have  $g_i <_N g_{n+1} <_N g_j$  for some  $i, j \leq n$ , with no other  $g_k$  lying in the interval  $(g_i, g_j)$  with respect to the ordering  $<_N$  for  $k \leq n$ . We then set  $\tau(g_{n+1}) = (\tau(g_i) + \tau(g_j))/2$ . It is clear that  $\tau$  is an injective function  $N \rightarrow \mathbb{Q}$  which preserves order. Moreover,  $N$  acts on  $\tau(N)$  by left translation, and this action extends to a faithful action on  $\mathbb{R}$ , as explained in [28].

We claim that  $t$  acts on  $\mathbb{R}$  without any fixed points. Observe that with this definition of  $\tau$ , we have that  $\tau(t^n) = n$ , which shows that the left translation action of  $t$  on  $\tau(N)$  restricts to translation on  $\mathbb{Z}$ , so that the action of  $t$  on  $\mathbb{R}$  cannot have any fixed points. To see that  $\tau(t^n) = n$ , we proceed by an easy induction, with the cases  $n \in \{-1, 0, 1\}$  being covered by the definition. Let  $g \in N$  be arbitrary, and let  $n$  be such that  $g \in C_n$ , where without loss of generality we assume that  $n \geq 0$ .

Assume first that  $g$  is not a power of  $t$ . Then by induction we have that  $\tau(t^{n+1})$  has been defined and is equal to  $n + 1$ . It follows that  $g$  is not larger than all the other elements on the list with respect to  $<_N$ , since  $t^{-n}g \in C_0$  and every element in  $C_0 = \phi^{-1}(K)$  is less than  $t^{-n}t^{n+1} = t$ . In particular,  $\tau(g)$  will take on some rational value in  $(-1, n + 1)$ .



Finally, we consider the case where  $g = t^{n+2}$ , where again without loss of generality we assume  $n \geq 0$ . We claim that  $g$  is larger than any other element on the list. The previous largest element on the list was  $t^{n+1}$ , and  $t^{n+2} \succ_N t^{n+1}$  by definition. Since  $\tau(t^{n+1}) = n + 1$ , we have  $\tau(t^{n+2}) = n + 2$ , which completes the induction.  $\square$

Theorem 4.5 and Lemma 5.3 trivially imply the following:

**Lemma 5.4.** *Each  $N \in \mathcal{N}$  embeds into the commutator subgroup of a minimal 3-chain group.*

*Proof of Theorem 1.6, in the case  $n \geq 3$ .* Note that the commutator groups arising from the conclusion of Lemma 5.4 are simple by Theorem 3.7. Let  $\mathcal{C}$  be the class of the isomorphism types of  $n$ -chain groups that have simple commutator subgroups, and  $\mathcal{C}_0 \subseteq \mathcal{C}$  be the subclass consisting of finitely presentable ones. Then  $\mathcal{C}$  is uncountable from Lemma 5.4 and from that each countable group admits countably many isomorphism types of finitely generated subgroups. Since  $\mathcal{C}_0$  is obviously countable, we have that  $\mathcal{C} \setminus \mathcal{C}_0$  is uncountable.  $\square$

*Proof of Corollary 1.7.* The commutator subgroup of each group in  $\mathcal{C}$  has at most countably many finitely generated subgroups. Since each  $N \in \mathcal{N}$  embeds into some group  $\mathcal{C}$ , we have the desired conclusion.  $\square$

**5.3. Smoothability of chain groups.** We conclude this section with a remark on the diversity of non- $C^2$ -smoothable chain groups. Recall the definition of the integral Heisenberg group:  $H = \langle x, y, z \mid [x, y]z^{-1} = [x, z] = [y, z] = 1 \rangle$ .

**Proposition 5.5.** *There exist uncountably many distinct isomorphism types of 3-chain groups which admit no faithful  $C^2$  actions on  $[0, 1]$ .*

*Proof.* Let  $X \subset \mathbb{Z} \setminus \{0\}$  be proper, and let  $G = \Gamma/N_X \in \mathcal{N}$ . Fix  $i \in \mathbb{Z} \setminus (X \cup \{0\})$ . Then we have a map  $j: H \rightarrow G$  defined by  $(x, y, z) \mapsto (s_0, s_i, [s_0, s_i])$ . Since the abelianization of  $\Gamma_0$  is torsion-free and freely generated by the generators  $\{s_i \mid i \in \mathbb{Z}\}$ , we have that  $\ker j \leq H' = Z(H) = \langle z \rangle$ . If  $j$  is not injective then the image of  $z$  in  $\Gamma_0$  is either trivial or has finite order. However,  $u_i$  has infinite order in  $\Gamma_0/N_X$ , so that  $j$  must be injective.

The Plante–Thurston Theorem [30] states that every nilpotent subgroup of  $\text{Diff}^2(I)$  is abelian. Since  $H$  is nonabelian nilpotent and embeds into  $G$ , we see  $G$  does not admit a faithful  $C^2$ -action on  $[0, 1]$ .  $\square$

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