

Chain groups of homeomorphisms of the interval and the circle

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ABSTRACT. We introduce and study the notion of a chain group of homeomorphisms of a one-manifold, which is a certain generalization of Thompson’s group F . Precisely, this is a group finitely generated by homeomorphisms, each supported on exactly one interval in a chain, and subject to a certain mild dynamical condition. The resulting class of groups exhibits a combination of uniformity and diversity. On the one hand, a chain group either has a simple commutator subgroup or the action of the group has a wandering interval. In the latter case, the chain group admits a canonical quotient which is also a chain group, and which has a simple commutator subgroup. Moreover, any 2-chain group is isomorphic to Thompson’s group F . On the other hand, every finitely generated subgroup of $\text{Homeo}^+(I)$ can be realized as a subgroup of a chain group. As a corollary, we show that there are uncountably many isomorphism types of chain group, as well as uncountably many isomorphism types of countable simple subgroups of $\text{Homeo}^+(I)$. We also study the restrictions on chain groups imposed by actions of various regularities, and show that there are uncountably many isomorphism types of chain groups which cannot be realized by C^2 diffeomorphisms.

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1. INTRODUCTION

In this paper, we introduce and study the notion of a chain group of homeomorphism of a connected one–manifold. A chain group can be viewed as a generalization of Thompson’s group F which sits inside of the group of homeomorphisms of the manifold in a particularly nice way.

Let M be a connected one–manifold, and let

$$\mathcal{J} = \{J_0, \dots, J_{n-1}\}$$

be a collection of nonempty open subintervals of M . We call \mathcal{J} an *chain of intervals* (or an *n–chain of intervals* if the cardinality of \mathcal{J} is important) if $J_i \cap J_k = \emptyset$ if $|i - k| > 1$, and if $J_i \cap J_{i+1}$ is a proper nonempty subinterval of J_i and J_{i+1} for $0 \leq i \leq n - 2$. If $M = S^1$, we also allow $J_i \cap J_{i+1}$ to be nonempty for all indices modulo n , in which case we call the resulting configuration an *ring of intervals*.

Given an chain or ring of intervals

$$\mathcal{J} = \{J_0, \dots, J_{n-1}\},$$

we consider a collection of homeomorphisms

$$\mathcal{F} = \{f_0, \dots, f_{n-1}\}$$

such that the *support* $\text{supp } f_i = J_i$. That is, $f_i(x) \neq x$ if and only if $x \in J_i$. We set

$$G = G_{\mathcal{F}} < \text{Homeo}(M)$$

to be the subgroup generated by \mathcal{F} . We call G a *prechain group* or a *pre-ring group*, and we write

$$\text{supp } G = \bigcup_{J \in \mathcal{J}} J.$$

We say that G is a *chain group* or *ring group* (sometimes *n–chain group* and *n–ring group*) if for all $f_i, f_j \in \mathcal{F}$ we have $[f_i, f_j] = 1$ whenever $|i - j| > 1$, and if $\langle f_i, f_{i+1} \rangle \cong F$ for $0 \leq i \leq n - 1$. Here, F denotes *Thompson’s group* F . Whereas this definition may seem rather unmotivated, part of the purpose of this article is to convince the reader that chain groups are natural objects.

In this paper, we will consider chain groups as both abstract groups and as groups with a distinguished finite generating set \mathcal{F} as above. Whenever we mention “the generators” of a chain group, we always mean the distinguished generating set \mathcal{F} which realizes the group as a group of homeomorphisms.

1.1. Main results. Elements of the class of chain groups enjoy many properties which are mostly independent of the choices of the homeomorphisms generating them, and at the same time can be very diverse. Moreover, chain groups are abundant in one–dimensional dynamics. Our first result establishes the naturality of chain groups:

Theorem 1.1. *Let \mathcal{J} be an n -chain (or ring) of intervals in M and let G be a n -prechain (or pre-ring) group generated by \mathcal{F} as above. If $M \cong S^1$, assume that $n \geq 3$ or that \mathcal{J} does not cover M . Then for all $N \gg 0$, the group*

$$G_N := \langle f^N \mid f \in \mathcal{F} \rangle < \text{Homeo}(M)$$

is a chain group (or a ring group).

Choosing a two-chain group whose generators are C^∞ diffeomorphisms of \mathbb{R} , we obtain the following immediate corollary, which is a complement to a result of Ghys–Sergiescu [16]:

Corollary 1.2. *Thompson’s group F can be realized as a subgroup of $\text{Diff}^\infty(M)$, the group of C^∞ orientation preserving diffeomorphisms of M .*

General chain groups have remarkably uncomplicated normal subgroup structure:

Theorem 1.3. *Let G be a chain group, and let $G' = [G, G]$ denote its derived subgroup. Then either:*

- (1) *The subgroup $G' < G$ is simple and every proper quotient of G is abelian;*
- (2) *The action of G has a wandering interval.*

Moreover, if $G' < G$ fails to be simple then there is a canonical surjection $G \rightarrow H$, where H is also a chain group, and where $H' < H$ is simple.

Here, we say that $J \subset \text{supp } G$ is a *wandering interval* if J is a nonempty open interval and if for each $g \in G$ we either have $g(J) = J$ or $g(J) \cap J = \emptyset$.

For ring groups, the analogue of Theorem 1.3 is a little more subtle (see Section 10):

Theorem 1.4. *Let*

$$\mathcal{F} = \{f_0, \dots, f_{n-1}\} \subset \text{Homeo}^+(S^1)$$

generate a ring group, where here $n \geq 5$. Suppose there is an i such that the action of $\langle f_i, f_{i+1} \rangle$ on its support has a dense orbit, where here the indices are considered modulo n . Then every proper quotient of G is abelian and G' is perfect.

A general chain group may fail to have a simple commutator subgroup, but it can be embedded in a larger chain group which does have a simple commutator subgroup:

Theorem 1.5. *For $n \geq 4$ there exists an n -chain group G such that $G' < G$ is not simple. However if G is an arbitrary chain group, then there exists an $(n+1)$ -chain group H such that $G < H$ and such that $H' < H$ is simple.*

We remark that Theorem 1.5 does not appear to have an immediate analogue in the context of ring groups. Even though the dense orbit versus wandering interval dichotomy still holds formally, it is unclear how to extract algebraic consequences from this fact.

As opposed to their normal subgroup structure, the subgroup structure of chain groups can be extremely complicated:

Theorem 1.6. *Let $H < \text{Homeo}^+(I)$ be an n -generated subgroup. Then there is an $(n + 2)$ -chain group G such that $H < G$.*

It is easy to see that Theorem 1.6 generalizes immediately to ring groups. Theorem 1.6 gives control over the number of generators in a chain group needed to accommodate a particular finitely generated subgroup of $\text{Homeo}^+(I)$. However, there does not seem to be a suitable notion of the “rank” of a chain group. Indeed, the next proposition shows that a given n -chain group not only contains m -chain groups for all m , but is in fact isomorphic to an m -chain group for all $m \geq n$:

Proposition 1.7. *Let G be an n -chain group, where $n \geq 2$. Then for all $m \geq n$, the group G is isomorphic to an m -chain group.*

Proposition 1.7 generalizes suitably to n -ring groups; see Proposition 5.7.

Theorem 1.6, together with certain strengthened versions of it, have several consequences which we note here. The following result stands in stark contrast to Theorem 1.1 in the case $n = 2$, which implies that there is only one isomorphism type of 2-chain groups:

Theorem 1.8. *For all $n \geq 3$, there exist uncountably many isomorphism types of n -chain groups.*

Corollary 1.9. *For all $n \geq 3$, there exist uncountably many isomorphism types of infinitely presented n -chain groups.*

In the case $n = 3$, Corollary 1.9 answers a question posed to the authors by J. McCammond.

Here in Corollary 1.9 and throughout this paper, we will say that a group is *infinitely presented* if it is not finitely presentable. For $n \geq 4$ we have that Theorem 1.8 and Corollary 1.9 follow from Theorem 1.6, once uncountable families of distinct isomorphism types of finitely generated subgroups of $\text{Homeo}^+(S^1)$ have been produced. To improve the bound to $n \geq 3$, we need to work a little harder. See Subsection 6.2.

We note that Theorem 1.8 and Corollary 1.9 immediately generalize to n -ring groups for $n \geq 5$.

A relatively slight improvement of Theorem 1.6 shows that one can embed uncountably many different isomorphism types of two-generated subgroups of the

homeomorphism group $\text{Homeo}^+(I)$ into the commutator subgroups of 5–chain groups. Combining this fact with Theorem 1.3, we obtain the following result:

Corollary 1.10. *For every one–manifold M , there exist uncountably many isomorphism types of countable simple subgroups of $\text{Homeo}^+(M)$. These simple groups can be realized as the commutator subgroups of 4–chain groups.*

The simple subgroups we produce in Corollary 1.10 will necessarily be infinitely generated, at least in the case where $M \cong I, \mathbb{R}$. We remark that it is not difficult to establish Corollary 1.10 for general n –manifolds, though we will not require such a statement here. See Section 7.

Theorem 1.8 and its corollaries above show that chain and ring groups can be very diverse. However, both chain and ring groups exhibit a remarkable phenomenon called *stabilization* (see Subsection 6.3 for precise definitions and discussion). We have the following results, which show that chain groups and ring groups acquire certain *stable isomorphism types*:

Proposition 1.11. *Let $G = G_{\mathcal{F}}$ be an n –chain group for $n \geq 2$. Then for all $N \gg 0$, the group*

$$G_N = \langle f^N \mid f \in \mathcal{F} \rangle$$

is isomorphic to the Higman–Thompson group F_n .

An immediate analogue in the case of ring groups holds:

Proposition 1.12. *Let $G = G_{\mathcal{F}}$ be an n –ring group for $n \geq 3$, with $\text{supp } G = S^1$. Then for all $N \gg 0$, the group*

$$G_N = \langle f^N \mid f \in \mathcal{F} \rangle$$

is isomorphic to the Higman–Thompson group T_n .

The Higman–Thompson groups $\{F_n\}_{n \geq 2}$ and $\{T_n\}_{n \geq 3}$ are defined and discussed in Subsection 2.2 below. We remark that Propositions 1.11 and 1.12 were observed independently (and in a different context) by Bleak–Brin–Kassabov–Moore–Zaremsky [5].

Proposition 1.11, combined with Proposition 1.7, implies the following corollary about the homology of chain groups:

Corollary 1.13. *For $n \in \mathbb{N}$ and all $2 \leq k \leq n$, there exists an n –chain group G_k such that $H_1(G_k, \mathbb{Z}) \cong \mathbb{Z}^k$.*

Observe that by considering the germs of the chain group at $\partial \text{supp } G$, we always have a surjective map $G \rightarrow \mathbb{Z}^2$. In particular, the rank of $H_1(G, \mathbb{Z})$ for any chain group G is at least two.

Another level of subtlety in the structure of chain groups concerns the degree of regularity of the generators of a chain group, as we have already suggested in

Corollary 1.2. For instance, one can strengthen the requirement that the generators be homeomorphisms and require instead that they are C^k for some $k \geq 1$, or even C^∞ . We will therefore say that a chain group G is *realized* by C^k diffeomorphisms if there is a chain group generated by C^k diffeomorphisms which is isomorphic to G . Before stating any results we adopt several conventions when talking about diffeomorphisms. We will denote by

$$\text{Diff}^k(I) < \text{Homeo}^+(I)$$

the group of C^k diffeomorphisms of I with k^{th} -order derivatives which exist at the endpoints but which can be arbitrary. We will denote by

$$\text{Diff}_0^k(I) < \text{Diff}^k(I)$$

the group of C^k diffeomorphisms of I which can be extended by the identity to C^k diffeomorphisms of \mathbb{R} . We remark that diffeomorphisms in $\text{Diff}_0^k(I)$ are sometimes called *tangent to the identity* up to order k , though we will not use this terminology here.

Proving that a given chain group generated by homeomorphisms cannot be smoothed seems rather difficult, but nevertheless we can (in a somewhat indirect way) prove the following:

Theorem 1.14. *For $n \geq 4$, there exist n -chain groups which are realized as subgroups of $\text{Diff}_0^1(I)$ but not as subgroups of $\text{Diff}_0^2(I)$. Moreover, there exist uncountably many distinct isomorphism types of chain groups which are realized by homeomorphisms but not as subgroups of $\text{Diff}_0^2(I)$.*

We immediately obtain the following corollary:

Corollary 1.15. *There exist uncountably many isomorphism types of 4-generated subgroups of $\text{Homeo}^+(I)$ which do not occur as subgroups of $\text{Diff}_0^2(I)$.*

We remark that one can weaken the C^2 hypothesis in both Theorem 1.14 and Corollary 1.15 to C^1 with derivatives of bounded variation. Finally, we have the following refinement of Theorem 1.6:

Theorem 1.16. *If $H < \text{Diff}_0^k(I)$ is n -generated then H embeds in an $(n+2)$ -chain subgroup of $\text{Diff}_0^k(I)$.*

It is easy to see that Theorem 1.14 and Corollary 1.15 generalize suitably to ring groups.

1.2. Notes and references.

1.2.1. *Motivations.* The original motivation for studying chain groups comes from the first and second authors' joint work with Baik [2] which studied right-angled Artin subgroups of $\text{Diff}^2(S^1)$. If

$$G = G_{\mathcal{F}} < \text{Homeo}^+(I)$$

is a prechain group, then one can form a graph whose vertices are elements of \mathcal{F} , and where an edge is drawn between two vertices if the supports are disjoint. The graph one obtains this way is an *anti-path*, i.e. a graph whose complement is a path. In light of general stabilization results for subgroups generated by “sufficiently high powers” of homeomorphisms or group elements or mapping classes as occur in right-angled Artin group theory and mapping class group theory (see for instance [12, 24, 21, 22, 23]), one might guess that after replacing elements of \mathcal{F} by sufficiently high powers, one would obtain the right-angled Artin group on the corresponding anti-path. It is the somewhat surprising stabilization of a 2-chain group at the isomorphism type of Thompson's group F instead of the free group on two generators that provides the key for systematically approaching the study of chain groups. Finally, Propositions 1.11 and 1.12 give the correct analogous general stabilization of isomorphism type result for general chain and ring groups.

1.2.2. *Uncountable families of countable simple groups.* Within any natural class of countable (and especially finitely generated) groups, it is typical to encounter a countable infinity of isomorphism types, but uncountable infinities of isomorphism types often either cannot be exhibited, or finding them is somewhat non-trivial. Within the class of finitely generated groups, the Neumann groups provide some of the first examples of an uncountable family of distinct isomorphism types of 2-generated groups. The reader may find a description of the Neumann groups in Section 6, where we use these groups allow us to find uncountably many isomorphism classes of chain (and ring) groups.

Uncountable families of simple groups, or more generally groups with a specified property, are also difficult to exhibit in general. Uncountably many distinct isomorphism types of finitely generated simple groups can be produced as a consequence of variations on [3]. For other related results on uncountable families of countable groups satisfying various prescribed properties, the reader is directed to [28, 31, 4, 27, 33, 7, 19, 25], for instance. To the authors' knowledge, the present work produces the first examples of uncountable families of distinct isomorphism types of countable subgroups of $\text{Homeo}^+(I)$, as well as an uncountable families of distinct isomorphism types of countable simple subgroups of $\text{Homeo}^+(I)$ (and more generally of $\text{Homeo}^+(M)$ for an arbitrary manifold M).

1.3. **Questions.** We conclude here with several questions, some undoubtedly rather difficult, which naturally arise out of this work.

Question 1.17. *Which chain groups are amenable? Which chain groups are sofic?*

We remark that the statement that all chain groups are nonamenable is equivalent to the statement that Thompson's group F is nonamenable. There are chain groups that are nonamenable and do not contain nonabelian free subgroups. In fact, the finitely presented example of a nonamenable group without a free subgroup, constructed by the third author in joint work with Moore, also admits a description as a 3-chain group:

Proposition 1.18. *The finitely presented group G_0 appearing in [26] admits a description as a 3-chain group.*

Proof. The group G_0 is generated by the map $a(x) = x + 1$ together with the maps

$$b(x) = \begin{cases} x & \text{if } x \leq 0 \\ \frac{x}{1-x} & \text{if } 0 \leq x \leq \frac{1}{2} \\ 3 - \frac{1}{x} & \text{if } \frac{1}{2} \leq x \leq 1 \\ x + 1 & \text{if } 1 \leq x \end{cases} \quad c(x) = \begin{cases} \frac{2x}{1+x} & \text{if } 0 \leq x \leq 1 \\ x & \text{otherwise} \end{cases}$$

One checks that the elements

$$ab^{-1}c^{-1}, cb(a^4b^{-1}a^{-4}), a^4ba^{-4} \in G_0$$

generate the group, with generators supported on

$$(-\infty, 2), (0, 6), (4, \infty)$$

respectively. Moreover, the supports of these elements form a 3-chain, and any pair of these generators either commutes or generates a copy of F , so that the resulting group is a 3-chain group. \square

Recall that a group G satisfies the finiteness property F_n if it admits a classifying space with finitely many cells in its n -skeleton.

Question 1.19. *Let $n \in \mathbb{N}$. Are there examples of chain groups that satisfy the finiteness property type F_n but not type F_{n+1} ?*

In the present work, we are able to embed a large class of finitely generated subgroups of $\text{Homeo}^+(I)$ into commutator subgroups of chain groups, which allows us to obtain results such as Corollary 1.10. The following more general question is unclear, however:

Question 1.20. *Can any finitely generated subgroup of $\text{Homeo}^+(I)$ be embedded in a commutator subgroup of a chain group?*

In a related vein:

Question 1.21. *Do there exist chain groups realized by C^k diffeomorphisms which are not finitely presentable, for $k \geq 2$?*

We know that there is only one isomorphism type of 2–chain groups. However, we can ask:

Question 1.22. *Can a 2–prechain group contain a nonabelian free group?*

2. PRELIMINARIES

We gather some well–known facts here which will be useful to us in the sequel.

2.1. Thompson’s group F . The classical reference for this section are the Cannon–Floyd–Parry notes [11]. The reader may also consult a contemporary treatment in Burillo’s in–progress book [9]. The group F was originally defined as the group of piecewise linear orientation preserving homeomorphisms of the unit interval with dyadic breakpoints, and where all slopes are powers of two.

Recall that each element f of $PL(I)$ can be represented by a *breakpoint notation*

$$((a_1, \dots, a_k), (b_1, \dots, b_k))$$

where $f(\partial I) = \partial I$ and $f(a_i) = b_i$ for each i , and moreover f is linear on each component of $I \setminus \{a_1, \dots, a_k\}$.

Thompson’s group F has the *standard presentation*

$$F = \langle a, b \mid [ab, b^a] = [ab, b^{a^2}] = 1 \rangle,$$

where

$$a = ((1/2, 3/4), (1/4, 1/2))$$

and where

$$b = ((1/2, 5/8, 3/4), (1/2, 3/4, 7/8));$$

see Figure 1. Here we use the convention

$$[f, g] = f g f^{-1} g^{-1}.$$

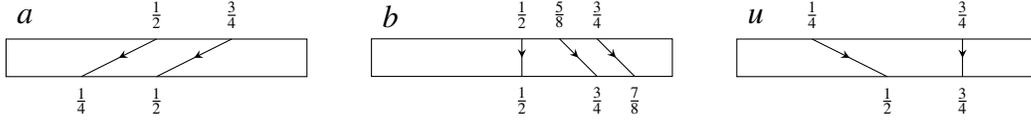
By the change of variables

$$u = (ab)^{-1} = ((1/4, 3/4), (1/2, 3/4))$$

we obtain another presentation

$$F = \langle u, b \mid [u, bub(bu)^{-1}] = [u, (bu)^2 b (bu)^{-2}] = 1 \rangle.$$

It follows immediately from the standard presentation that $F^{ab} \cong \mathbb{Z}^2$. The group F contains no nonabelian free groups and satisfies no law, by a result of Brin and Squier [6]. It follows immediately from the work of Brin and Squier together with the definition of a chain group that chain groups also satisfy no law whenever $n \geq 2$ since they contain copies of F , though in general they may contain nonabelian free groups by Theorem 1.6. We remark that groups of piecewise linear homeomorphisms with infinite numbers of breakpoints are significantly less restricted than F and do not seem particularly amenable to systematic study; see [1].

FIGURE 1. Generators of the Thompson's group F .

We shall use the piecewise projective and piecewise linear descriptions of F acting on the real line. In the 1970s, Thurston observed that the group generated by $a(x) = x + 1$ together with

$$b(x) = \begin{cases} x & \text{if } x \leq 0 \\ \frac{x}{1-x} & \text{if } 0 \leq x \leq \frac{1}{2} \\ 3 - \frac{1}{x} & \text{if } \frac{1}{2} \leq x \leq 1 \\ x + 1 & \text{if } 1 \leq x \end{cases}$$

is isomorphic to the group of piecewise $\mathrm{PSL}_2(\mathbb{Z})$ projective homeomorphisms of the real line with breakpoints at rational values, and is isomorphic to Thompson's group F .

The group F has the following remarkable property:

Proposition 2.1. *The commutator subgroup $[F, F]$ is an infinitely generated simple group. Moreover, the center of F is trivial.*

The simplicity of the commutator subgroup and the triviality of the center of F imply that every proper quotient of F is abelian. We will use Proposition 2.1 to prove Theorem 1.3, so that Proposition 2.1 is not a consequence of Theorem 1.3.

Proposition 2.1 allows for a strengthening of the piecewise $\mathrm{PSL}_2(\mathbb{Z})$ projective homeomorphism description of F . Indeed, it is easy to verify that the group generated by $a(x) = x + 1$ together with

$$b(x) = \begin{cases} x & \text{if } x \leq 0 \\ g(x) & \text{if } 0 \leq x \leq 1 \\ x + 1 & \text{if } 1 \leq x \end{cases}$$

where $g(x)$ is any homeomorphism of $[0, 1]$ onto $[0, 2]$, is isomorphic to Thompson's group F . To see this, one check that the two relations in the standard presentation

$$\langle a, b \mid [ab^{-1}, b^a], [ab^{-1}, b^{a^2}] \rangle$$

hold and since this group is not abelian, and since every proper quotient of F is abelian, our group is isomorphic to F . We remark that this presentation is obtained by a notational modification of the presentation for the interval mentioned above, in particular, b^{-1} replaces b here.

Thompson's group F has a very diverse array of subgroups, including subgroups $G_n < F$ which satisfy the finiteness property F_n but not F_{n+1} . An example of a finitely generated subgroup of F which is not finitely presented is the lamplighter group $\mathbb{Z} \wr \mathbb{Z}$. We record this fact for use later in the paper:

Lemma 2.2. *Thompson's group F contains a copy of the lamplighter group $\mathbb{Z} \wr \mathbb{Z}$.*

Proof. There are many copies of $\mathbb{Z} \wr \mathbb{Z}$ inside of F . For instance, we can realize $F < \text{Homeo}^+(\mathbb{R})$ as above in a way that $a(x) = x + 1$ lies in F , and we set $f \in F$ to be any nontrivial element with support contained in a compact subinterval of \mathbb{R} . Then for all $N \gg 0$, it is easy to check that $\langle a^N, f \rangle \cong \mathbb{Z} \wr \mathbb{Z}$. \square

2.2. The Higman–Thompson groups. The Higman–Thompson groups $\{F_n\}_{n \geq 2}$ and $\{T_n\}_{n \geq 3}$ are certain generalizations of Thompson's group F , which first appear in [18] (see also [8]). Their relationship to Thompson's group F is most evident from the following well-known presentation, which the reader may find in [9] for instance:

$$F \cong \langle \{g_i\}_{i \in \mathbb{Z}_{\geq 0}} \mid g_j^{g_i} = g_{j+1} \text{ if } i < j \rangle.$$

The group F_n can be defined by the following infinite presentation:

$$F_n = \langle \{g_i\}_{i \in \mathbb{Z}_{\geq 0}} \mid g_j^{g_i} = g_{j+n-1} \text{ if } i < j \rangle.$$

Observe that F_2 is just Thompson's group F . The groups $\{F_n\}$ share many algebraic features with Thompson's group F :

Lemma 2.3. *The groups $\{F_n\}$ are finitely presented for each $n \geq 2$. Moreover, the centers of these groups are trivial and their commutator subgroups are simple groups.*

The following proposition follows easily from the presentation for F_n :

Proposition 2.4. *We have $H_1(F_n, \mathbb{Z}) \cong \mathbb{Z}^n$.*

Sketch of proof. Observe that the abelianization of F_n is a quotient of

$$\bigoplus_{i=0}^{\infty} \mathbb{Z}_i,$$

where we think of \mathbb{Z}_i as generated by g_i . The conjugation relations in F_n show that $\{g_0, g_1, \dots, g_{n-1}\}$ are distinct in the abelianization of F_n . However, for $j \geq n$ we have $g_j = g_i$ in the abelianization of F_n , for some $i \leq n - 1$. \square

The groups $\{T_n\}_{n \geq 3}$ are certain finitely presented simple groups which are closely related to the groups $\{F_n\}$, but whose presentations are somewhat more complicated; the reader is directed to [8] for a detailed discussion.

2.3. Higman's Theorem. The main technical tool we will require to prove Theorem 1.3 is Higman's Theorem [17]. To properly state this result, consider a group G acting on a set X . For $g \in G$, we will retain standard notation and write

$$\text{supp } g = \{x \in X \mid g(x) \neq x\}.$$

Theorem 2.5. *Let G be a group acting faithfully on a set X , and suppose that for all triples $g_1, g_2, g_3 \in G \setminus \{1\}$ there is an element $h \in G$ such that*

$$(\text{supp } g_1 \cup \text{supp } g_2) \cap (g_3^h(\text{supp } g_1 \cup \text{supp } g_2)) = \emptyset.$$

Then the commutator subgroup $G' = [G, G]$ is simple.

We shall use the following consequence of Theorem 2.5 which is adapted to our setup.

Corollary 2.6. *Let G be a group of compactly supported orientation preserving homeomorphisms of $(0, 1)$. Suppose that for each pair of compact subintervals $I_1, I_2 \subset (0, 1)$, there is an $f \in G$ such that $f(I_1) \subset I_2$. Then G' is simple.*

Proof. We would like to show that for all

$$g_1, g_2, g_3 \in G \setminus \{1\},$$

there is a $h \in G$ such that

$$h(\text{supp } g_1 \cup \text{supp } g_2) \cap (g_3 h(\text{supp } g_1 \cup \text{supp } g_2)) = \emptyset.$$

It is straightforward to check that there is an interval $A \subset \text{supp } g_3$ such that $g_3(A) \cap A = \emptyset$. Let

$$J = \text{supp } g_1 \cup \text{supp } g_2 \subset (0, 1).$$

By our hypothesis, there is an $h \in G$ such that $h(J) \subseteq A$, which establishes the claim of the corollary. \square

2.4. Orderability and subgroups of $\text{Homeo}^+(\mathbb{R})$ and $\text{Homeo}^+(I)$. Recall that a group G is *left orderable* if there exists a total ordering \leq of G which is left invariant, i.e. for all $g, h, k \in G$ we have $h \leq k$ if and only if $gh \leq gk$. While orderability is an algebraic property of a group, it has a very useful dynamical interpretation. The reader may find the following fact as Theorem 2.2.19 of [30] (see also [14]):

Lemma 2.7. *Let G be a countable group. We have that $G < \text{Homeo}^+(\mathbb{R})$ if and only if G is left orderable.*

In our proofs of Theorems 1.8 and its corollaries, we will need a recipe for assembling orderable groups from orderable pieces. We have the following well-known fact:

Lemma 2.8. *Let*

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

be an extension of groups. Suppose that the groups Q and N are both left orderable. Then G is left orderable.

Proof. We let $<_Q$ and $<_N$ be the given orders on Q and N respectively, and we use them to build a left order on G . Let $g, h \in G$ be distinct elements, and consider $g^{-1}h$. We set $g <_G h$ if and only if either $1 \neq g^{-1}h$ in Q and $1 <_Q g^{-1}h$, or if $1 = g^{-1}h$ in Q and $1 <_N g^{-1}h$. It is straightforward to check that $<_G$ is indeed a left order on G . \square

We will pass freely between orientation preserving homeomorphisms of the real line and of the interval, and we will often use the following fact implicitly:

Lemma 2.9. *There are natural injective homomorphisms $\text{Homeo}^+(\mathbb{R}) \rightarrow \text{Homeo}^+(I)$ and $\text{Homeo}^+(I) \rightarrow \text{Homeo}^+(\mathbb{R})$.*

Proof. The existence of the second injective homomorphism follows immediately by including the unit interval into the real line, and extending homeomorphisms by the identity outside of I . To see the first homomorphism, we first realize a natural isomorphism $\text{Homeo}^+(\mathbb{R}) \cong \text{Homeo}^+((0, 1))$ realized by taking any homeomorphism between \mathbb{R} and $(0, 1)$. If $f \in \text{Homeo}^+((0, 1))$ then we can extend f to an element of $\text{Homeo}^+(I)$ by defining $f(0) = 0$ and $f(1) = 1$. For any such f , this is clearly the unique extension to I . \square

2.5. Regularity of group actions in one dimension. If M is a one-manifold, especially a compact one-manifold, there are significant algebraic differences between the groups of homeomorphisms of M , the group of once-differentiable diffeomorphisms of M , and the group of diffeomorphisms of M with higher degrees of differentiability. Thus when considering prechain and chain groups (as well as pre-ring and ring groups), it is natural to consider actions with various degrees of regularity. Indeed, there are certain groups which occur as subgroups of chain groups of homeomorphisms but not as subgroups of chain groups of diffeomorphisms of sufficiently high regularity. We will note our main example here, with discussion to follow in Section 9:

Theorem 2.10 (See [15, 20] and [32]). *Let N be a finitely generated residually torsion-free nilpotent group. Then for every one-manifold M , we have $N < \text{Diff}_0^1(M)$, the group of C^1 orientation preserving diffeomorphisms of M whose derivatives*

agree with the identity at the endpoints of M if $M \cong I$ or at infinity if $M \cong \mathbb{R}$. If $N < \text{Diff}^2(M)$ and M is compact, then N is abelian.

In the first part of Theorem 2.10, Farb and Franks show that for a torsion-free nilpotent subgroup N of $\text{Diff}^1(I)$, the generators can be chosen arbitrarily close to the identity and the derivatives of the generators agree with those of the identity at the endpoints of I , so that the action of N can be extended to a differentiable action of N on \mathbb{R} which is supported within I .

3. TWO-CHAIN GROUPS ARE ISOMORPHIC TO THOMPSON'S GROUP F

For an interval $J \subseteq \mathbb{R}$, let us denote the left- and the right-endpoints of J by $\partial^- J$ and $\partial^+ J$, respectively. An n -chain is a ordered n -tuple of intervals (J_1, J_2, \dots, J_n) such that

$$\partial^- J_i < \partial^- J_{i+1} < \partial^+ J_i < \partial^+ J_{i+1}$$

for each $i < n$ and such that $J_i \cap J_j = \emptyset$ for $|i - j| > 1$.

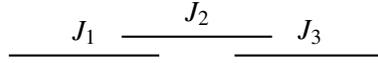


FIGURE 2. A chains of three intervals.

Theorem 3.1. *Let $f, g \in \text{Homeo}^+(\mathbb{R})$, and suppose that $(\text{supp } f, \text{supp } g)$ forms a chain. Then for all but finitely many $p > 0$, the group $\langle f^p, g^p \rangle$ is isomorphic to Thompson's group F .*

Proof. Let $\text{supp } f = (x, z)$ and $\text{supp } g = (y, w)$ such that

$$x < y < z < w.$$

By considering inverses if necessary, we may assume $f(y) > y$ and $g(z) > z$. Then for all sufficiently large $p > 0$, we have

$$z \leq g^p f(y) \leq g^p f^p(y) < g^{2p} f^p(y).$$

We claim $\langle f^p, g^p \rangle \cong F$. After replacing f and g by their p^{th} powers, we may assume $p = 1$. Note that

$$\begin{aligned} \text{supp } f \cap (gf) \text{supp } g &= (x, z) \cap (gf(y), w) = \emptyset, \\ \text{supp } f \cap (gf)^2 \text{supp } g &= (x, z) \cap (g^2 f(y), w) = \emptyset. \end{aligned}$$

Applying Tietze transformations, we see F has the presentation

$$F = \langle u, b \mid [u, bub(bu)^{-1}] = [u, (bu)^2 b (bu)^{-2}] = 1 \rangle.$$

So we have a homomorphism

$$\Phi: F \rightarrow \text{Homeo}^+(I)$$

given by $\Phi(u) = f$ and $\Phi(b) = g$. The image $G = \Phi(F)$ is nonabelian, for we have

$$gf(\text{supp } g) = (gf(y), w)$$

and

$$fg(\text{supp } g) = f(y, w) = (f(y), w).$$

Since every proper quotient of F is abelian, we see that Φ is injective. \square

We extract the following useful fact from the proof of Theorem 3.1 above:

Lemma 3.2. *Let $f, g \in \text{Homeo}^+(\mathbb{R})$ be such that $\text{supp } f = (x, z)$ and $\text{supp } g = (y, w)$, with*

$$x < y < z < w.$$

If $g \circ f(y) \geq z$, then we have

$$F = \langle f, g \rangle < \text{Homeo}^+(\mathbb{R}).$$

Lemma 3.2 can be viewed as a dynamical condition which guarantees that a 2-prechain group is in fact a 2-chain group.

We can now obtain Theorem 1.1:

Proof of Theorem 1.1. Let $G = G_{\mathcal{F}}$ be a prechain group. By replacing the elements of \mathcal{F} by sufficiently high powers, we may assume that any pair either commutes or generates a copy of F , by Theorem 3.1. The resulting group generated by these powers of homeomorphisms will therefore be a chain group. \square

We close this section by extracting the following dynamical fact about two-chain groups for its independent interest:

Proposition 3.3. *Let $f, g \in \text{Homeo}^+(\mathbb{R})$ be such that $(\text{supp } f, \text{supp } g)$ forms a chain. Assume $f(t) > t$ for all $t \in \text{supp } f$ and $g(t) > t$ for all $t \in \text{supp } g$. Then for all but finitely many $p > 0$, we have*

$$\text{supp}[f^p, g^{-p}] = \text{supp } f \cap \text{supp } g.$$

Proof. Let $\text{supp } f = (x, z)$ and $\text{supp } g = (y, w)$. As in the proof of Theorem 3.1, we may assume $z < gf(y)$.

If $t \leq y$, then $g(t) = t$ and $f^{-1}(t) < y$. So we have

$$[f, g^{-1}](t) = fg^{-1}f^{-1}(t) = ff^{-1}(t) = t.$$

If $t \geq z$, then $g(t) > z$ and so,

$$[f, g^{-1}](t) = fg^{-1}g(t) = f(t) = t.$$

So, we have that $\text{supp}[f, g^{-1}] \subseteq (y, z)$.

If $y < t \leq g^{-1}f(y)$, then $f^{-1}g(t) \leq y$ and hence,

$$[f, g^{-1}](t) = ff^{-1}g(t) = g(t) > t.$$

If $g^{-1}f(y) < t \leq g^{-1}(z)$, then $y < f^{-1}g(t)$ and so,

$$g^{-1}z < f(y) = fg^{-1}(y) < [f, g^{-1}](t)$$

and $[f, g^{-1}](t) > t$. If $g^{-1}(z) < t < z$, then $g(t) > z$ and hence,

$$[f, g^{-1}](t) = fg^{-1}g(t) = f(t) > t.$$

It follows that

$$(y, z) = (y, g^{-1}f(y)] \cup (g^{-1}f(y), g^{-1}(z)] \cup (g^{-1}f(y), g^{-1}(z)] \subseteq \text{supp}[f, g^{-1}]. \quad \square$$

4. THE NORMAL SUBGROUP STRUCTURE OF n -CHAIN GROUPS

For this section, we let $\mathcal{F} = \{f_1, \dots, f_n\} \subset \text{Homeo}^+(I)$ and we suppose that

$$G = G_{\mathcal{F}} = \langle f_1, \dots, f_n \rangle$$

is a chain group. We choose indices so that

$$(\text{supp } f_1, \dots, \text{supp } f_n)$$

forms a chain of intervals in I . For simplicity of notation, we assume that

$$\text{supp } G = \bigcup_i \text{supp } f_i = (0, 1).$$

The first main result of this section is the following.

Theorem 4.1. *Let G be a chain group. Then either:*

- (1) *The commutator subgroup $G' < G$ is simple;*
- (2) *The natural action of G on I has a wandering interval.*

Recall that a *wandering interval* for the action of G is a nonempty open interval $J \subset \text{supp } G$ such that for each $g \in G$, either $g(J) = J$ or $g(J) \cap J = \emptyset$.

The conclusions of Theorem 4.1 are not mutually exclusive, as can easily be seen by blowing up an orbit of a point in the case of a minimal action. Less trivially, there exist chain groups whose commutator groups are not simple. See Section 8 for more detail. Theorem 4.1 will follow from a sequence of lemmas.

The main engine behind the first conclusion of Theorem 4.1 is Higman's Theorem (see Theorem 2.5).

4.1. The center of a chain group is trivial. We first prove the following fairly easy proposition:

Proposition 4.2. *Let G be an n -chain group for $n \geq 2$. The center $Z(G) < G$ is trivial.*

Proof. The case $n = 2$ follows from Proposition 2.1. For the general case (which also gives a self-contained proof of the case $n = 2$), let $1 \neq g \in G$ be a candidate element of the center. Logically, we either have $\text{supp } g = \text{supp } G$ or $\text{supp } g \subsetneq \text{supp } G$.

In the first case, we assume $\text{supp } g = (0, 1)$, so that g has no fixed points in this interval. If $f \in \mathcal{F} \subset G$ is a generator, then f has a fixed point x in the interior of $(0, 1)$. Replacing g by its inverse if necessary and taking a sufficiently large integer N , the conjugate f^{g^N} either does not fix the point x , or we have that

$$\text{supp } f \cap \text{supp } f^{g^N} = \emptyset,$$

so that g is not centralized by f and consequently $g \notin Z(G)$.

The second case again bifurcates into two subcases: either $\text{supp } g$ is contained in a compact subinterval of $(0, 1)$, or $\text{supp } g$ accumulates at an endpoint of $[0, 1]$, say 0. In the first subcase, it is straightforward to produce an element $h \in G$ such that

$$\text{supp } h = \text{supp } G = (0, 1).$$

It is then easy to see that for $N \gg 0$,

$$\text{supp } g \cap \text{supp } g^{h^N} = \emptyset.$$

For the other subcase, we may again conjugate g by a large enough power of h (or h^{-1}) so that $\text{supp } g^h$ contains all of $(0, 1)$ except for a small pre-chosen neighborhood of the point 1. Here, we are using that the germ of G at 0 is a cyclic group. So, taking the generator $f \in \mathcal{F}$ with $\text{supp } f = (0, x)$, we have that $[f, g^{h^N}] \neq 1$. This can be seen by observing that f conjugated by either g^{h^N} or its inverse will not fix x . Thus in either subcase we obtain $g \notin Z(G)$. \square

4.2. The case of dense orbits. We begin with the case where our chain group action has dense orbits. We first note the following relatively straightforward observation:

Lemma 4.3. *Let G be an n -chain group, with $n \geq 2$. Then we have $G' = G''$. In particular, G/G'' is abelian.*

Proof. Recall that $F' = [F, F]$ is simple, by Proposition 2.1. Since it follows that $F' = F''$, we have

$$[f_i, f_{i+1}] \in \langle [f_i, f_{i+1}] \rangle' = \langle [f_i, f_{i+1}] \rangle'' \subseteq G''.$$

So

$$G' = \langle \langle \{ [f_i, f_{i+1}] : 1 \leq i < n \} \rangle \rangle \leq G'',$$

thus establishing the lemma. \square

Next, we prove two technical lemmas concerning actions of chain groups.

Lemma 4.4. *Let $G = G_{\mathcal{F}}$ be an n -chain group for some $n \geq 2$, let $g \in \mathcal{F}$, and let $x \in (0, 1)$ be an endpoint of $\text{supp } g$. Then for every closed interval $A \subset (0, 1)$ and every neighborhood J of x in $(0, 1)$, there is an element $f \in G$ such that $f(A) \subset J$.*

Proof. Without loss of generality, we may assume that $g = f_i$ and x is a left endpoint of the support of g , since the general case follows by an easy modification of our argument here. We set

$$h_1 = f_n \cdots f_1.$$

It is straightforward to see that there is an integer $m_1 > 0$ such that $h_1^{m_1}(A) \subset (x, 1)$. Now, let

$$h_2 = f_n f_{n-1} \cdots f_i.$$

There is an integer $m_2 < 0$ such that $h_2^{m_2}(h_1^{m_1}(A)) \subset J$, which proves the lemma. \square

Lemma 4.5. *For all $f \in G$ and all closed intervals $A \subset (0, 1)$, there is an element $g \in G'$ such that the actions of f and g agree as functions on A .*

Proof. We fix

$$h = f_n \cdots f_1$$

throughout the proof of this lemma. We first show that there is an element $g \in G$ such that the maps $g \cdot f$ and f agree on A , and such that $\text{supp}(g \cdot f) \subset (0, 1)$.

Let U_1 and U_2 be neighborhoods of 0 and 1 respectively, that are disjoint from A . There are positive integers m_1 and m_2 such that the maps

$$g_1 = h^{-m_1} f_1 h^{m_1}$$

and

$$g_2 = h^{m_2} f_n h^{-m_2}$$

are supported inside of U_1 and U_2 respectively, and whose germs at 0 and 1 agree with f_1 and f_n , respectively.

Note that there exist integers ℓ_1 and ℓ_2 such that the germs of f at 0 and 1 agree with $f_1^{\ell_1}$ and $f_2^{\ell_2}$ respectively. We set

$$g = g_1^{-\ell_1} g_2^{-\ell_2}.$$

It now suffices now to prove the statement of the lemma for $g \cdot f$ rather than for f , since these two homeomorphisms agree on A . Observe that by construction, $\text{supp}(g \cdot f)$ is contained in a compact subinterval of $(0, 1)$. It follows that there is a positive integer m such that

$$\text{supp}(h^{-m} \cdot (g \cdot f)^{-1} \cdot h^m) \subset (0, 1) \setminus A,$$

and such that

$$\text{supp}(h^m \cdot (g \cdot f)^{-1} \cdot h^{-m}) \subset (0, 1) \setminus A.$$

It follows that the actions of $g \cdot f$ and $[g \cdot f, h^{\pm m}]$ agree on A pointwise. \square

Lemma 4.6. *Let $G = G_{\mathcal{F}}$ be an n -chain group for some $n \geq 2$, let $f \in \mathcal{F}$, and let $x \in (0, 1)$ be an endpoint of $\text{supp } f$. If the orbit $G(x)$ is dense in $[0, 1]$ then G' is simple.*

Proof. We will deduce that G'' is simple by applying Corollary 2.6 to the group G' . In combination with our observation in Lemma 4.3 that $G' = G''$, we shall conclude that G' is simple.

Elements of G' have the property that their supports are contained in compact subintervals of $(0, 1)$, since the germs of G at the global fixed points 0 and 1 are abelian quotients of G . Hence, G' consists of compactly supported homeomorphisms of $(0, 1)$. By Corollary 2.6, it suffices to check that for any pair of compact intervals $A_1, A_2 \subset (0, 1)$, there is an element $f \in G'$ such that $f(A_1) \subset A_2$. By Lemma 4.5, it suffices to find an $f \in G$ such that $f(A_1) \subset A_2$.

Let $f_1 \in G$ be such that $f_1(x)$ lies in the interior of A_2 . There is a neighborhood J of x such that $f_1(J) \subset A_2$. By Lemma 4.4, there is an element $f_2 \in G$ such that $f_2(A_1) \subset J$. It follows that $f_1 f_2(A_1) \subset A_2$. \square

The first conclusion of Theorem 4.1 follows immediately from Lemma 4.6. Applying Proposition 4.2, we get the following:

Corollary 4.7. *Let G be a chain group with a dense orbit. Every proper quotient of G is abelian, and every finite index subgroup of G is normal.*

4.3. Cantor sets and canonical quotients. In this subsection, we complete the proof of Theorem 4.1.

Consider the standard PL action of Thompson's group F on the real line \mathbb{R} , given by $a(x) = x + 1$ and the homeomorphism $b(x)$ which is the identity for $x \leq 0$, which is $2x$ for $x \in [0, 1]$, and which is $x + 1$ for $x \geq 1$. Applying a homeomorphism between \mathbb{R} and $(0, 1)$ and compactifying to the interval I , we see that this action of the Thompson's group F on I has dense orbits in the interior of the support. In particular, the image of the dyadic rationals forms a single orbit. However, we may not always have dense orbits for a given n -chain group, we noted above. In such a case, we will *minimalize* the action to guarantee dense orbits. We first prove the following general result:

Lemma 4.8. *Let $G = G_{\mathcal{F}}$ be an n -chain group for some $n \geq 2$, let $f \in \mathcal{F}$, and let $x \in (0, 1)$ be an endpoint of $\text{supp } f$. Then the closure of the orbit $G(x)$ is either the whole interval $[0, 1]$ or it is a Cantor set $\Lambda \subset [0, 1]$. In the latter case, the action of G on Λ is minimal.*

Proof. We follow the same idea as in [29, Section 2.1.2]. We consider the collection \mathcal{P} of closed G -invariant subsets of I containing x , partially ordered by inclusion. Since $I \in \mathcal{P}$, we have that $\mathcal{P} \neq \emptyset$. The intersection of nonempty nested compact sets is nonempty and compact, so Zorn's Lemma implies that \mathcal{P} has a minimal

element Λ , which must contain the closure of $G(x)$. Since $x \notin \{0, 1\}$, we see that $G(x)$ is infinite and every point in $G(x)$ is an accumulation point.

Note that the minimality of the choice of Λ implies either that $\partial\Lambda = \{0, 1\}$ in which case $\Lambda = [0, 1]$, or that $\partial\Lambda$ is equal to the set of accumulation points of Λ , in which case Λ has empty interior. The case $\Lambda = [0, 1]$ is ruled out by assumption, so that $\Lambda = \partial\Lambda$ is a Cantor set. The minimality of Λ also implies that Λ must coincide with the closure of $G(x)$.

The minimality of the action of G on Λ follows more or less by definition, since Λ is the closure of the orbit of one point. \square

Observe that Λ in Lemma 4.8 is the closure of the orbit of any endpoint of the support of any $f \in \mathcal{F}$, provided this point is neither 0 nor 1. Following standard terminology from foliation theory (cf. [29]), we call Λ the *minimal invariant Cantor set*.

Lemma 4.9. *Suppose G has a minimal invariant Cantor set. Then there exists a monotone increasing map $h: I \rightarrow I$ and an injective map $\Phi: G \rightarrow \text{Homeo}^+(I)$ such that $h \circ g = \Phi(g) \circ h$ for all $g \in G$ and such that $\Phi(G)$ has dense orbits.*

Proof. Let Λ be the minimal invariant Cantor set. Set $h: [0, 1] \rightarrow [0, 1]$ as the devil's staircase map [10, Example 2.15] which contracts the closure of each component of $(0, 1) \setminus \Lambda$ to one point. This furnishes a monotone, continuous, surjective map $h_\Lambda: \Lambda \rightarrow I$ which is either one-to-one or two-to-one at every point of Λ , and which commutes with the action of G . Let $\Phi: G \rightarrow \text{Homeo}^+(I)$ be induced by the action of G on Λ . We see that $\Phi(G)$ has dense orbits since the action of G on Λ is minimal. \square

Lemma 4.10. *Let G be a chain group and let $\Phi(G)$ be its minimalization as in Lemma 4.9. Then $\Phi(G) < \text{Homeo}^+(I)$ is naturally a chain group.*

Proof. Let $G = G_{\mathcal{F}}$. For every $f \in \mathcal{F}$, we have that $h(\text{supp } f)$ is again an interval in I , where here h is the devil's staircase map (and h_Λ is its restriction to the minimal invariant Cantor set). Since f has no fixed points in the interior of $\text{supp } f$ and since h is order preserving, we have that $\Phi(f)$ has no fixed points in the interior of $h(\text{supp } f)$. Thus, $\Phi(f)$ is a prechain group.

If $f_i, f_j \in \mathcal{F}$, then either f_i and f_j commute, or $\langle f_i, f_j \rangle \cong F$. If f_i and f_j commute the $\text{supp } f_i$ and $\text{supp } f_j$ are disjoint. Moreover, since h_Λ is at most two-to-one and since the endpoints of $\text{supp } f_i$ and $\text{supp } f_j$ lie in Λ , we have that $h(\text{supp } f_i)$ and $h(\text{supp } f_j)$ are disjoint and possibly share one boundary point. In particular, $\Phi(f_i)$ and $\Phi(f_j)$ commute.

Suppose now that $\langle f_i, f_j \rangle \cong F$. Since every proper quotient of F is abelian, it suffices to see that

$$h(\text{supp } f_i) \neq h(\text{supp } f_j)$$

and that these intervals are not disjoint, since then $\Phi(f_i)$ will not commute with $\Phi(f_j)$. More is true: $h(\text{supp } f_i)$ and $h(\text{supp } f_j)$ in fact form a chain of intervals. We assume that $\text{supp } f_i = (x, w)$ and $\text{supp } f_j = (y, z)$, where

$$x < y < w < z.$$

Note that all four of these points are accumulation points of Λ . It is straightforward to check that Λ accumulates on x and y from the right and on w and z from the left, since these are the pairs of left and right endpoints of supports of the homeomorphisms f_i and f_j , respectively. Since h_Λ is at most two-to-one and order preserving, we cannot have $h_\Lambda(x) = h_\Lambda(y)$, nor $h_\Lambda(y) = h_\Lambda(w)$, nor $h_\Lambda(w) = h_\Lambda(z)$. The first two of these equalities is ruled out by x and y being accumulation point of Λ with accumulation from the right, and the third is ruled out by z being an accumulation point of Λ from the left. See Figure 3.

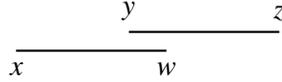


FIGURE 3. The supports of f_i and f_j .

In particular, we obtain

$$h_\Lambda(x) < h_\Lambda(y) < h_\Lambda(w) < h_\Lambda(z),$$

the desired conclusion. \square

Lemma 4.10 completes the proof of Theorem 1.3. We remark that the homeomorphism Φ which minimalizes the action of G may not be injective. See Section 8.

4.4. Embedding general chain groups in chain groups with simple commutator subgroups. In this subsection we prove half of Theorem 1.5:

Lemma 4.11. *Let $f \in \text{Homeo}^+(\mathbb{R})$ satisfy $\text{supp } f = (-\infty, 1)$. Then there exist $g \in \text{Homeo}^+(\mathbb{R})$ such that $\text{supp } g = (0, \infty)$ and $\langle f, g \rangle$ is a 2-chain group with dense orbits.*

The proof is immediate by conjugating f to a standard generator of F as we have seen.

Proposition 4.12. *Let G be an n -chain group. There exists an $(n + 1)$ -chain group H which has dense orbits, and such that $G < H$. In particular, the derived subgroup $H' < H$ is simple.*

Proof. We realize

$$G < \text{Homeo}^+([0, 1/2])$$

by scaling appropriately. Let

$$\{f_1, \dots, f_n\}$$

be the standard generating set for G . By Lemma 4.11, we can choose $g \in \text{Homeo}^+(I)$ with

$$\text{supp } g = (1/2 - \epsilon, 1)$$

for a sufficiently small $\epsilon > 0$, and such that $H = \langle G, g \rangle$ is an $(n + 1)$ -chain group and such that $\langle f_n, g \rangle$ has a dense orbit.

Note that for any interval $A \subset \text{supp } G$, there is an element $h \in H$ such that $h(A) \subset \text{supp } g$. Since $F = \langle f_n, g \rangle$ has dense orbits, we have that any dense orbit of the F action intersects A under the action of H . Therefore, H has dense orbits. \square

5. FINITELY GENERATED SUBGROUPS OF $\text{Homeo}^+(I)$ EMBED INTO CHAIN GROUPS

In this section, we prove Theorem 1.6, which is fundamental for establishing many of the remaining results claimed in the introduction.

5.1. The continuous category. Let

$$\{\phi_1, \dots, \phi_n\} \subset \text{Homeo}^+(I),$$

where here we implicitly identify I with a subset of \mathbb{R} . Let $a \in \text{Homeo}^+(\mathbb{R})$ be translation by 1, so that $a(x) = x + 1$ for all $t \in \mathbb{R}$. Let $\{g_1, \dots, g_n\}$ be given by conjugating $\{\phi_1, \dots, \phi_n\}$ by the affine maps

$$\{\alpha_i(x) = x/2 + 4i\}_{1 \leq i \leq n}$$

of \mathbb{R} , so that g_i is conjugate to ϕ_i by α_i and is supported on the interval $[4i, 4i + 1/2]$. We next define a homeomorphism h of \mathbb{R} and corresponding conjugates $\{h_i\}$ of h by the map a^{4i} which satisfies the following properties:

- (1) For $x \leq 0$, we set $h(x) = x$;
- (2) For $x \geq 1$, we set $h(x) = x + 1$;
- (3) For $x \in (0, 1)$, we have $x < h(x) < x + 1$;
- (4) For each i , we have $x < h_i g_i(x) < x + 1$ for $x \in (4i, 4i + 1)$;

It is a straightforward verification to show that such a homeomorphism h exists.

For $1 \leq i \leq n$, we set $b_i = h_i g_i$ and $c = h_{n+1}$. Finally, we define $f_1 = a^{-1} b_1$, we set $f_i = b_i^{-1} b_{i-1}$ for $2 \leq i \leq n$, we set $f_{n+1} = c^{-1} b_n$ and we set $f_{n+2} = c$, and we write $\mathcal{F} = \{f_1, \dots, f_{n+2}\}$. To prevent confusion, we remind the reader that homeomorphisms act from the left, so that $gf = g \circ f$.

Theorem 5.1. *The group G generated by \mathcal{F} is a chain group which contains a group isomorphic to $\langle \phi_1, \dots, \phi_n \rangle < \text{Homeo}^+(I)$.*

Clearly, Theorem 5.1 implies Theorem 1.6.

Proof of Theorem 5.1. First, we easily see that $G = \langle a, b_1, \dots, b_n, c \rangle$. By conjugating c^{-1} by an appropriate power of a and precomposing with b_i , we see that $g_i \in G$. By conjugating each g_i by a suitable power of a , we obtain a copy of the group $\langle \phi_1, \dots, \phi_n \rangle$, scaled by a factor of two.

It remains to show that G is in fact a chain group. We first check that each f_i has connected support. Consider first $f_1(x) = b_1(x) - 1$. We clearly have $f_1(x) = x - 1$ for $x \leq 4$ and $f_1(x) = x$ for $x \geq 5$. For $4 \leq x < 5$ we have $b_1(x) < x + 1$, so that $b_1(x) - 1 < x$. In particular, $f_1(x)$ has no fixed points for $x < 5$.

It is clear that f_{n+2} is the identity for $x \leq 4n + 4$ and that f_{n+2} has no fixed points for $x > 4n + 4$.

The analysis for the cases $2 \leq i \leq n + 1$ is uniform. Clearly $f_i(x) = x$ for $x \geq 4i + 1$ and for $x \leq 4i - 4$. For the intermediate values of x , we first apply b_{i-1} . The result is some real number greater than x , which is precisely equal to $x + 1$ if $x \geq 4i - 3$. We then apply b_i^{-1} (or c^{-1} in the case $i = n + 1$). Since $b_i(x) < x + 1$ for $x < 4i + 1$, we have that $b_i^{-1}(y) > y - 1$ for $y < 4i + 1$, by a simple application of the inverse function theorem. It follows that for $4i \leq x < 4i + 1$, we have $b_i^{-1}(x + 1) > x$, so that f_i has no fixed points in $(4i - 4, 4i + 1)$.

We now have to check that for all pairs $i \neq j$, the homeomorphisms f_i and f_j either commute or generate a copy of F . Note that

$$\text{supp } f_1 = (-\infty, 5),$$

that

$$\text{supp } f_{n+2} = (4n + 4, \infty),$$

and that

$$\text{supp } f_i = (4i - 4, 4i + 1)$$

for $2 \leq i \leq n + 1$. If $|i - j| > 2$ we easily see that

$$\text{supp } f_i \cap \text{supp } f_j = \emptyset,$$

so that f_i and f_j commute.

Now consider $\langle f_i, f_{i+1} \rangle$ for $1 \leq i \leq n + 1$. We check the dynamical criterion of Lemma 3.2 to show that this group is Thompson's group F . Consider the element $f_{i+1}f_i$ (and the element $f_2f_1^{-1}$ in the case $i = 1$). It is easy to see then that this element is $b_2^{-1}a$ for $i = 1$, that it is $b_{i+1}^{-1}b_{i-1}$ for $2 \leq i \leq n - 1$, that it is $c^{-1}b_{n-1}$ for $i = n$, and that it is b_n for $i = n + 1$. We evaluate the element $f_{i+1}f_i$ on the left endpoint $x = 4i$ of $\text{supp } f_{i+1}$ and show that the value is at least as large as the right endpoint $x = 4i + 1$ of $\text{supp } f_i$, which will suffice to apply Lemma 3.2 and prove that $\langle f_i, f_{i+1} \rangle \cong F$.

Consider the point $4i$ for $1 \leq i \leq n + 1$. Applying b_{i-1} (or a in the case $i = 1$), we see that

$$b_{i-1}(4i) = 4i + 1.$$

In the case $i = n + 1$, we are done. Now applying b_{i+1}^{-1} (or c^{-1} in the case $i = n$), we note that these elements are the identity for $x \leq 4i + 4$, so that the point $x = 4i + 1$ is fixed, which shows that

$$f_{i+1}f_i(4i) \geq 4i + 1,$$

as we wanted.

Finally, conjugating the entire setup by an order preserving homeomorphism $\mathbb{R} \rightarrow (0, 1)$ realizes G as a chain subgroup of $\text{Homeo}^+(I)$, completing the proof of the result. \square

5.2. The smooth category. To complete the proof of Theorem 1.6, we just need to argue that the construction in Subsection 5.1 can be appropriately smoothed. Within the class of smooth diffeomorphisms of \mathbb{R} , there is little to check. It is an easy exercise to produce an h as in Subsection 5.1 which is C^∞ , so that if $\{\phi_1, \dots, \phi_n\}$ are all C^k diffeomorphisms for some $1 \leq k \leq \infty$ then the corresponding homeomorphisms $\{f_1, \dots, f_n\}$ will all be C^k . The only potential difficulty is smoothing at the two points at infinity in \mathbb{R} . Note, however, that the germs of homeomorphisms in G at infinity form an abelian group of rank two, generated by $a(x) = x + 1$ near both points at infinity, acting independently of each other. It thus suffices to check that a is C^∞ at $\{\pm\infty\}$, which will show that G can be extended to the usual two point compactification of \mathbb{R} in a way that is C^∞ at the new points.

The map a naturally extends to an analytic diffeomorphism of the circle, being naturally realized as an element in $\text{PSL}_2(\mathbb{R})$. The map a is infinitely differentiable at infinity, so that after cutting S^1 open at infinity, we can realize a as a C^∞ diffeomorphism of the interval. We thus obtain:

Theorem 5.2. *Let $\langle \phi_1, \dots, \phi_n \rangle < \text{Diff}_0^k(I)$ be an n -generated subgroup of C^k diffeomorphisms of I . Then there exists an $(n + 2)$ -chain group $G < \text{Diff}^k(I)$ which contains a subgroup isomorphic to $\langle \phi_1, \dots, \phi_n \rangle$.*

Combining Theorem 5.2 and Theorem 5.1, we nearly obtain the full statement of Theorem 1.16.

One can improve the conclusions of Theorem 5.2 slightly, replacing $\text{Diff}^k(I)$ by $\text{Diff}_0^k(I)$. Indeed, for all $1 \leq k \leq \infty$, we have that every finitely generated subgroup of $\text{Diff}^k(I)$ is conjugate into $\text{Diff}_0^k(I)$. The authors are grateful to A. Navas for pointing out a reference for this fact [34]. Thus, we obtain the following corollary, which gives us the full statement of Theorem 1.16:

Corollary 5.3. *Let $\langle \phi_1, \dots, \phi_n \rangle < \text{Diff}_0^k(I)$ be an n -generated subgroup, for some $k \geq 1$. Then there exists an $(n + 2)$ -chain group G which contains a subgroup isomorphic to $\langle \phi_1, \dots, \phi_n \rangle$ and which lies in $\text{Diff}_0^k(I)$.*

5.3. Isomorphisms between chain groups. In this subsection, we prove Proposition 1.7, which follows fairly easily from the ideas in Subsection 5.1. We establish Proposition 1.7 through a related proposition here.

Proposition 5.4. *Let $G < \text{Homeo}^+(I)$ be an n -chain group for $n \geq 2$, and let $m \geq n$. Then there exists an m -chain group $H < \text{Homeo}^+(I)$ which is isomorphic to G .*

The reader will note that, via the considerations of Subsection 5.2, the construction of H in Proposition 5.4 can be arranged with the same degree of regularity as is enjoyed by G .

Proof of Proposition 5.4. We treat the case $n = 2$ separately. Since any 2-chain group is isomorphic to F , we may choose any realization of $F < \text{Homeo}^+(I)$ that we like. We replace I by \mathbb{R} , we set $a(x) = x + 1$, and we set $b(x)$ to be the identity for $x \leq 0$, the function $2x$ on $[0, 1]$, and $x + 1$ for $x \geq 1$. By the same argument as in Theorem 5.1, we have that $\langle a, b \rangle \cong F$. Setting $b_1 = b$ and $b_i = a^{4i}ba^{-4i}$ for $2 \leq i \leq n$ furnishes $n + 1$ homeomorphisms which together generate the original copy of F . As in Theorem 5.1, we set $f_1 = a^{-1}b_1$, we set $f_i = b_i^{-1}b_{i-1}$ for $2 \leq i \leq n$, and we set $f_{n+1} = b_n$. It is straightforward to see that $\{f_1, \dots, f_n\}$ generate an $(n + 1)$ -chain group isomorphic to F .

Now let G be an n -chain group of the interval for $n \geq 3$. We will show that G is isomorphic to an $(n + 1)$ -chain group, which will establish the proposition by an easy induction. We consider the rightmost three intervals in the chain and we write $\{a, b, c\}$ for the three homeomorphisms supported on them, which after replacing them by their inverses if necessary, move points to the left. Thus, we write the standard generating set for G as

$$\{f_1, \dots, f_{n-3}, a, b, c\}.$$

We set

$$d = b^{ac} = (ac)^{-1}b(ac).$$

Notice that the left endpoint of $\text{supp } d$ is strictly to the right of the left endpoint of $\text{supp } b$, and similarly for the right endpoints of the supports of $\text{supp } d$ and $\text{supp } b$. Notice that for all $n \neq 0$, we have

$$\langle d^n, c \rangle \cong \langle d^n, a \rangle \cong F,$$

since the homeomorphisms a and c commute with each other. We then set $e = a^{b^{-m}}$, where m is chosen large enough so that $\text{supp } e \cap \text{supp } d = \emptyset$. Notice that again we

have

$$\langle a, b \rangle \cong \langle e, b \rangle \cong F.$$

Choose $n \gg 0$ so that $\langle b, d^n \rangle \cong F$, which exists by Lemma 3.2. Replacing n by an even large exponent if necessary, we select n so that $d^n(\text{supp } b) \cap \text{supp } c = \emptyset$, and we set $f = b^{d^{-n}}$. Notice that since $\text{supp } e \cap \text{supp } d = \emptyset$, we have that

$$\langle e, f \rangle \cong \langle e, b \rangle \cong F.$$

We then take H to be the group generated by $\{e, f, d^n, c\}$, together with the original homeomorphisms $\{f_1, \dots, f_{n-3}\}$. Note that

$$\{\text{supp } e, \text{supp } f, \text{supp } d, \text{supp } c\}$$

form a chain of intervals. See figure 4.

Observe that if $\{f_i, f_j\}$ are generators of G for $i, j \leq n-3$ then we immediately have f_i and f_j either commute or generate a copy of F , since they are untouched by our modifications. Moreover, $\langle f_{n-3}, e \rangle \cong F$ since $[f_{n-3}, b] = 1$. It is clear by our construction and subsequent choice of n that all other pairs of generators either commute or generate F , so that H is indeed an $(n+1)$ -chain group.

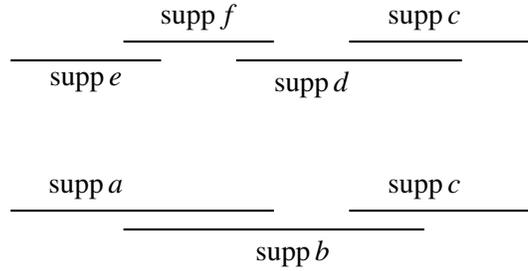


FIGURE 4. Converting an n -chain group to an $(n+1)$ -chain group.

Finally, we claim that $H \cong G$. It is immediate that $H < G$. We have that $b \in H$ since b is a conjugate of f by d^n . We also have that $a \in H$ since a is a conjugate of e by a power of b . Thus, all the generators of G lie in H , so that $G = H$. \square

Let \mathcal{G}_n denote the class of (isomorphism types of) n -chain groups.

Corollary 5.5. *For all $m \geq n$, we have $\mathcal{G}_n \subset \mathcal{G}_m$.*

Corollary 5.6. *For all $n \geq 2$, there exists an n -chain group which is isomorphic to F .*

Note that Proposition 1.7 follows immediately from Proposition 5.4. Observe now that if G is an n -ring group with $n \geq 3$ then G contains a 2-chain group. In particular, every n -ring group contains an m -chain group for every $m \geq 2$. Any easy modification of the proof of Proposition 5.4 yields the following:

Proposition 5.7. *Let $G < \text{Homeo}^+(S^1)$ be an n -ring group, for some $n \geq 3$. Then for all $m \geq n$, there exists an m -ring group $H < \text{Homeo}^+(S^1)$ such that $G \cong H$.*

Proof. Consider the setup of the proof of Proposition 5.4, so that now

$$\text{supp } c \cap \text{supp } f_1 \neq \emptyset.$$

Replacing b at the start by a conjugate by a power of c if necessary, we may still assume that

$$\text{supp } d \cap \text{supp } f_1 = \emptyset,$$

where again $d = b^{ac}$. We then define e and f identically as before. The necessary verifications are straightforward. \square

Corollary 5.8. *For $3 \leq n < m$, the class of n -ring groups is contained in the class of m -ring groups.*

6. UNCOUNTABLY MANY INFINITELY PRESENTED CHAIN GROUPS

In this section, we establish Theorem 1.8 as well as Corollary 1.9, and we discuss stabilization of isomorphism type of chain groups.

6.1. Uncountable families of isomorphism types. We first establish the existence of uncountably many isomorphism types of subgroups of $\text{Homeo}^+(\mathbb{R})$.

Lemma 6.1. *There exists a 2-generated left-orderable group Γ and a collection of normal subgroups $\{N_i\}_{i \in I}$ of Γ with the following properties:*

- (1) *The collection $\{N_i\}_{i \in I}$ is uncountable;*
- (2) *For each i , the group $N_i < \Gamma$ is central;*
- (3) *For each $i \in I$, the quotient $\Gamma_i = \Gamma/N_i$ is left orderable.*

The group Γ and its subgroups as in Lemma 6.1 appear in III.C.40 of de la Harpe's book [13]. Here, we merely observe that Γ and the quotients Γ_i are all left orderable.

Proof of Lemma 6.1. To establish the first two claims, we reproduce the argument given by de la Harpe nearly verbatim. Let $S = \{s_i\}_{i \in \mathbb{Z}}$, and let

$$R = \{[[s_i, s_j], s_k] = 1\}_{i,j,k \in \mathbb{Z}} \cup \{[s_i, s_j] = [s_{i+k}, s_{j+k}]\}_{i,j,k \in \mathbb{Z}}.$$

Define $\Gamma_0 = \langle S \mid R \rangle$ and let $\Gamma = \Gamma_0 \rtimes \mathbb{Z}$, where the conjugation action of $\mathbb{Z} = \langle t \rangle$ is given by $t^{-1}s_it = s_{i+1}$. For each i , we set $u_i = [s_0, s_i]$. Note the following easy observations:

- (1) The group $[\Gamma_0, \Gamma_0]$ is central in Γ , is generated by $\{u_i\}_{i \in \mathbb{Z}}$, and is isomorphic to an infinite direct sum of copies of \mathbb{Z} ;
- (2) The group Γ is generated by s_0 and t ;
- (3) The quotient group $\Gamma/[\Gamma_0, \Gamma_0]$ is isomorphic to the lamplighter group $\mathbb{Z} \wr \mathbb{Z}$.

For each subset $X \subset \mathbb{Z} \setminus \{0\}$, we can consider the group

$$N_X = \langle u_i \mid i \in X \rangle.$$

Evidently these groups are distinct for distinct subsets of $\mathbb{Z} \setminus \{0\}$, and they are all normal because $[\Gamma_0, \Gamma_0]$ is central in Γ . We thus establish the first two claims of the lemma.

For the third claim, note that the lamplighter group $\mathbb{Z} \wr \mathbb{Z}$ is left orderable since it lies as a subgroup of F (see Lemma 2.2). For each $X \subset \mathbb{Z} \setminus \{0\}$, the groups $[\Gamma_0, \Gamma_0]/N_X$ are all free abelian and therefore left orderable. By Lemma 2.8, it follows that the group Γ/N_X is left orderable. \square

Lemma 6.2. *There exist uncountably many isomorphism types of two-generated subgroups of $\text{Homeo}^+(\mathbb{R})$.*

Proof. By Lemma 2.7, it suffices to prove that there are uncountably many isomorphism types of two-generated left orderable groups. To this end, suppose there exist only countably many isomorphism types of two-generated left orderable groups. Then the class of groups

$$\mathcal{N} = \{\Gamma/N_X\}_{X \subset \mathbb{Z}}$$

furnished by Lemma 6.1 consists of only countably many isomorphism types. It follows that there is an element $N \in \mathcal{N}$ and uncountably many surjective homomorphisms $\Gamma \rightarrow N$. Since N and Γ are both finitely generated, this is a contradiction. \square

We can now prove Theorem 1.8 in nearly its full generality:

Proof of Theorem 1.8 in the case $n \geq 4$. Fix $n \geq 4$. By Theorem 1.6, for every two-generated subgroup $H < \text{Homeo}^+(I)$ there exists an n -chain group G such that $H < G$. Suppose for a contradiction that there were only countably many isomorphism types of n -chain groups. Then there would only be countably many isomorphism types of two-generated subgroups of n -chain groups. Lemma 6.2 shows that there are uncountably many isomorphism classes of two-generated subgroups of $\text{Homeo}^+(I)$ and hence of n -chain groups, which is a contradiction. \square

Proof of Corollary 1.9. Fix $n \geq 4$, and suppose that every n -chain group is finitely presented. Then there would be only countably many isomorphism types of n -chain groups, which contradicts Theorem 1.8. \square

6.2. Uncountably many isomorphism types of 3-chain groups. We now prove Theorem 1.8 in the case $n = 3$. Corollary 1.9 will follow immediately in this case, just as in Subsection 6.1.

We retain notation from Subsection 6.1 and write

$$\mathcal{N} = \{\Gamma/N_X\}_{X \subset \mathbb{Z}}.$$

As before, each $N \in \mathcal{N}$ is generated by two elements, $s(= s_0)$ and t .

Lemma 6.3. *Let $N \in \mathcal{N}$ be generated by elements $s, t \in N$. There exists a faithful action of N on \mathbb{R} such that the element $t \in N$ acts without fixed points.*

Proof. Let $<_N$ be a left-invariant ordering on N as furnished by Lemma 2.8, using the surjection

$$\phi: N \rightarrow N/([\Gamma_0, \Gamma_0]/N_X) \cong \mathbb{Z} \wr \mathbb{Z}.$$

In this ordering, every positive element which is nontrivial under the surjection to the lamplighter group $\mathbb{Z} \wr \mathbb{Z}$ is strictly larger than any element in $[\Gamma_0, \Gamma_0]/N_X$.

We first specify the ordering on $\mathbb{Z} \wr \mathbb{Z}$ which we use. Write K for the kernel of the map

$$\mathbb{Z} \wr \mathbb{Z} \rightarrow \mathbb{Z}$$

given by $s \mapsto 0$. Observe now that we can realize $\mathbb{Z} \wr \mathbb{Z}$ as acting on \mathbb{R} , where t acts by $x \mapsto x + 1$ and where s is an arbitrary homeomorphism supported on the interval $(0, 1/2)$. This action of $\mathbb{Z} \wr \mathbb{Z}$ on \mathbb{R} gives rise to an ordering $<_{\mathbb{Z} \wr \mathbb{Z}}$ on $\mathbb{Z} \wr \mathbb{Z}$ with $k <_{\mathbb{Z} \wr \mathbb{Z}} t$ for any $k \in K$, and we use this ordering to build an ordering $<_N$ on N via Lemma 2.8.

To obtain an action of N on \mathbb{R} , we follow the Ghys' dynamical realization (see [30]). We define a map

$$\tau: N \rightarrow \mathbb{Q} \subset \mathbb{R}$$

inductively. First, we set $\tau(1) = 0$, we set $\tau(t^{-1}) = -1$, and we set $\tau(t) = 1$. We now enumerate the remaining elements of N arbitrarily, subject to the following requirement: let $C_n \subset N$ be the coset of $\phi^{-1}(K) < N$ corresponding to t^n . If $n \geq 0$ then we require t^n and t^{n+1} to appear on the list before any other element of C_n appears. Similarly, if $n \leq 0$ then we require t^n and t^{n-1} to appear on the list before any other element of C_n appears. Moreover, if t^m appears before t^n on the list then we require $|m| \leq |n|$.

With this enumeration, we extend τ to all of N . Having defined

$$\{\tau(g_1), \dots, \tau(g_n)\},$$

we consider the next element g_{n+1} on the list. If

$$g_{n+1} >_N \max\{g_1, \dots, g_n\}$$

then we set

$$\tau(g_{n+1}) = \max\{\tau(g_1), \dots, \tau(g_n)\} + 1.$$

Similarly, if

$$g_{n+1} <_N \min\{g_1, \dots, g_n\}$$

then we set

$$\tau(g_{n+1}) = \min\{\tau(g_1), \dots, \tau(g_n)\} - 1.$$

Otherwise, we have

$$g_i <_N g_{n+1} <_N g_j$$

for some $i, j \leq n$, with no other g_k lying in the interval (g_i, g_j) with respect to the ordering $<_N$ for $k \leq n$. We then set

$$\tau(g_{n+1}) = (\tau(g_i) + \tau(g_j))/2.$$

It is clear that τ is an injective function $N \rightarrow \mathbb{Q}$ which preserves order. Moreover, N acts on $\tau(N)$ by left translation, and this action extends to a faithful action on \mathbb{R} , as explained in [30].

We claim that t acts on \mathbb{R} without any fixed points. Observe that with this definition of τ , we have that $\tau(t^n) = n$, which shows that the left translation action of t on $\tau(N)$ restricts to translation on \mathbb{Z} , so that the action of t on \mathbb{R} cannot have any fixed points. To see that $\tau(t^n) = n$, we proceed by an easy induction, with the cases $n \in \{-1, 0, 1\}$ being covered by the definition. Let $g \in N$ be arbitrary, and let n be such that $g \in C_n$, where without loss of generality we assume that $n \geq 0$.

Assume first that g is not a power of t . Then by induction we have that $\tau(t^{n+1})$ has been defined and is equal to $n + 1$. It follows that g is not larger than all the other elements on the list with respect to $<_N$, since $t^{-n}g \in C_0$ and every element in $C_0 = \phi^{-1}(K)$ is less than $t^{-n}t^{n+1} = t$. In particular, $\tau(g)$ will take on some rational value in $(-1, n + 1)$.

Finally, we consider the case where $g = t^{n+2}$, where again without loss of generality we assume $n \geq 0$. We claim that g is larger than any other element on the list. The previous largest element on the list was t^{n+1} , and $t^{n+2} >_N t^{n+1}$ by definition. Since $\tau(t^{n+1}) = n + 1$, we have $\tau(t^{n+2}) = n + 2$, which completes the induction. \square

Conjugating the construction of Lemma 6.3 above by a suitable homeomorphism, we obtain a faithful action of N on the interval I for which the element $t \in N$ has no fixed points in $(0, 1)$.

Lemma 6.4. *For any*

$$g \in \text{Homeo}^+([1/4, 1/2]),$$

there exists a 3-chain group G supported on the real line such that the following are satisfied:

- (1) *The group G contains as a subgroup the group of piecewise linear homeomorphisms of \mathbb{R} that have breakpoints at dyadic rationals, and such that all slopes are powers of 2;*
- (2) *There is an element $f \in G$ that fixes the complement of $(1/4, 1/2)$ pointwise, and such that the restriction of f to $(1/4, 1/2)$ agrees with g .*

Proof. Let $a(x) = x + 1$,

$$b(x) = \begin{cases} x & \text{if } x \leq 0 \\ 2x & \text{if } 0 \leq x \leq 1 \\ x + 1 & \text{if } 1 \leq x \end{cases}$$

Note that $\langle a, b \rangle$ is the standard PL copy of Thompson's group F defined on the real line.

Denote by g_1 as the homeomorphism that fixes the complement of $(1/4, 1/2)$ pointwise and which agrees with g on the interval $[1/4, 1/2]$, and let

$$g_2 = a^4 g_1 a^{-4}.$$

Now define

$$c(x) = \begin{cases} x & \text{if } x \leq 4 \\ 2 \cdot g_2(x) - 4 & \text{if } 4 \leq x \leq 5 \\ x + 1 & \text{if } 5 \leq x \end{cases}$$

It is easy to check that

$$G := \langle ab^{-1}, bc^{-1}, c \rangle$$

is a 3-chain group, and that it contains the required copy of Thompson's group F . The proof is completed by observing that the element

$$b^{-1} a^{-4} c a^4$$

is precisely the required element f . \square

Lemma 6.5. *Let*

$$H = \langle f_1, f_2 \rangle < \text{Homeo}^+([1/4, 1/2])$$

be such that f_1 has no fixed points in the interval $(1/4, 1/2)$. Then the group H embeds in a 3-chain group.

Proof. Let g be an element of the standard PL copy of F such that

$$\text{supp } g = (1/4, 1/2).$$

The construction of such elements of F is fairly straightforward. Since f_1 has no fixed point in $(1/4, 1/2)$, we have that f_1 is topologically conjugate to g within

$$\text{Homeo}^+([1/4, 1/2]) < \text{Homeo}^+(\mathbb{R}).$$

In particular, there is an element $h \in \text{Homeo}^+(\mathbb{R})$ such that $g = h f_1 h^{-1}$. We now let $k = h f_2 h^{-1}$. We have thus applied a topological conjugacy to H which realizes one generator of f_1 inside of the usual PL copy of Thompson's group F ,

and where the other generator is sent to some element of $\text{Homeo}^+([1/4, 1/2])$. We can therefore apply Lemma 6.4 to the homeomorphism k , thus realizing

$$H^h = \langle g, k \rangle$$

as a subgroup of a 3-chain group G . \square

Combining Lemma 6.5 with Lemma 6.3, we obtain the following immediate corollary:

Corollary 6.6. *Let $N \in \mathcal{N}$. Then N embeds inside a 3-chain subgroup of $\text{Homeo}^+(I)$.*

Theorem 1.6 now follows in the case $n = 3$ as in Subsection 6.1. We have the following natural question:

Question 6.7. *Let $H < \text{Homeo}^+(I)$ be a two-generated subgroup. Does H embed in a 3-chain group?*

6.3. Stabilization. There are several senses in which one can discuss stabilization of isomorphism types of chain groups. For one, one can consider eventual stabilization of isomorphism type as generators are raised to powers, and one can ask if two chain groups become isomorphic after raising generators to sufficiently high powers.

Let $G = G_{\mathcal{F}}$ be a prechain group. We say that G *stabilizes* if for all $N \gg 0$, the groups

$$\{G_N = \langle \{f^N \mid f \in \mathcal{F}\} \rangle\}$$

form a single isomorphism class, the *stable type* of G . A set $\{G_i\}_{i \in \mathbb{N}}$ of prechain groups is said to *stabilize* if for each $i \in \mathbb{N}$, the group G_i stabilizes and if the stable types are isomorphic for all i .

We first note the following:

Proposition 6.8. *A 2-prechain group always stabilizes. The class of all 2-prechain groups stabilizes. The stable type is Thompson's Group F .*

Proposition 6.8 is merely a restatement of Theorem 3.1. However, chain groups and ring groups always stabilize. It is easy to see that the following result implies Proposition 1.11:

Proposition 6.9. *Let $\mathcal{F} \subset \text{Homeo}^+(I)$ generate a chain group $G = G_{\mathcal{F}}$, and suppose that for some*

$$x \in \text{supp } f_1 \setminus \bigcup_{i=2}^n \text{supp } f_i,$$

we have

$$f_n \cdots f_1(x) \in \text{supp } f_n \setminus \bigcup_{i=1}^{n-1} \text{supp } f_i.$$

Then $G \cong F_n$, where here F_n denotes the corresponding Higman–Thompson group.

Proof. We find a surjective map from F_n to G . Since the image is not abelian, and since F_n has trivial center and simple commutator subgroup (cf. Subsection 2.2), this will show that $G \cong F_n$. For $0 \leq i \leq n-1$, we set

$$h_i = f_n \cdot f_{n-1} \cdots f_{i+1}.$$

It is clear that $\langle h_0, \dots, h_{n-1} \rangle \cong G$. We then set

$$h_{k(n-1)+i} = h_i^{h_0^k}$$

for $k \geq 1$.

The map $F_n \rightarrow G$ is defined by sending $g_i \mapsto h_i$ for all $i \in \mathbb{Z}_{\geq 0}$. It is a straightforward verification to see that this map is well-defined, since all the relations defining F_n also hold in G . Moreover this map is obviously surjective. This completes the proof. \square

Combining Proposition 6.9 with Proposition 5.4, we have the following:

Corollary 6.10. *For all $m \geq n$, there exists an m -chain group G_m such that $G_m \cong F_n$*

In particular, we obtain Corollary 1.13:

Corollary 6.11. *For all $2 \leq k \leq n$, there exists an n -chain group G_k such that $H_1(G_k, \mathbb{Z}) \cong \mathbb{Z}^k$.*

A proof Proposition 1.12 can be given in a manner identical to that of Proposition 6.9. We omit the details.

7. UNCOUNTABLY MANY COUNTABLE SIMPLE SUBGROUPS OF $\text{Homeo}^+(M)$

In this section, we establish Corollary 1.10. As remarked in the introduction (and suggested to the authors by A. Navas), the construction given here can be generalized to a general n -manifold. Indeed, a closed n -disk D^n can be viewed as

$$S^{n-1} \times (0, 1] \cup \{0\}.$$

If $G < \text{Homeo}^+(I)$ then $G < \text{Homeo}^+(D^n)$ by acting as G on $x \times (0, 1] \cup \{0\}$ for each $x \in S^{n-1}$ (and where each interval of the form $x \times (0, 1]$ shares the point $\{0\}$). It is easy to show that this construction is continuous (and in fact smoothable; cf. [34]).

The basic idea is the same as in the proof of Theorem 1.8. Retaining the notation of Section 6, we write

$$\mathcal{N} = \{\Gamma/N_X\}_{X \subset \mathbb{Z}}$$

for the set of left orderable subgroups of $\text{Homeo}^+(\mathbb{R})$ furnished by Lemma 6.1.

Let $N \in \mathcal{N}$ be generated by elements f and g . Since N is left orderable, we may realize N as a group of homeomorphisms on the interval $[0, 1/2] \subset \mathbb{R}$, and we set a to be the homeomorphism of \mathbb{R} defined by $x \mapsto x + 1$. Let $K = \langle f, g, a \rangle$.

Lemma 7.1. *The group N can be embedded in the commutator subgroup of K .*

Proof. Consider the commutators

$$[f, a] = f a f^{-1} a^{-1}$$

and

$$[g, a^{-1}] = g a^{-1} g^{-1} a.$$

Observe that the first of these is a composition of two homeomorphisms with supports $(0, 1/2)$ and $(1, 3/2)$ respectively, acting by f on $(0, 1/2)$ and by f^{-1} on $(1, 3/2)$. Similarly the second homeomorphism acts by g on $(0, 1/2)$ and by g^{-1} on $(-1, -1/2)$. We claim that

$$\langle [f, a], [g, a^{-1}] \rangle \cong \langle f, g \rangle.$$

This follows from the fact that any relations in the group N automatically lie in the commutator subgroup of N , so that if w is a word in f and g which represents the identity in N then the exponent sum in both f and g must be zero. Indeed, this is because for any $N \in \mathcal{N}$, we have that N surjects to the lamplighter group $\mathbb{Z} \wr \mathbb{Z}$, whose abelianization is free abelian of rank two.

Substituting $[f, a]$ and $[g, a^{-1}]$ for f and g respectively in w , we see that the resulting homeomorphism will be the identity outside of $(0, 1/2)$, and within the interval $(0, 1/2)$, the homeomorphisms $[f, a]$ and $[g, a^{-1}]$ coincide with f and g , respectively. \square

Proof of Corollary 1.10. Let K be as in the statement of Lemma 7.1 and Γ as in Lemma 6.1 (and throughout Section 6). It suffices to embed K into a chain group. Indeed, suppose that commutator subgroups of chain groups form only countably many isomorphism types. Then for some chain group G there are uncountably many different homomorphisms $\Gamma \rightarrow [G, G]$. Since $[G, G]$ is countable and Γ is finitely generated, this is a contradiction.

The group K can be realized as a 3-generated group of homeomorphisms of the interval I . Corollary 6.6 above shows that K can be embedded in an n -chain group for any $n \geq 3$, and hence in the commutator subgroup of an n -chain group with simple commutator group for any $n \geq 4$, by Proposition 4.12. \square

8. WANDERING INTERVALS

In this section, we produce a family of examples which complete the proof of Theorem 1.5 and show that there exist chain groups whose commutator groups are not simple. The basic idea is to mimic the Denjoy examples and blow up an orbit

of a point and to insert groups of arbitrary complexity into the wandering interval which results.

The construction is a modification of the construction given in Subsection 5.1, for a two-generated subgroup of $\text{Homeo}^+(I)$ which is generated by two copies of the same homeomorphism, and thus isomorphic to \mathbb{Z} . We retain the notation from Subsection 5.1 and we write ϕ_1, ϕ_2 for two copies of an arbitrary fully supported homeomorphism on I . The construction of Subsection 5.1 furnishes four homeomorphisms $\{f_1, \dots, f_4\}$ which generated a 4-chain group G (whose natural domain is \mathbb{R}) and which contains a copy of the group generated by ϕ_1 and ϕ_2 on a scaled subinterval of \mathbb{R} . Appropriate Tietze transformations furnish four homeomorphisms $\{a, b_1, b_2, c\}$ which generate G , where $a(x) = x + 1$, and where b_i is a composition of an a -conjugate of c with a scaled and translated copy g_i of ϕ_i , for $i = 1, 2$. With this notation, the support of g_i is exactly the interval $(4i, 4i + 1/2)$.

Let $y \in (4, 4.5)$ be an arbitrary point in the interior and let $\mathcal{O}(y)$ be the orbit of y . By modifying the choice of y slightly, we may assume that

$$\{\mathbb{Z}[1/2]\} \cap \mathcal{O}_y = \emptyset.$$

We now let $\{I_z\}_{z \in \mathcal{O}_y}$ be a collection of closed intervals with nonempty interior, such that

$$\sum_{z \in \mathcal{O}_y} \ell(I_z) < \infty,$$

where here ℓ denotes Euclidean length. We identify I_z with the interval $[0, \ell(I_z)]$, and for each pair $z_1, z_2 \in \mathcal{O}_y$, we have a canonical homeomorphism

$$\lambda_{z_1, z_2}: I_{z_1} \rightarrow I_{z_2}$$

which is scaling by $\ell(I_{z_2})/\ell(I_{z_1})$.

We now replace each point in $z \in \mathcal{O}_y$ by the corresponding interval I_z , which has the effect of sewing countably many intervals of finite total length into \mathbb{R} , which results in a manifold homeomorphic to \mathbb{R} . The action of G is extended to this new copy of \mathbb{R} as follows: the element $g \in G$ acts as before on points outside of \mathcal{O}_y . If $z_1, z_2 \in \mathcal{O}_y$ and $g(z_1) = z_2$ then g sends I_{z_1} to I_{z_2} via λ_{z_1, z_2} . We will call this operation *blowing up* the orbit of y . To prevent confusion, we will write \mathbb{R}_y for \mathbb{R} with the orbit of y blown up.

Lemma 8.1. *After blowing up the orbit of y , the action of G on \mathbb{R}_y is still as a chain group.*

Proof. The proof of this claim reduces to showing that the actions of the elements

$$\{a^{-1}b_1, b_2^{-1}b_1, c^{-1}b_2, c\}$$

are each supported on one of four intervals in a chain, and that there are no fixed points in the interiors of these intervals. By the choice of y , the endpoints of the

support on \mathbb{R} of each of these homeomorphisms is the corresponding endpoint of the support in \mathbb{R}_y . The proof that there are no fixed points in the interiors of these intervals is a straightforward verification. Proving that pairs of generators with overlapping support generate a copy of F is also a straightforward application of Lemma 3.2. \square

Recall that the homeomorphisms g_1 and g_2 , which are affine conjugates of the homeomorphisms ϕ_1 and ϕ_2 respectively, occur as elements of the group G . We set $\mathcal{O}_1 \subset \mathcal{O}_y$ to be the $\langle g_1 \rangle$ -orbit of y , and we set \mathcal{O}_2 to be $a^4(\mathcal{O}_1)$. Observe that \mathcal{O}_1 consists of countably many points in $(4, 4.5)$ and \mathcal{O}_2 consists of countably many points in $(8, 8.5)$. Moreover, \mathcal{O}_2 is exactly the $\langle a^4 g_1 a^{-4} \rangle$ -orbit of $a^4(y)$, which is to say the $\langle g_2 \rangle$ -orbit of $a^4(y)$.

We now choose an arbitrary countable collection

$$\{\psi_z \mid z \in \mathcal{O}_2\} \subset \text{Homeo}^+(I).$$

We modify the action of g_2 on \mathbb{R}_y by precomposing it with

$$\prod_{z \in \mathcal{O}_2} \psi_z,$$

where this product is interpreted as applying ψ_z , appropriately scaled, on the interval I_z . We use g_y to denote the resulting homeomorphism of \mathbb{R}_y .

We now consider the group G_y of homeomorphisms of \mathbb{R}_y generated by

$$\{a, b_1, b_y, c\},$$

where $\{a, b, c\}$ are unmodified versions of the corresponding homeomorphisms of \mathbb{R} canonically extended to act on \mathbb{R}_y , and b_y is the homeomorphism obtained from b_2 by replacing the role of g_2 with g_y . Applying suitable Tietze transformations as before:

Lemma 8.2. *The group*

$$G_y = \langle a^{-1}b_1, b_y^{-1}b_1, c^{-1}b_y, c \rangle < \text{Homeo}^+(\mathbb{R}_y)$$

is a chain group.

Proof. The proof is identical to that of Lemma 8.1. The verifications that the generators have connected supports are straightforward and follow from the definitions of the homeomorphisms as homeomorphisms of \mathbb{R}_y . The verification that this group is a 4-chain group follows easily from Lemma 3.2. \square

Note now that the group $\langle g_1, a^{-4}g_y a^4 \rangle$ lies as a subgroup of G_y . Moreover, this group is supported on the interval $(4, 4.5)$, and the generators are just g_1 and a copy of g_y which has been translated over to have the same domain as g_1 . For compactness of notation, we write s and t for g_1 and $a^{-4}g_y a^4$, respectively.

Lemma 8.3. *Let $m \in \mathbb{Z}$. Then*

$$\text{supp } s^m t^{-m} \subset \bigcup_{z \in \mathcal{O}_1} I_z.$$

Moreover, if ψ_z is nontrivial for some $z \in \mathcal{O}_2$, then $\text{supp } s^m t^{-m} \neq \emptyset$.

Proof. This is immediate from the definition, using the fact that s and t agree outside of the blowup of the orbit of y . \square

Let K be the subgroup of G_y which consists of all elements whose support is contained in \mathcal{I} . Proposition 8.4 below finishes the proof of Theorem 1.5:

Proposition 8.4. *The subgroup $1 \neq K < G_y$ is normal. The quotient G_y/K is canonically isomorphic to G . The derived subgroup $G'_y < G_y$ is not simple.*

Proof. The nontriviality of K follows from Lemma 8.3. That K is normal follows from the fact that the blowup of the G -orbit of y is invariant under the action by G , and K is exactly the subgroup of G_y consisting of elements which fix each component of the blown up orbit. Finally, the quotient map $G_y \rightarrow G_y/K$ is realized topologically by collapsing each component of the blown up orbit in a G -equivariant fashion. The result is the original action of G on the real line, so that $G_y/K \cong G$.

For the last claim, it suffices to show that $K \cap G'_y \neq 1$, since the quotient G is nonabelian. Note that \mathcal{O}_1 is contained in a compact subset of \mathbb{R} . We choose a nontrivial element of K as furnished by Lemma 8.3 and take a commutator with a sufficiently high power of a . If this power of a takes \mathcal{O}_1 off of itself, then this commutator will be nontrivial and will lie in $K \cap G'_y$. \square

9. DEGREES OF REGULARITY

We now consider the structural constraints on chain groups imposed by regularity of the generators, culminating in a proof of Theorem 1.14.

Lemma 9.1. *Let N be a finitely generated, residually torsion-free nilpotent group generated by n elements. Then there exists an $(n + 2)$ -chain subgroup of $\text{Diff}_0^1(I)$ which contains N as a subgroup. If N is nonabelian then no chain subgroup of $\text{Diff}_0^2(I)$ contains N .*

Proof. Theorem 2.10 shows that $N < \text{Diff}_0^1(I)$. By Corollary 5.3, we have that

$$N < G < \text{Diff}_0^1(I)$$

for some $(n + 2)$ -chain group, which establishes the first part of the lemma. For the second part, we have that N cannot be a subgroup of $\text{Diff}_0^2(I)$ if N is nonabelian by Theorem 2.10 again, which establishes the second part. \square

Let

$$\mathcal{N} = \{\Gamma/N_X\}_{X \subset \mathbb{Z}}$$

as in Sections 6 and 7. We will show here that most of the groups in \mathcal{N} cannot admit C^2 (or even C^1 with first derivatives of bounded variation) actions on any compact one–manifold.

Lemma 9.2. *Let $X \subset \mathbb{Z} \setminus \{0\}$ be proper. Then the group Γ_0/N_X contains a copy of the integral Heisenberg group.*

Recall the definition of the integral Heisenberg group:

$$H = \langle x, y, z \mid [x, y]z^{-1} = [x, z] = [y, z] = 1 \rangle.$$

Proof of Lemma 9.2. Recall that $[\Gamma_0, \Gamma_0] < \Gamma_0$ is a central subgroup which is a free abelian group on the elements

$$\{u_i = [s_0, s_i] \mid i \in \mathbb{Z}\}.$$

Let $i \in \mathbb{Z} \setminus \{0, X\}$, so that $u_i \notin N_X$. We claim that the subgroup of Γ_0/N_X generated by $\{s_0, s_i\}$ is isomorphic to H .

Note that we may define a map

$$j: H \rightarrow \langle s_0, s_i \rangle < \Gamma_0/N_X$$

by sending $x \mapsto s_0$ and $y \mapsto s_i$, which forces $z \mapsto u_i$. We claim that this map is injective. Since the abelianization of Γ_0 is torsion–free and freely generated by the generators $\{s_i \mid i \in \mathbb{Z}\}$, we have that $\ker j < \langle z \rangle < H$. If j is not injective then the image of z in Γ_0 is either trivial or has finite order. However, u_i has infinite order in Γ_0/N_X , so that j must be injective. \square

Lemma 9.3. *Let $X \subset \mathbb{Z} \setminus \{0\}$ be proper. Then the group $\Gamma/N_X \in \mathcal{N}$ admits no C^2 action on a compact one–manifold.*

Proof. Suppose the contrary. For some $i \in \mathbb{Z} \setminus \{0\}$ we have $u_i \notin N_X$, so that Γ/N_X contains the subgroup

$$\langle s_0, s_i, u_i \mid [s_0, s_i] = u_i, [u_i, s_0] = [u_i, s_i] = 1 \rangle.$$

By Lemma 9.2, this group is the integral Heisenberg group, a nonabelian torsion–free nilpotent group. By Theorem 2.10, the integral Heisenberg group admits no C^2 action on a compact one–manifold, so that neither does Γ/N_X . \square

We can now prove Theorem 1.14:

Proof of Theorem 1.14. The first part of the theorem is the content of Lemma 9.1. For the second part of the theorem, we argue as in Sections 6 and 7. Every group in \mathcal{N} can be embedded in an n –chain group for $n \geq 4$, by Theorem 1.6. This gives rise to uncountably many isomorphism types of n –chain groups which contain

elements of \mathcal{N} , since every element of \mathcal{N} is two-generated and each chain group is countable. By Lemma 9.3, a chain group which contains an element of \mathcal{N} defined by a proper subset of $\mathbb{Z} \setminus \{0\}$ admits no C^2 action on a compact one-manifold, which completes the proof. \square

10. COMMUTATOR SUBGROUPS OF RING GROUPS

Let

$$\mathcal{J} = \{J_0, \dots, J_{n-1}\}$$

be an n -ring of intervals in S^1 . To guarantee a distinct setup from the case of chain groups, we assume

$$\bigcup_{J \in \mathcal{J}} J = S^1.$$

Moreover, in order to prevent certain pathological situations, we assume that $n \geq 3$. We let f_i be a homeomorphism of S^1 which satisfies $\text{supp } f_i = J_i$.

Recall that the group generated by

$$\mathcal{F} = \{f_0, \dots, f_{n-1}\}$$

is called a *pre- n -ring group*, and that a pre-ring group is a *ring group* if every pair of generators either commutes or generates a copy of Thompson's group F . Theorem 1.1 applies to ring groups, so that after replacing the generators of a pre-ring group by sufficiently high powers, the result will be a ring group. We now state our main theorem about the normal subgroup structure of ring groups.

Theorem 10.1. *Let $G = \langle f_0, \dots, f_{n-1} \rangle$ be an n -ring group for $n \geq 5$. Assume moreover that there is an i such that the action of $\langle f_i, f_{i+1} \rangle$ on its support has a dense orbit. Then every proper quotient of G is abelian, and G' is perfect.*

The remainder of the section is dedicated to proving Theorem 10.1. For what follows, we assume that $G = \langle f_0, \dots, f_{n-1} \rangle$ is an n -ring group for $n \geq 5$, and there is an i such that the action of $\langle f_i, f_{i+1} \rangle$ on its support has dense orbits.

Lemma 10.2. *Let J_1 and J_2 be proper subintervals of S^1 . Then there is an element $f \in G$ such that $f(J_1) \subset J_2$.*

Proof. Let r be an end point of the support of generator f_j and let f_{j+1} be the generator whose support has a nontrivial intersection with the support of f_j , but does not contain r . By our density assumption, there exists an element $g_1 \in G$ such that $g_1(r) \in J_1^c$, where J_1^c denotes the complement of J_1 . For sufficiently large positive integers n_1 and n_2 , the set

$$f_{j+1}^{n_2}(f_j^{n_1}(g_1^{-1}(J_1^c)))$$

contains the support of f_j . It follows that

$$J_3 = f_{j+1}^{n_2} (f_j^{n_1} (g_1^{-1}(J_1)))$$

is contained in the interior of the support of a chain subgroup H of our ring group.

Let s be an end point of a generator f_k of H . By the density of the action, there is an element $g_2 \in G$ and a neighborhood $K \ni s$ such that $g_2(K) \subset J_2$. Now we can product an element g_3 of the chain group that maps J_3 into K . The element

$$f = g_2 g_3 f_{j+1}^{n_2} f_j^{n_1} g_1^{-1}$$

does the required job. \square

Lemma 10.3. *For each $g \in G$, there is an element $h \in G$ such that hgh^{-1} maps an interval*

$$J \subset \text{supp}\langle f_i, f_{i+1} \rangle$$

into $\text{supp}\langle f_i, f_{i+1} \rangle$, and so that

$$hgh^{-1}(J) \cap J = \emptyset.$$

Proof. Write

$$K = \text{supp}\langle f_i, f_{i+1} \rangle.$$

Let $K_1 \subset K$ be an interval such that $g(K_1) \cup K_1$ does not cover S^1 and such that

$$g(K_1) \cap K_1 = \emptyset.$$

Now let $h_1 \in G$ be an element furnished by Lemma 10.2 such that

$$h_1(g(K_1) \cup K_1) \subseteq K.$$

This shows that $g_1 = h_1 g h_1^{-1}$ maps the interval $h_1(K_1) \subset K$ inside K , and

$$g_1(h_1(K_1)) \cap h_1(K_1) = \emptyset,$$

as required. \square

Lemma 10.4. *For each $g \in G$, there are elements $h_1, h_2 \in G$ such that $k = h_1 g h_1^{-1}$ satisfies the following conditions:*

- (1) *The element $[h_2, k]$ is nontrivial;*
- (2) *We have*

$$\text{supp}[h_2, k] \subset \text{supp}\langle f_i, f_{i+1} \rangle.$$

Proof. Let h_1 be the element obtained for g as furnished by Lemma 10.3. We know that $k = h_1 g h_1^{-1}$ maps an interval

$$J \subset K = \text{supp}\langle f_i, f_{i+1} \rangle$$

into K , so that $k(J) \cap J = \emptyset$. We can easily construct an element h_2 of $\langle f_i, f_{i+1} \rangle$ such that $\text{supp} h_2 \subset J$. It follows that the support of $[h_2, k]$ is contained in K . By construction, this element is nontrivial and supported in $J \cup k(J)$. \square

Proof of Theorem 10.1. For the rest of the proof we use H to denote the group $\langle f_i, f_{i+1} \rangle$. Let K be an $(n - 1)$ -chain subgroup of G , generated by $n - 1$ of the n generators of G , and containing H . We denote the closure of the support of K by $U \subset S^1$. Define G_U as the group of elements of G whose support is contained in the interior of U . Note that $H' \subset G_U$. The action of H , and hence that of H' , has dense orbits (see Lemma 4.5). A routine application of Higman's Theorem (Theorem 2.5) shows that G'_U is simple, as in the proof of Theorem 4.1. Since G_U has trivial center (cf. Proposition 4.2), it follows that every proper quotient of G_U is abelian. Furthermore, since the commutators of pairs of generators of H also lie in the second term of the derived series of H , it follows that they are all contained in G'_U .

Now let N be a normal subgroup of G , and let $g_1 \in N \setminus \{1\}$. By Lemma 10.4, there are elements $h_1, h_2 \in G$ such that $[h_2, h_1 g_1 h_1^{-1}]$ is a non trivial element of G_U . It follows that $N \cap G_U \neq \{1\}$. In particular, we have that $H' < G'_U < N$. Since the choice of chain subgroup H containing $\langle f_i, f_{i+1} \rangle$ was arbitrary and since $n \geq 5$, it follows that every proper quotient of G is abelian. \square

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