A Cryptographic Application of the Thurston norm

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Abstract. We discuss some applications of 3-manifold topology to cryptography. In particular, we propose a public-key and a symmetric-key cryptographic scheme based on the Thurston norm on the first cohomology of hyperbolic manifolds.

1 Introduction

Geometric group theory and low dimensional topology have developed many powerful tools for studying groups, and many group theoretic ideas have been productive in group-based cryptography. Here, we propose importing ideas from hyperbolic geometry to build new cryptoschemes with certain security advantages.

Specifically, we consider the Thurston norm on $H^1(M, \mathbb{R})$, the first cohomology of a finite volume hyperbolic 3–manifold, as introduced in [34]. The Thurston norm measures the Euler characteristic of the simplest surface in $M$ which represents a second homology class which is Poincaré dual to an integral cohomology class, and extends it to the entirety of $H^1(M, \mathbb{R})$. The Thurston norm has a remarkable linear nature which makes computations with it tractable, and as outlined below, organizes the fibrations of a fibered hyperbolic 3–manifold.

We will use the Thurston norm to build a new symmetric-key cryptographic scheme. Combined with a certain group based public key exchange, we obtain a public-key cryptoscheme which has two levels of security and in which all communications are over public channels.

National Security Agency (NSA) announced plans to upgrade current security standards in 2015; the goal is to replace all deployed cryptographic protocols

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with quantum secure protocols, due to the increasing possibility of quantum attacks. This transition requires a new, post-quantum, security standard to be accepted by the National Institute of Standards and Technology (NIST). Proposals for quantum secure cryptosystems and protocols have been submitted for the standardization process. There are six main primitives currently proposed to be quantum-safe: (1) lattice-based (2) code-based (3) isogeny-based (4) multivariate-based (5) hash-based, and (6) group-based cryptographic schemes. Applications to post-quantum group-based cryptography could be shown if the underlying security problem is NP-complete or unsolvable; ideally one could analyze the relationship of the problems under consideration here to the hidden subgroup problem (HSP), then analyze Grover’s search problem. As for the relationship to the HSP, the groups under consideration are infinite and so a practical way to process them needs to be developed.

In [21], a practical cryptanalysis of WalnutDSA was proposed, a platform which was given in 2016 in [3] as a post-quantum cryptosystem using braid groups and conjugacy search problem, submitted to NIST competition in 2017.

There are other group-theoretic problems and classes of groups which have been proposed for post-quantum group-based cryptography, as we summarize here. The pioneers in this field were Wagner-Magyarik, who in [36] used a right-angled Artin group as a platform, relying on the word problem and the word choice problem; this approach was later improved by Levy-Perret in [27]. At the same time, Birget-Magliveras-Sramka [7] proposed a new protocol based on different groups that share some properties with Higman-Thompson groups. Later on, Flores-Kahrobaei and Flores-Kahrobaei-Koberda proposed right-angled Artin groups as a platform for various cryptographic protocols [16, 17]. Eick and Kahrobaei proposed polycyclic groups as a platform, using the conjugacy search problem as a basis for security [15]. Gryak-Kahrobaei proposed other group-theoretic problems for consideration for polycyclic group platforms [18]. Kahrobaei-Koupparis [23] proposed a post-quantum digital signature using polycyclic groups. Kahrobaei-Khan [22] proposed a public-key cryptosystem using polycyclic groups [22]. Habeeb-Kahrobaei-Koupparis-Shpilrain proposed public key exchanges using semidirect products of semigroups in [19]. Thompson’s groups have been considered by Shpilrain-Ushakov, with cryptoschemes based on the decomposition search problem [32]. Hyperbolic groups have been proposed by Chatterji-Kahrobaei-Lu, relying on properties of subgroup distortion and the geodesic length problem [12]. Cavallo-Habeeb-Kahrobaei-Shpilrain proposed using small cancellation groups for secret sharing scheme [10, 20]. Free metabelian groups have been proposed as a platform by Shpilrain-Zapata, with the scheme based on the subgroup membership search problem [33]. Kahrobaei-Shpilrain proposed free nilpotent $p$-groups as a platform for a semidirect product public key [25]. Linear groups were proposed by Baumslag-Fine-Xu [6], and Grigorchuk’s group have been proposed in [30]. Finally in [24], Arithmetic groups were proposed as platform for a symmetric-key cryptographic scheme.
2 Hyperbolic 3–manifolds and the Thurston norm

We review some well–known background about hyperbolic 3–manifolds and the Thurston norm [34], concentrating on the case of fibered hyperbolic 3–manifolds.

2.1 Generalities about the Thurston norm

Let $M = M_\psi$ be a fibered hyperbolic 3–manifold. That is to say, there is an orientable surface $S$ with negative Euler characteristic and a mapping class $\psi \in \text{Mod}(S)$ such that $M$ is the mapping torus of $\psi$. Observe that the rank of $H^1(M, \mathbb{R})$ is at least one, since $\pi_1(M)$ surjects to $\mathbb{Z}$. Integral cohomology classes of $M$ which correspond to fibrations of $M$ are called fibered cohomology classes of $M$.

A fibered 3–manifold $M$ is called atoroidal if it does not contain a non-peripheral incompressible torus. Here, this means that if $T \subset M$ is a $\pi_1$-injective copy of the torus, then $T$ can be pushed into a cusp of $M$. It is a famous result of Thurston that a fibered 3–manifold admits a finite volume hyperbolic metric if and only if it is atoroidal, which in turn will happen if and only if no power of the mapping class $\psi$ fixes the homotopy class of a simple closed loop on $S$. Here, a simple closed loop on $S$ is an essential copy of $S$ which is not parallel to a puncture or boundary component of $S$. Such a mapping class $\psi$ is called pseudo-Anosov.

It is a standard result from foliation theory that if the rank of $H^1(M, \mathbb{R})$ is at least two, then small perturbations of $\phi$ will give new fibrations of $M$ over the circle which are inequivalent to $\phi$. Thurston’s work [34] organized the fibrations of $M$ by defining a norm $\| \cdot \|_T$ on $H^1(M, \mathbb{R})$, called the Thurston norm. The norm of a cohomology class $\| \phi \|_T$ is given by $\min_{S} |\chi(S\phi)|$, where this minimum is taken over surfaces which are Poincaré dual to $\phi$. The following summarizes the relevant features of the Thurston norm:

Theorem 1. Let $M$ be a compact atoroidal 3–manifold with $\chi(M) = 0$, and let $\| \cdot \|_T$ be the Thurston norm.

1. $\| \cdot \|_T$ is a nondegenerate norm on the vector space $H^1(M, \mathbb{R})$;
2. The unit norm ball is a convex polytope, all of whose vertices lie at rational points in $H^1(M, \mathbb{R})$;
3. Let $\phi \in H^1(M, \mathbb{Z})$ be a fibered cohomology class of $M$. Then there is a maximum dimensional face $F$ of the unit ball of $\| \cdot \|_T$ such that $\phi \in \mathbb{R} \cdot F$.
   Moreover, every primitive integral cohomology class $\phi \in \mathbb{R} \cdot F$ is fibered. The face $F$ is called a fibered face of the unit norm ball.

Here, an integral cohomology class if called primitive if it is nonzero and if it is not an integer multiple of another integral cohomology class. Viewed as a tuple of vectors, an integral cohomology class is primitive if and only if the entries of the tuple are relatively prime and not all zero.

Let $\psi \in H^1(M, \mathbb{Z})$ be a fibered cohomology class. Then $M$ fibers over the circle with fiber $S = S_\psi$, where $\pi_1(S) < \pi_1(M)$ is identified with $\ker \psi$. The
following proposition is standard and we include its proof for the convenience of the reader.

**Proposition 1.** Suppose $M$ is hyperbolic, and let $S$ be the fiber of a fibration of $M$ over $S^1$. Then $\pi_1(S) < \pi_1(M)$ is exponentially distorted, and the membership problem for $\pi_1(S)$ is solvable in linear time.

**Proof.** We have that $\pi_1(M)$ is a semidirect product of $\mathbb{Z}$ with $\pi_1(S)$, where the conjugation action of a generator $t$ of $\mathbb{Z}$ is given by a pseudo-Anosov mapping class of $\pi_1(S)$. If $1 \neq \gamma \in \pi_1(S)$ and $\psi$ is a pseudo-Anosov mapping class which has been lifted to an automorphism of $\pi_1(S)$, then the length of the shortest representative in the conjugacy class of $\psi^n(\gamma)$ grows like $\lambda^n$, where $\lambda > 1$ is a real number called the stretch factor of $\psi$. Since conjugation by $t$ acts on $\pi_1(S)$ by $\psi$, we have that $\psi^n(\gamma) = t^{-n}\gamma t^n$, a word whose length is linear in $n$. It follows that $\pi_1(S) < \pi_1(M)$ is exponentially distorted.

Now suppose that $g \in \pi_1(M)$ is a given element, and we wish to determine if $g \in \pi_1(S)$. The group $\pi_1(M)$ surjects to $\mathbb{Z}$ by a homomorphism $\phi$, and the kernel of this map is exactly $\pi_1(S)$. If $\pi_1(M) = \langle g_1, \ldots, g_k \rangle$, then $\phi$ is determined by its values on the generators of $\pi_1(M)$. If $g \in \pi_1(M)$ is a product of $N$ generators of $\pi_1(M)$, we compute the value of $\phi$ on the $N$ generators needed to represent $g$ and add them up, which requires computational resources bounded by a linear function in $N$. If the resulting sum is zero, then $g \in \pi_1(S)$. If the sum is nonzero then $g \notin \pi_1(S)$.

### 2.2 Examples

From the general description, the Thurston norm seems very difficult to compute and hence unwieldy for many practical applications. However, there are many situations in which the Thurston norm can be computed, at least in a cone over a fibered face.

In [29], McMullen defined the Teichmüller polynomial associated to a fibered face and gave a practically implementable algorithm for computing it. From the Teichmüller norm (computed from the Teichmüller polynomial) and the Alexander norm (computed from the Alexander polynomial) on $H^1(M, \mathbb{R})$ for a fibered hyperbolic 3-manifold $M$, one can compute a fibered face of the unit Thurston norm ball, the cone over which contains the given fibered cohomology class. He indicates how to carry out these computations for certain pseudo-Anosov braids of the 4-times punctured sphere.

A very detailed computation of the Alexander and Teichmüller norm for a particular two-cusped hyperbolic 3-manifold, namely the sibling of the Whitehead link complement, was carried out by Aaber–Dunfield [1]. The authors of that paper explicitly computed the Alexander and Teichmüller polynomials of the sibling of the Whitehead link complement, as well as the whole Thurston norm ball, which is a square with unit side length centered at the origin. They show that all four faces of the square are fibered, and so that every primitive cohomology class of the manifold is fibered except the ones lying on the two lines passing through the corners of the square.
For other 3–manifolds presented as mapping tori of multiply punctured disks, an algorithm for computing the Teichmüller polynomial (as well as further topological data associated to other fibrations of the manifold) was proposed by Lanneau–Valdez [26]. Thus, the theory we apply in this paper is rich with computationally tractable examples.

3 An application to public-key cryptography

We now propose a cryptographic scheme which uses the Thurston norm. Alice and Bob are communicating over an insecure channel. The following information is public:

- A fibered hyperbolic 3–manifold \( M \) with \( H^1(M, \mathbb{R}) \) of rank at least two, a fixed finite presentation of \( \pi_1(M) \), and a fibered face \( F \) of the Thurston norm ball. For instance, one could use Thurston’s example of the simplest pseudo-Anosov braid, or alternatively Aaber–Dunfield’s example, as described above.
- For every primitive integral cohomology class \( \phi \in \mathbb{R}_+ \cdot F \), a standard presentation \( \mathcal{P}_\phi \) of the corresponding fiber group \( \pi_1(S_\phi) \), a stable letter \( t_\phi \in \pi_1(M) \), and an automorphism \( \psi_\phi \) of \( \pi_1(S_\phi) \) such that \( \pi_1(M) \) is the semidirect product of \( \pi_1(S_\phi) \) with \( \mathbb{Z} \) with the conjugation action of \( t_\phi \) given by \( \psi_\phi \). An explicit isomorphism between \( \pi_1(M) \) with its fixed presentation and this semidirect product presentation corresponding to the fibered class \( \phi \) is included in the data. We write \( \text{Prim}(\mathbb{R}_+ \cdot F) \) for the collection of such primitive integral cohomology classes. Here, the choice of the stable letter is not strictly necessary, but it precludes the need for an extra conjugacy decision later on in the cryptoscheme.
- A finitely generated group \( G \) suitable for a public-key cryptographic scheme such as the Anshel-Anshel-Goldfeld protocol (see [2]) and an efficiently computable function \( f : G \rightarrow \text{Prim}(\mathbb{R}_+ \cdot F) \).

Here, a standard presentation for a fiber group is either a presentation of a free group with finitely many generators and no relations, or the standard presentation of a closed surface group of genus \( g \):

\[
\pi_1(S_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle.
\]

The scheme is as follows:

1. Alice and Bob use a public-key cryptographic scheme in order to produce a shared private key \( g \in G \).
2. Using the public function \( f \), Alice and Bob agree on the fibered cohomology class \( \phi = f(g) \) with corresponding fiber group \( \pi_1(S) \) with presentation \( \mathcal{P} \) and automorphism \( \psi \).
3. Alice chooses an arbitrary positive integer \( N \) and computes the length of \( \psi^N(s) \) for every generator of \( \pi_1(S) \) in the presentation \( \mathcal{P} \). The key is the maximum length \( \ell_{\text{max}} \) obtained in this way, on the generator \( s_{\text{max}} \) in \( \mathcal{P} \).
4. Over a public channel, Alice sends a finite collection \( \{x_1, \ldots, x_t\} \subset \pi_1(M) \), written in terms of the fixed generators for \( \pi_1(M) \), and of length comparable to \( N \). She chooses these elements in such a way that if \( X_\phi \) is the generating set in the presentation \( P_\phi \), then \( \psi^N(X_\phi) \subset \{x_1, \ldots, x_t\} \), but so that most of the elements in \( \{x_1, \ldots, x_t\} \) do not belong to \( \pi_1(S) \).

5. Bob checks which of these elements lie in \( \pi_1(S) \).

6. Bob uses the fact that \( \psi \) is given by conjugation by an element of \( \pi_1(M) \) to recover \( N \).

7. Bob computes the length of \( \psi^N(s) \) for each generator of \( \pi_1(S) \) and recovers \( \ell_{\text{max}} \).

We remark that the value of \( \ell_{\text{max}} \) is uniquely determined by Alice's initial choice of \( N \).

4 An explicit example

We consider the example of a fibered 3-manifold coming from the simplest pseudo-Anosov braid, as worked out in [29].

4.1 The 3-manifold

The initial fiber is \( S_0 \), which is identified with the thrice punctured disk, and whose mapping class group is identified with the three–stranded braid group \( B_3 \). In the standard braid generating set \( \{\sigma_1, \sigma_2\} \), the simplest pseudo-Anosov braid is given by \( \beta = \sigma_1 \sigma_2^{-1} \).

We have that \( \pi_1(S_0) = \langle x, y, z \rangle \cong F_3 \), where these generators are identified with small based loops about the three punctures of \( S_0 \). We have

\[
\sigma_1 : (x, y, z) \mapsto (y, y^{-1}xy, z)
\]

and

\[
\sigma_2^{-1} : (x, y, z) \mapsto (x, yzy^{-1}, y).
\]

Thus, we have

\[
\beta : (x, y, z) \mapsto (yzy^{-1}, yz^{-1}y^{-1}xzy^{-1}, y).
\]

A presentation for the fundamental group of the fibered 3-manifold associated to \( \beta \) can be written as

\[
\pi_1(M) = \langle t, x, y, z \mid t^{-1}xt = yzy^{-1}, t^{-1}yt = yz^{-1}y^{-1}xzy^{-1}, t^{-1}zt = y \rangle.
\]

It is easy to see that \( \beta \) acts transitively on the punctures of \( S_0 \), and so that \( H_1(M, \mathbb{Z}) \cong \mathbb{Z}^2 \). If \( \phi \in H^1(M, \mathbb{Z}) \), then \( \phi \) is determined by its values on \( t \) and on \( x \), so that we may write \( \phi = (a, b) \in \mathbb{Z}^2 \) for the cohomology class which satisfies \( \phi(t) = a \) and \( \phi(x) = b \). McMullen computes the Thurston norm on \( H^1(M, \mathbb{R}) \) and shows that it is given by

\[
\|\phi\|_T = \max \{|2a|, |2b|\}.
\]

Each face of the Thurston unit norm ball is fibered. The face \( F \) whose cone contains \((1, 0)\) is therefore given by \( F = \{1/2\} \times [-1/2, 1/2] \).
4.2 From a public key to a new fibration

Let $G$ be a finitely generated group suitable for the Anshel-Anshel-Goldfeld protocol, with a fixed normal form. Then we can construct a computable function which associates to elements of $G$ various fibrations of $M$. If $g \in G$, we write $g$ in a normal form and let $|g|$ denote the length of $g$ in this normal form. Then, we may set

$$f(g) = D(g) \left( \left\{ \frac{1}{2} \right\}, \left\{ \frac{|g|}{|g|+1} - \frac{1}{2} \right\} \right),$$

where here $D(g)$ is chosen to be the smallest positive integer so that $f(g) \in \mathbb{Z}^2$. Note that $D(g) \leq \text{lcm}\{2, |g| + 1\}$. Thus, we have associated to $g \in G$ a new cohomology class $f(g)$ which is defined by

$$f(g)(t) = \frac{D(g)}{2}$$

and

$$f(g)(x) = |g| \cdot \frac{D(g)}{|g|+1} - \frac{D(g)}{2}.$$  

This cohomology class will represent a new fibration provided $|g| \neq 0$. We remark that the function $f$ constructed here is just one possible example which would suit our purposes. There are many other suitable candidates for $f$.

4.3 New fiber subgroups

Given $\phi \in H^1(M, \mathbb{Z}^2)$. We have that the new fiber subgroup $\pi_1(S_\phi)$ is given by the kernel of $\phi$, viewed as the composition

$$\pi_1(M) \to H_1(M, \mathbb{Z}) \to \mathbb{Z},$$

where the first map is the abelianization map and where the second map is $\phi$. The group $\pi_1(S_\phi)$ will always be a finite rank free group, and its rank can be computed as $||\phi||_T + 1$, since $||\phi||_T$ denotes the absolute value of the Euler characteristic of the fiber. Finding a presentation for the fiber subgroup is sometimes possible [9, 14], though in general it may be difficult. This is why we assume that free presentations for fiber subgroups are part of the public data.

4.4 Distortion of lengths

The advantage of the scheme we propose is that a very large integer is encoded by a relatively small one. Alice picks $N$, and her key is the maximal length of $\psi_\phi^N$ applied to elements of the free generating set of the fiber subgroup. To communicate $N$ over the channel, she applies $\psi_\phi^N$, viewed as conjugation by the element of $t_\phi \in \pi_1(M)$. The resulting elements of $\pi_1(M)$ will have lengths which are linear in $N$. When Alice sends information over the channel, Bob applies $t_\phi$ successively to generators of $\pi_1(S_\phi)$ and checks to see if the generating set lies in $\{x_1, \ldots, x_t\}$. This will require linearly many calculations in $N$ and $t$, since the word problem in hyperbolic 3-manifold groups has linear complexity. Once Bob
finds the first such $N$, this will be the same value of $N$ as chosen by Alice, since
$\psi_\phi$ is pseudo-Anosov and is therefore not periodic.

The key that Alice and Bob wish to share is instead $\ell_{\max}$, which is the
maximal length of $\psi_\phi^N$ applied to a generator of $\pi_1(S_\phi)$, viewed as an element
of $\pi_1(S_\phi)$. Alice and Bob now both know $N$, and thus can recover $\ell_{\max}$, since
$\psi_\phi$ is a public automorphism of the free group $\pi_1(S_\phi)$. The size of $\ell_{\max}$ will
be exponential in $N$. The precise distortion can be computed from the Teichmüller
polynomial. McMullen computes the Teichmüller polynomial for that fibered
face to be

$$\Theta(t, u) = 1 - t(1 + u + u^{-1}) + t^2.$$ 

Here, the polynomial $\Theta(t, u)$ is viewed as an element of the group ring $\mathbb{Z}[H, t^{-1}]$, where here $H = \langle u \rangle$ denotes the $\psi_\phi$-invariant homology of $S_\phi$. In this case, the
generator $t$ can be identified with the stable letter $t$ of the fibration defining
$M$, and we can write the other generator as $u = [x] + [y] + [z]$, the sum of
the homology classes of the three punctures. Then, we can write

$$\Theta(t, u) = \sum_{g \in H_1(M, \mathbb{Z})} a_g g,$$

with $a_g \in \mathbb{Z}$. If $\phi \in H^1(M, \mathbb{Z})$, then we view $\phi$ as an element of $\text{Hom}(\pi_1(M), \mathbb{Z})$
and we can write

$$\Theta(k) = \sum_{g \in H_1(M, \mathbb{Z})} a_g k^\phi(g).$$

We set $\lambda_\phi$ to be the largest root of $\Theta(k)$, which will always be a real number
which is greater than one. Then we have $\ell_{\max} \sim \lambda_\phi^N$.

5 An application to symmetric-key cryptography

The scheme developed in Sections 3 and 4 can be simplified somewhat to yield
a symmetric-key cryptographic scheme. Such a scheme would eschew the need
for an initial private shared key. The setup for this scheme would be as follows.

**Public information:** The fibered hyperbolic manifold $M$, presented as a
mapping torus, would be public as before. A presentation for $\pi_1(M)$ would also
be public.

**Private information:** Alice and Bob would agree beforehand on a fibered
cohomology class $\phi$ of $M$. This would yield a private fibered subgroup $\pi_1(S_\phi)$
with a preferred presentation, and a private automorphism $\psi_\phi$ of $\pi_1(S_\phi)$.

The implementation of the cryptoscheme would be as follows:

1. Alice chooses an arbitrary integer $N$.
2. Alice communicates the integer $N$ to Bob over a public channel.
3. Alice and Bob both compute $\psi_\phi^N$ on the generators of $\pi_1(S_\phi)$ in the preferred
   presentation, and compute the lengths of the resulting words.
4. The shared key is $\ell_{\max}$, the longest length of a word obtained by applying
   $\psi_\phi^N$ to the generators.
6 Remarks on security

There are several layers of security which are built into the schemes we propose. In the public-key scheme, the first layer of security lies in similar security assumptions of the public key-exchange scheme à la Anshel-Anshel-Goldfeld (AAG). We recall that this is used by Alice and Bob to agree on a fibered cohomology class. We note that there have been several proposals for the platform group for the AAG protocol, such as braid groups, polycyclic groups, Grigorchuk groups, certain classes of right-angled Artin groups and their subgroups, in which the simultaneous conjugacy search problem is difficult. Apart from a few proposed attacks (see [35, 31] for attacks adapted to braid group and linear group platforms as well as potential ripostes), the AAG scheme remains secure for suitable choices of parameters. In the symmetric-key scheme, the use of AAG is unnecessary, thus removing any vulnerabilities to attacks on that protocol.

In both schemes, the key that Alice and Bob share will be much larger than any of the public data over the channel. Thus, even if the eavesdropper is able to guess $N$, the key $\ell_{\text{max}}$ is exponentially longer than $N$ and would therefore be much more difficult to guess, short of knowing which cohomology class Alice and Bob are using.

For Alice and Bob, computations inside the group $\pi_1(S)$ are easy since this group is either free or a surface group, and hence computing word lengths and solving the word problem is relatively easy (using small cancellation theory, for instance) from the standard presentations given in the public data.

References

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