Integrability of moduli and regularity of Denjoy counterexamples

SANG-HYUN KIM AND THOMAS KOBERDA

Abstract. We study the regularity of $C^{1,\alpha}$ exceptional diffeomorphisms of the circle, i.e. those without periodic points and which have a wandering interval, and whose first derivatives are continuous with concave modulus of continuity $\alpha$. We prove that if the function $1/\alpha$ is integrable near zero, then there exists a $C^{1,\alpha}$ exceptional diffeomorphism of the circle. This result accounts for all previously known moduli of continuity for derivatives of exceptional diffeomorphisms. As a partial converse, we prove that if $\ell_i$ denotes the length of the $i$th component of a maximal wandering set of a $C^{1,\alpha}$ exceptional diffeomorphism $f$ and if $\ell_{i+1}/\ell_i \to 1$, then $\alpha^{-1}(1/i)$ controls $\ell_i$ from below. Thus, if such an $f$ exists, then the function $\alpha^{-1}(t)/t^2$ is integrable near zero. These results are related to a long-standing question of D. McDuff concerning the length spectrum of exceptional $C^1$ diffeomorphisms of the circle.

1. Introduction

Let $f \in \text{Homeo}^+(S^1)$ be an orientation preserving homeomorphism of the circle without any periodic points. It is well-known that in this case $f$ has an irrational rotation number $\theta$. Here, we make the identification $S^1 = \mathbb{R}/\mathbb{Z}$, and we have the rotation number

$$\text{rot}(f) = \lim_{n \to \infty} \frac{F^n(x) - x}{n} \pmod{1} \in \mathbb{R}/\mathbb{Z},$$

where $F$ is any lift of $f$ to $\mathbb{R}$ and $x \in \mathbb{R}$ is arbitrary. It is well-known that $\text{rot}(f)$ is independent of $x$ and of the choice of lift.

Throughout this paper, we shall always assume that $f \in \text{Homeo}^+(S^1)$ has irrational rotation number $\theta$ unless otherwise noted. A standard fact going back to Poincaré asserts that if $f$ has a dense orbit, then $f$ is topologically conjugate to an irrational rotation by $\theta$, denoted $T(\theta)$. If $f$ does not have a dense orbit then it must have a wandering interval, which is to say a nonempty interval $J$ such that $f^n(J) \cap J = \emptyset$ for $n \neq 0$.

Date: August 21, 2019.
2010 Mathematics Subject Classification. Primary: 37E10; Secondary: 37C05, 37C15.
Key words and phrases. Exceptional circle diffeomorphism, moduli of continuity, length spectrum.
A classical result of Denjoy asserts that if \( f \) is twice differentiable (or if in fact the derivative of \( f \) has bounded variation), then \( f \) is topologically conjugate to a rotation by \( \theta \). In lower levels of regularity, this fact ceases to hold. It is easy to produce continuous examples which are not topologically conjugate to a rotation, and Denjoy showed that one can construct differentiable examples for every rotation number \( \theta \). Such examples will be called *exceptional* diffeomorphisms, since they have a so-called exceptional minimal set, which in this case will be homeomorphic to a Cantor set (see Theorem 2.1.1 of \([9]\)).

### 1.1. Main results.

In this paper, we consider the problem of determining which moduli of continuity can be imposed on the derivatives of exceptional diffeomorphisms of the circle. Here, a *modulus of continuity* is a homeomorphism

\[
\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}
\]

which is concave as a function. The function \( \alpha \) will oftentimes be called a *concave modulus*. A function \( g : S^1 \to \mathbb{R} \) is said to be \( \alpha \)-continuous if it satisfies

\[
\sup_{x \neq y} \frac{|g(x) - g(y)|}{\alpha(|x - y|)} \leq \infty.
\]

Here, we interpret \(|x - y|\) by identifying \( S^1 \) with \( \mathbb{R}/\mathbb{Z} \) and computing this difference modulo 1. The value of this supremum is sometimes called the \( \alpha \)-norm of \( g \), and is written \([g]_\alpha\). We say that \( f \in \text{Homeo}^+(S^1) \) is \( C^{1,\alpha} \) if \( f \) is differentiable and if \([f']_\alpha < \infty\). Commonly, one writes \( f \in \text{Diff}_{+}^{1,\alpha}(S^1) \). Here, we always implicitly assume that \( f^{-1} \in \text{Diff}_{+}^{1,\alpha}(S^1) \) as well.

Note that if \( \alpha(x) = x \), then \( \alpha \)-continuity is just Lipschitz continuity. More generally, if \( \alpha(x) = x^\tau \) for some \( 0 < \tau \leq 1 \) then \( \alpha \)-continuity is \( \tau \)-Hölder continuity. For every \( 0 < \tau < 1 \), it is known that there exist \( C^{1,\tau} \) exceptional diffeomorphisms of the circle for arbitrary \( \theta \) (see \([4]\), also Theorem 3.1.2 of \([9]\)). In fact, Herman \([4]\) proved that for arbitrary \( \theta \) there exist \( C^{1,\alpha} \) exceptional diffeomorphisms of the circle for

\[
\alpha(x) = x(\log 1/x)^{1+\varepsilon}
\]

for all \( \varepsilon > 0 \), which implies the corresponding conclusion for Hölder moduli. Moreover, he proved that such diffeomorphisms can be chosen arbitrarily \( C^1 \)-close to a rotation by \( \theta \), and with uniformly bounded \( C^{1,\alpha} \)-norms. It is a well-known open problem to determine whether or not there exist \( C^{1,\alpha} \) exceptional diffeomorphisms of the circle for \( \alpha(x) = x \log 1/x \).

Before stating our results, we introduce some notation and terminology. We will consider a maximal wandering set of open intervals for \( f \) in the circle and write them as \( \{J_i\}_{i \in \mathbb{Z}} \). These intervals are characterized by the property that \( f(J_i) = J_{i+1} \), and that any interval properly containing \( J_i \) for any index \( i \) must meet the exceptional minimal set of \( f \). We will write \( \ell_i \) for the length of the interval \( J_i \).
and we will oftentimes refer to the collection \( \{J_i\}_{i \in \mathbb{Z}} \) as a \textit{maximal wandering set}. Observe that a maximal wandering set may not be unique.

**Theorem 1.1.** Let \( \theta \) be arbitrary, and suppose that we have

\[
\int_0^1 \frac{1}{\alpha(x)} \, dx < \infty.
\]

Then there exists a \( C^{1,\alpha} \) exceptional diffeomorphism \( f \) of \( S^1 \) with rotation number \( \theta \). Moreover, we may arrange for \( \ell_{i+1}/\ell_i \to 1 \) as \( i \to \infty \), and for \( f \) to be arbitrarily \( C^1 \)–close to a rotation by \( \theta \).

Theorem 1.1 recovers all previously known possible moduli of continuity for derivatives of exceptional diffeomorphisms of the circle. In particular, Theorem 1.1 recovers the fact that there are \( C^{1,\tau} \) exceptional diffeomorphisms for every irrational rotation number for all \( 0 < \tau < 1 \), as well as Herman’s corresponding result for \( \alpha(x) = x(\log 1/x)^{1+\epsilon} \). Moreover, we can immediately assert the existence of exceptional \( C^{1,\alpha} \) diffeomorphisms for moduli which were previously unrecorded in the literature, such as

\[
\alpha(x) = x(\log 1/x)(\log \log 1/x)^{1+\epsilon}.
\]

Theorem 1.2 may be compared with Lemma 3.1.3 of [9]. We immediately obtain the following consequence of Theorem 1.2, by a straightforward change of variables.

**Corollary 1.3.** Suppose \( f \) is an exceptional \( C^{1,\alpha} \) diffeomorphism, and suppose that \( \ell_{i+1}/\ell_i \to 1 \) as \( i \to \infty \). Then

\[
\int_0^1 \frac{\alpha^{-1}(t)}{t^2} \, dt < \infty.
\]

Notice that since \( \alpha \) is a homeomorphism of the non-negative reals, we may make sense of the notation \( \alpha^{-1} \) as a function. Theorem 1.2 recovers the fact that if \( \ell_{i+1}/\ell_i \to 1 \) as \( i \to \infty \), then \( f' \) cannot be \( \alpha \)–continuous for \( \alpha(x) = x \log 1/x \) (see Exercise 4.1.26 and the examples in section 4.1.4 of [9]).

The final main result is as follows, and provides a partial converse to Theorem 1.2.
Theorem 1.4. Let \( \theta \) be arbitrary. Suppose that
\[
\int_0^1 \frac{\alpha^{-1}(t)}{t^2} \, dt < \infty,
\]
and that
\[
\sup_{t > 0} \frac{\alpha(t)}{ta'(t)} < \infty.
\]
Then there exists a \( C^{1,\alpha} \) exceptional diffeomorphism \( f \) with rotation number \( \theta \), and such that \( \ell_{i+1}/\ell_i \to 1 \) as \( i \to \infty \).

Here, the assumption that \( \alpha \) is differentiable does not result in any loss of generality [8]. We remark that the supremum in Theorem 1.4 is bounded below by 1 as follows from the standard concavity estimate \( ta'(t) \leq \alpha(t) \). Theorem 1.4 is closest to a converse for Theorem 1.2, since the hypotheses will be shown to imply that \( 1/\alpha(x) \) becomes integrable near zero.

1.2. Denjoy counterexamples beyond moduli of continuity. Identifying the precise conditions under which a homeomorphism \( f \in \text{Homeo}^+(S^1) \) with irrational rotation number is necessarily topologically conjugate to an irrational rotation is tantalizing. Some regularity of \( f^{-1} \) is necessary, as was demonstrated by Hall [3]. Even a characterization of moduli of continuity for which there exist exceptional diffeomorphisms does not appear to be the end of the discussion. For instance, Sullivan proved that if the logarithm of the derivative of \( f \) satisfies the Zygmund condition (also called the “big” Zygmund condition), then \( f \) is topologically conjugate to an irrational rotation. More generally, Hu–Sullivan [5] show that if the derivative of \( f \) has bounded quadratic variation and bounded Zygmund variation, then \( f \) is topologically conjugate to an irrational rotation. In the same paper, Hu–Sullivan show that if the logarithm of the cross ratio distortion of \( f \) has bounded variation, then \( f \) cannot have any wandering intervals and hence must by topologically conjugate to an irrational rotation. None of these conditions on \( f \) or \( f' \) seem to be expressible in terms of moduli of continuity for \( f' \).

2. Endpoint derivatives, the fundamental estimate, and McDuff’s question

In this section, we investigate the successive ratios of lengths of intervals in the wandering set of an exceptional diffeomorphism \( f \).

2.1. Endpoint derivatives and the fundamental estimate. As before, we write \( \{J_i\}_{i \in \mathbb{Z}} \) for a maximal wandering set of \( f \), so that \( f(J_i) = J_{i+1} \), and so that \( \ell_i \) is the length of \( J_i \). The following lemma is well-known, and is crucial for our work in this article.

Lemma 2.1. Suppose that \( \ell_{i+1}/\ell_i \to 1 \) as \( i \to \infty \). Then \( f'(y) = 1 \) for \( y \in \partial J_i \).
Here, we need only assume this limit as \( i \to \infty \), and not necessarily \( i \to \pm \infty \). Note that Lemma 2.1 immediately implies that \( f' \) is identically 1 on the exceptional minimal set of \( f \).

**Proof of Lemma 2.1.** Let \( y \in \partial J_i \). Then there is a sequence \( n_k \to \infty \) of indices and points \( y_k \in \partial J_{n_k} \) such that \( y_k \to y \). The Mean Value Theorem implies that there exists a point \( z_k \in J_{n_k} \) such that \( f'(z_k) = \ell_{n_k+1}/\ell_{n_k} \), and this ratio converges to 1 as \( k \to \infty \), whence we have that \( f'(z_k) \to 1 \). Finally, we have that \( |z_k - y_k| \leq \ell_{n_k} \to 0 \). Combining these observations, we obtain \( f'(y) = 1 \) by continuity of \( f' \). \( \square \)

From Lemma 2.1, we obtain the following crucial fact.

**Lemma 2.2 (Fundamental Estimate).** Suppose that \( f \) is a \( C^{1,\alpha} \) exceptional diffeomorphism, and that \( \ell_{i+1}/\ell_i \to 1 \) as \( i \to \infty \). Then

\[
\sup_i \frac{1}{\alpha(\ell_i)} \left( 1 - \frac{\ell_{i+1}}{\ell_i} \right) < \infty.
\]

**Proof.** Let \( J_i \) be fixed. The Mean Value Theorem implies that there is a point \( z_i \in J_i \) such that \( f'(z_i) = \ell_{i+1}/\ell_i \). From Lemma 2.1 we have that \( f'(y) = 1 \) for \( y \in \partial J_i \). Since \( f' \) is \( \alpha \)-continuous, there exists a constant \( K \) such that \( |f'(y) - f'(z_i)| \leq K\alpha(|y - z_i|) \). We thus obtain

\[
1 - \frac{\ell_{i+1}}{\ell_i} \leq K\alpha(|y - z_i|) \leq K\alpha(\ell_i).
\]

The lemma follows immediately. \( \square \)

2.2. **Remarks on McDuff’s Question.** It is evident from the discussion in this section that the assumption \( \ell_{i+1}/\ell_i \to 1 \) for the lengths of the successive wandering intervals implies the strong conclusion that \( f' = 1 \) on the endpoints of the wandering intervals. This is a somewhat restrictive phenomenon, and the methods of this paper are not suited to address the possibility that \( f' \) is not identically 1 on the endpoints. For instance, the fundamental estimate, the 1 appearing in the expression

\[
\left( 1 - \frac{\ell_{i+1}}{\ell_i} \right)
\]

must be replaced by the corresponding endpoint derivative. This will be destructive in certain inductive procedures involved in the proof of Theorem 1.2, since certain products which we need to be bounded away from zero will become products of derivatives of \( f \) at endpoints.

There is a long-standing open question about exceptional diffeomorphisms of the circle due to D. McDuff [7, 1]. In her work, she considers the ratio spectrum for a maximal wandering set of an exceptional diffeomorphism. Namely, she rearranges the lengths of the intervals to form a decreasing sequence \( \{\lambda_i\}_{i \in \mathbb{N}} \), and considers the
possible accumulation points of the set $\lambda_i/\lambda_{i+1}$. She proves that this set is bounded and that 1 is an accumulation point, and asks if 1 is in fact the only accumulation point. One can naturally strengthen her question as follows:

**Question 2.3.** Let $U \subset S^1$ be an open interval meeting the exceptional minimal set, and let $\{\lambda_i\}_{i \in \mathbb{Z}}$ be the lengths of intervals in a maximal wandering set lying in $U$, arranged in decreasing order. Does it follow that $\lambda_i/\lambda_{i+1} = 1$?

Even assuming a positive answer to this strengthened version of McDuff’s question, and even after possibly assuming strong Diophantine properties for the rotation number $\theta$, it seems impossible to remove the assumptions $\ell_{i+1}/\ell_i \to 1$ in our results. The reason for this is the mismatch which occurs upon rearranging the lengths of intervals. One may control the first return time of a wandering interval to $U$ via Diophantine properties, but the index rearrangement map

\[ i \mapsto \ell_i \mapsto \lambda_j \mapsto j \]

may be so badly behaved that one may not be able to relate $\ell_{i+1}/\ell_i$ to $\lambda_{j+1}/\lambda_j$ in a way which would be useful for our purposes.

### 3. Integrability of $1/\alpha$

In this section, we prove Theorem 1.1. Here and for the rest of the paper, we use $\lambda$ to denote Lebesgue measure.

#### 3.1. Herman’s construction of exceptional diffeomorphisms.

In this subsection, our discussion is modeled on the construction of exceptional diffeomorphisms due to Herman [4]. Here and in the sequel, we will suppress the differential when writing integrals when there is no danger of confusion.

**Proposition 3.1.** Let $\alpha$ be a concave modulus, and let $\{(x_i, y_i)\}_{i \in \mathbb{Z}}$ be a disjoint collection of intervals in $S^1$. Suppose that there exists a positive, $\alpha$–continuous map $g$ on $S^1$ satisfying the following for all $i \in \mathbb{Z}$:

- $\int_{S^1} g = 1$;
- $\int_{x_{i-1}}^{x_i} g = |x_{i+1} - x_{i+1}|$;
- $\int_{x_{i-1}}^{x_i} g = |x_{i+1} - x_i|$.

Then the map

\[ f(x) := x_1 + \int_{x_0}^{x} g \]

is a $C^{1,\alpha}$ diffeomorphism of $S^1$ such that $f(x_i, y_i) = (x_{i+1}, y_{i+1})$ for all $i$. 


Proof. It is obvious that \( f \) is bijective and \( C^{1,\alpha} \). After dividing the cases that
\[
x_{i-1} \leq x \leq x_i
\]
and that
\[
x_{i-1} \leq x_i \leq x,
\]
one can easily verify the equality
\[
x_i + \int_{x_{i-1}}^{x} g = x_{i+1} + \int_{x_i}^{x} g.
\]
The proof is then immediate. \( \square \)

It is not very difficult to construct a \( g \) satisfying the hypotheses of Proposition \( \text{3.1} \) with the extra condition \( g(x) = 1 \) on \( S^1 \setminus \bigcup_i [x_i, y_i] \). The idea is that we set \( g \) to be constantly 1 except for some suitable bumps (up or down) inside each interval \([x_i, y_i]\). We remark that the method of this subsection, however, does not extend to the case when \( \alpha(x) = x \log(1/x) \); cf. \([9, \text{Exercise 4.1.26}]\) and \([6, \text{Chapter 12}]\).

For disjoint compact intervals \( A = [a, a'], B = [b, b'], C = [c, c'] \) of \( S^1 \), we write \( A < B < C \) if we have the relation
\[
a \leq a' < b \leq b' < c \leq c' < a,
\]
where here the relation \( < \) is interpreted in the circular order on \( S^1 \).

**Definition 3.2.** Let \( \{J_i\}_{i \in \mathbb{Z}} \) be a sequence of compact intervals in \( S^1 \). Suppose we have the relation
\[
J_i < J_j < J_k
\]
if and only if we have
\[
J_{i+1} < J_{j+1} < J_{k+1}.
\]
Then we say that the sequence \( \{J_i\} \) is *circular order preserving*.

We use the notation
\[
\langle x \rangle = x - \lfloor x \rfloor \in [0, 1)
\]
for \( x \in \mathbb{R} \).

**Example 3.3.** Let \( \theta \) be a given irrational number, and let \( \{\ell_i\}_{i \in \mathbb{Z}} \) be a positive sequence such that
\[
L := \sum_i \ell_i \in (0, 1].
\]
Using the Dirac measure \( \delta_p \) for \( p \in S^1 \), we define a measure \( \mu \) on \( S^1 \) as
\[
\mu := (1 - L)\lambda + \sum_i \ell_i \delta_{\langle \ell_i \rangle \theta}.
\]
We let \( x_i := \mu[0, \langle i \theta \rangle) \) for \( i \in \mathbb{Z} \). Then the set
\[
\{ J_i := [x_i, x_i + \ell_i] \}_{i \in \mathbb{Z}}
\]
is a disjoint, circular order preserving collection of compact intervals in \( S^1 \). We call \( \{ J_i \} \) a blow-up of the sequence \( \{ \langle i \theta \rangle \} \subseteq S^1 \).

**Proposition 3.4.** Let \( \alpha \) be a concave modulus. Suppose we have a disjoint, circular order preserving sequence \( \{ J_i = [x_i, y_i] \}_{i \in \mathbb{Z}} \) of compact intervals in \( S^1 \) such that
\[
\lambda \left( [x_{i-1}, x_i] \setminus \bigcup_{k \in \mathbb{Z}} J_k \right) = \lambda \left( [x_i, x_{i+1}] \setminus \bigcup_{k \in \mathbb{Z}} J_k \right)
\]
for all \( i \). For \( \ell_i := |J_i| \), we also assume that
\[
\sup_i \frac{1}{\alpha(\ell_i)} \left( 1 - \frac{\ell_{i+1}}{\ell_i} \right) < \infty.
\]
Then there exists a \( C^{1,\alpha} \) diffeomorphism \( f \) of \( S^1 \) satisfying \( f(J_i) = J_{i+1} \) for all \( i \).

The reader will note the appearance of the conclusion of the fundamental estimate (Lemma 2.2) occurring in the statement of the proposition.

**Proof of Proposition 3.4.** Let \( \chi_J \) denote the indicator function of \( J \subseteq S^1 \). We let \( \rho_i \) be an arbitrary smooth function supported on \([0, 1]\) such that \( \int \rho_i = 1 \) and such that
\[
1 - \left( 1 - \frac{\ell_{i+1}}{\ell_i} \right) \rho_i(x) > 0
\]
for all \( x \). It is a straightforward exercise to produce such a function \( \rho_i \) for each \( i \), and in fact one may assume that \( \rho_i \) is bounded independently of \( i \), since we have that
\[
\inf_i \frac{\ell_{i+1}}{\ell_i} > 0
\]
by the fundamental estimate (Lemma 2.2). Define a positive function
\[
g(x) := 1 - \sum_i \left( 1 - \frac{\ell_{i+1}}{\ell_i} \right) \chi_{J_i}(x) \rho_i \left( \frac{x - x_i}{\ell_i} \right).
\]
Such functions have been constructed by many authors; see Section 12.2 of [6], and particularly X.3 of [4], for instance. It is obvious that the hypotheses of Proposition 3.1 are satisfied.

For distinct points \( x, y \in J_i \), we have
\[
\sup_{x, y \in J_i} \frac{|g(x) - g(y)|}{\alpha(x - y)} = \left( 1 - \frac{\ell_{i+1}}{\ell_i} \right) \sup_{s, t \in [0, 1]} \frac{|\rho_i(s) - \rho_i(t)|}{\alpha(\ell_i(s - t))} \leq \frac{1}{\alpha(\ell_i)} \left( 1 - \frac{\ell_{i+1}}{\ell_i} \right) \| \rho_i \|_{\text{Lip}}.
\]
Note that here we used the fact that \( x/\alpha(x) \) is monotone increasing. It follows that \( [g]_{J_i, \alpha} \) is bounded. Since \( g = 1 \) outside \( \bigcup_i J_i \), it follows that \( [f]_{\alpha} = [g]_{\alpha} \) is bounded. \( \square \)
Corollary 3.5. Let $\alpha$ be a concave modulus. If a positive sequence $\{\ell_i\}_{i \in \mathbb{Z}}$ satisfies that $\sum \ell_i \leq 1$ and that
\[
\sup_i \frac{1}{\alpha(\ell_i)} \left( 1 - \frac{\ell_{i+1}}{\ell_i} \right) < \infty,
\]
then there exists an exceptional $C^{1,\alpha}$ diffeomorphism $f$ of $S^1$ with a wandering interval $J \subseteq S^1$, and such that $|f^i(J)| = \ell_i$ for all $i$.

Proof. The corollary is a simple consequence of Example 3.3 and Proposition 3.4. \qed

3.2. Integrability of moduli. In order to establish Theorem 1.1, it now suffices to translate the integrability of the function $1/\alpha$ near zero into a sequence of interval lengths $\ell_i$ which allow us to apply Corollary 3.5.

Proof of Theorem 1.1. As follows from the work of Medvedev \cite{8}, we lose no generality with the assumption that $\alpha$ is smooth. We put
\[
K := \max \{ 2, 1/\alpha(1) \},
\]
and set
\[
v(x) := x^2 \alpha(1/x).
\]
For all $t \geq 1$, we have
\[
v(x/t) = (x^2/t^2) \cdot \alpha(t/x) \geq (x^2/t^2) \cdot \alpha(1/x) = v(x)/t^2.
\]
Since $x/\alpha$ is monotone increasing, we have
\[
(v/x)' = (\alpha(1/x)/(1/x))' \geq 0.
\]
In particular, whenever $x \geq K$ we have
\[
v(x) \geq x \cdot v(1)/1 \geq x/K.
\]

We also note from the estimate
\[
0 \geq (\alpha/x)' = (x\alpha' - \alpha)/x^2,
\]
we have that $0 < x\alpha' \leq \alpha$ for all $x$. So, we get
\[
v = x^2 \alpha(1/x) \geq x\alpha'(1/x) > 0.
\]
For all $x \geq K$, we obtain that
\[
xv' = x(2x\alpha(1/x) - \alpha'(1/x)) = 2v - x\alpha'(1/x) \in [v, 2v].
\]

We now set
\[
\ell_i := 1/v(|i| + K)
\]
for all $i \in \mathbb{Z}$. Since
\[
\int_K^\infty 1/v = \int_0^{1/K} 1/\alpha < \infty,
\]
we see that $\sum \ell_i \leq 1$, possibly increasing $K$ if necessary.
Let \( i \in \mathbb{Z} \) and set \( j = |i| \). Note that
\[
v(j + K) \geq (j + K)/K,
\]
and that
\[
v(j + K \pm 1) \geq v((j + K)/2) \geq v(j + K)/4.
\]
For some \( y_0 \) between \( j + K \) and \( j + K \pm 1 \), we have
\[
\frac{1}{\alpha(\ell_j)} \left| 1 - \frac{\ell_{j+1}}{\ell_j} \right| = \frac{1}{\ell_j^2 v(1/\ell_j)} \left| 1 - \frac{v(j + K)}{v(j + K \pm 1)} \right|
\]
\[
= \frac{v(j + K)^2}{v \circ v(j + K)} \cdot \frac{v'(y_0)}{v(j + K \pm 1)} = \frac{v(j + K)}{v \circ v(j + K)} \cdot \frac{v(j + K)}{v(j + K \pm 1)} \cdot v'(y_0)
\]
\[
\leq \frac{j/K + 1}{v(j/K + 1)} \cdot 4 \cdot \frac{2v(y_0)}{y_0} \leq \frac{j/K + 1}{v(2j + 2K)/(2K)^2} \cdot \frac{8v(y_0)}{y_0}
\]
\[
= 32K^2 \frac{j + K}{y_0} \cdot \frac{v(y_0)}{v(2j + 2K)} \leq 64K.
\]
Hence the conditions of Corollary 3.5 are met.

By choosing \( L = \sum_i \ell_i \approx 0 \) in the proof of Proposition 3.4 we may require that
\[
\|f\|_1 = \max(\|f - T(\theta)\|, \|f' - 1\|)
\]
is as small as desired. To see that \( \|f - T(\theta)\| \to 0 \) as \( L \to 0 \), we may simply choose the index shift \( K \) to be very large, whence \( f \) will converge to rotation by \( \theta \). To see that \( \|f' - 1\| \to 0 \), we note that \( \|f' - 1\| \) vanishes if \( x \notin J_i \) for some \( i \), and is equal to
\[
\left(1 - \frac{\ell_{i+1}}{\ell_i}\right) \rho_i \left(\frac{x - x_i}{\ell_i}\right)
\]
for \( x \in J_i \). Since \( \rho_i \) is bounded independently of \( i \) and since \( 1 - \frac{\ell_{i+1}}{\ell_i} \to 0 \) as \( i \to \infty \) by the fundamental estimate, we see that \( \|f' - 1\| \) indeed tends to 1. \( \square \)

**Remark 3.6.** For the estimates \( \|f\|_1 = \max(\|f - T(\theta)\|, \|f' - 1\|) \) in the proof of Theorem 1.1 the reader may compare our arguments with ones proposed in [4]. It is not difficult to extend the argument carried out here to show that \([f']_a\) may be chosen to be uniformly bounded even as \( \|f\|_1 \to 0 \), though we are unable to force \([f']_a \to 0 \) as well.

### 3.3. The Yoccoz equivariant family

Here we record an explicit construction of vector fields \( \rho_i \) and derivative \( g = f' \) conditioned to a collection of intervals \( \{J_i\}_{i \in \mathbb{Z}} \) constructed by Yoccoz. This family has the additional feature that it is equivariant. We follow the definition given in Section 4.1.4 of [9].

One begins with
\[
\varphi_a(x) := -\frac{1}{a} \cot \left(\frac{\pi x}{a}\right),
\]
defined on \((0, a)\). Then the map

\[ \varphi_{a,b} := \varphi_b^{-1} \circ \varphi_a \]

extends to a \(C^1\) diffeomorphism from \([0, a]\) to \([0, b]\), which is tangent to the identity at the endpoints. Let us reconstruct this family \(\{\varphi_{a,b}\}\) from integrating against a vector field.

For each \(r \in (0, 1]\), let us set

\[ h(r, x) := \frac{1 + r}{1 + r^2 \cot^2(\pi x)} \]

A straightforward calculation shows that \(h(r, x) \in C^1_{[0,1]}(\mathbb{R})\). It is easy to see that

\[ \int_0^1 h(r, x) \, dx = 1. \]

Moreover, for \(a > b > 0\) and for

\[ u := \varphi_{a,b}(x), \quad r := b/a, \]

one verifies as in [9] that

\[ \varphi_{a,b}'(x) = \frac{u^2 + 1/a^2}{u^2 + 1/b^2} = 1 - \frac{1 - r^2}{1 + (bu)^2} = 1 - \frac{1 - r^2}{1 + r^2 \cot^2(\pi x/a)} = 1 - (1-r)h(r, x/a). \]

So, the vector field

\[ \rho_i(x) := h(\ell_{i+1}/\ell_i, x) \]

induces the desired function \(g = f'\) in Theorem 1.1.

**Remark 3.7.** It is natural to consider whether or not Theorem 1.1 can be suitably generalized for higher rank free abelian group actions on the circle. The natural hypothesis in this case would be for

\[ \int_0^1 \frac{1}{\alpha_d} \, dx < \infty, \]

and the transform \(\nu\) would be defined by the formula \(\nu(x) = x^{d+1} \alpha(1/x)^d\). Then, for each multi-index \((i_1, \ldots, i_d) \in \mathbb{Z}^d\), one could assign length values

\[ \ell_{(i_1, \ldots, i_d)} = \frac{1}{\nu(|i_1| + \cdots + |i_d| + K)} \]

for some suitable \(K \gg 0\). The construction of \(d\) pairwise commuting diffeomorphisms then becomes more delicate than the construction of a single diffeomorphism, though the Yoccoz equivariant family provides a suitable construction.
Then, one is left with the task of showing that the derivatives of these diffeomorphisms are indeed \( \alpha \)-continuous. By Lemma 4.1.25 of [9], this reduces to verifying the inequality
\[
\left| \frac{\ell_{i+e_j}}{\ell_i} - 1 \right| \frac{1}{\alpha(\ell_i)} \leq C
\]
for some universal constant \( C \), where here \( i \) denotes a multi-index and \( e_j \) denotes a standard unit multi-index.

In verifying this equality, most estimates in the proof of Theorem 1.1 carry over with suitable modifications, including the estimate \( (v(x)/x)' > 0 \), the estimate \( v(Kx) \leq K^{d+1}v(x) \), and the estimate
\[
v(x/K) \geq \frac{v(x)}{K^{d+1}}.
\]

One then encounters the expression
\[
\left( \frac{v(j + K)}{v \circ v(j + K)} \right)^{1/d} \cdot \frac{v(j + K)}{v(j + K \pm 1)} \cdot v'(y_0),
\]
which needs to be controlled by a constant. In the proof of Theorem 1.1, we have \( d = 1 \) and hence all the terms in this expression cancel to make this expression dominated by a constant, though the moment \( d > 1 \) one loses this control.

With some fairly straightforward manipulations, one obtains that the desired bound holds, as long as
\[
\liminf_{x \to 0} \frac{x}{\alpha(x^{d+1}/\alpha(x)^d)} > 0.
\]

We thus obtain the following proposition:

**Proposition 3.8.** Let \( d > 1 \) and let \( \alpha \) be a concave modulus such that
\[
\int_0^1 \frac{1}{\alpha^d} \, dx < \infty.
\]

Then there exists a free \( C^{1,\alpha} \) action of \( \mathbb{Z}^d \) on \( S^1 \) with exceptional minimal set, provided that the limit
\[
\liminf_{x \to 0} \frac{x}{\alpha(x^{d+1}/\alpha(x)^d)} > 0
\]
holds.

We remark that this limit holds for the modulus
\[
\alpha(x) = (x \log 1/x)^{1+d\epsilon})^{1/d}
\]
for \( \epsilon > 0 \) as considered by Deroin–Kleptsyn–Navas [2], and for Hölder moduli smaller than \( 1/d \).
4. Bounding lengths from below

In this section, we establish Theorem 1.2 and Theorem 1.4.

4.1. Bounding the values of $\alpha(\ell_i)$. It is evident that passing from the fundamental estimate (Lemma 2.2) to a priori bounds on the values of $\{\ell_i\}_{i \in \mathbb{Z}}$ is a natural approach to a converse to Theorem 1.1, and this is precisely the sort of control given to us in Theorem 1.2 above.

Proof of Theorem 1.2. From the fundamental estimate (Lemma 2.2), we see that

$$\ell_{i+1} \geq \ell_i (1 - K\alpha(\ell_i))$$

for a fixed constant $K$ and for all $i$. We define the function $h(x) = x(1 - K\alpha(x))$, so that we obtain $\ell_{i+1} \geq h(\ell_i)$. It is easy to show that $h$ is increasing for small values of $x$, and that in fact $h'(0) \geq 1$.

Inductively, we suppose that $\alpha(\ell_i) \geq A/i$. After possibly increasing the index to some $i \geq i_0$, we obtain

$$\ell_{i+1} \geq h(\ell_i) \geq h(\alpha^{-1}(A/i)) = \alpha^{-1} \left( \frac{A}{i} \right) \left( 1 - \frac{AK}{i} \right),$$

where the second inequality holds because $\alpha$ is a homeomorphism and because $h$ is increasing near zero. We apply $\alpha$ to this string of inequalities now. Since $\alpha$ is concave, we have that for any $0 < c < 1$, there is an inequality $\alpha(cx) \geq c\alpha(x)$. Thus, after applying $\alpha$ to the right–most hand side, we get a quantity bounded below by

$$\left( 1 - \frac{AK}{i} \right) \frac{A}{i}.$$ 

If one can show that this quantity is bounded below by $A/(i + 1)$, then one obtains $\alpha(\ell_{i+1}) \geq A/(i + 1)$, the desired conclusion. A straightforward manipulation shows that we obtain

$$\left( 1 - \frac{AK}{i} \right) \frac{A}{i} \geq \frac{A}{i + 1}$$

provided that

$$i \geq \frac{KA}{1 - KA}.$$ 

Thus, letting $A$ be a positive constant so that $\alpha(\ell_k) \geq A/k$ for $0 < k \leq i_0$ and so that

$$1 \geq \frac{KA}{1 - KA},$$

we obtain the desired conclusion. □
The reader will note that whereas this proof apparently argues for positive indices, the same argument will automatically imply the same conclusion for negative indices after a suitable re-indexing. Since $\ell_{i+1}/\ell_i \to 1$ as $i \to \infty$, we have that $f'$ is identically 1 on the endpoints of the wandering intervals in the maximal wandering set under consideration. It therefore follows that $\ell_{i-1}/\ell_i \to 1$ as $i \to -\infty$, and the same fundamental estimate continues to hold, so that we still obtain $\alpha(\ell_i) \geq -A/i$ for a suitably small $A > 0$ and $i$ negative.

**Example 4.1.** Let $f$ be an exceptional diffeomorphism such that $\ell_{i+1}/\ell_i \to 1$ for intervals in a maximal wandering set. Then it follows that $f'$ is not $\alpha$-continuous for $\alpha = x \log 1/x$ (cf. [9,6]). Indeed, it suffices to show that if $A$ is a fixed positive constant and if $\alpha(\ell_i) \geq A/i$ for all $i \geq 2$, then $\ell_i \to \infty$. We set

$$x_i = \frac{A}{K \cdot j \log j},$$

where $A$ is chosen once and for all and $K$ is a positive constant which will be fixed later. Observe that $\sum_i x_i = \infty$, so it suffices to show that $\alpha(x_i) \leq A/i$ for all $i \geq 2$. Applying $\alpha$, we see

$$\alpha(x_i) = \frac{A}{K \cdot i \log i} \left( \log K + \log i + \log \log i + \log A^{-1} \right).$$

This simplifies to

$$\frac{A(\log K + \log A^{-1})}{K} \cdot \frac{1}{i \log i} + \frac{A}{K \cdot i} + \frac{A \cdot \log \log i}{K \cdot i \log i}.$$  

Since $1/(i \log i) < 1/i$ for $i \geq 3$, since $(\log \log i)/\log i < 1$ for $i \geq 2$, and since $A$ is a fixed positive constant, it is clear that we can choose a sufficiently large $K$ to make this entire expression less than $A/i$ for $i \geq 2$, which is what we set out to show.

It is not difficult to generalize this previous computation to show the same conclusion for the moduli

$$\alpha(x) = x(\log 1/x)(\log \log 1/x),$$

and indeed for

$$\alpha(x) = x(\log 1/x)(\log \log 1/x)(\log \log \log 1/x) \cdots (\log^n 1/x).$$
4.2. A converse to Theorem 1.2. We now investigate the degree to which $\alpha(\ell_i) \geq A/i$ is a sufficient condition for the existence of a $C^{1, \alpha}$ exceptional diffeomorphism with $\ell_{i+1}/\ell_i \to 1$ as $i \to \infty$. The precise statement we prove is Theorem 1.4, though we prove something slightly different here.

**Lemma 4.2.** Let $\alpha$ be smooth concave modulus such that

$$\sup_{x > 0} \frac{\alpha(x)}{x \alpha'(x)} < \infty.$$ 

Then if

$$\int_0^1 \frac{\alpha^{-1}(x)}{x^2} \, dx < \infty,$$

we have that

$$\int_0^1 \frac{1}{\alpha(x)} \, dx < \infty.$$ 

Theorem 1.4 follows immediately from Lemma 4.2 by applying Theorem 1.1.

**Proof of Lemma 4.2.** We first consider the substitution $x = \alpha(t)$ and $dx = \alpha'(t) \, dt$. Then we have

$$\int_0^1 \frac{\alpha^{-1}(x)}{x^2} \, dx = \int_0^{\alpha^{-1}(1)} \frac{t \alpha'(t)}{(\alpha(t))^2} \, dt,$$

and these integrals are finite. On the other hand, we have that

$$\int_0^{\alpha^{-1}(1)} \frac{1}{\alpha(t)} \, dt = \int_0^{\alpha^{-1}(1)} \frac{t \alpha'(t)}{(\alpha(t))^2} \cdot \frac{\alpha(t)}{t \alpha'(t)} \, dt.$$ 

On the right hand side, we have that

$$\sup_{t > 0} \frac{\alpha(t)}{t \alpha'(t)}$$

is finite by hypothesis, and hence the corresponding term in the product can be bounded on $(0, \alpha^{-1}(1)]$. The integral of $t \alpha'(t)/(\alpha(t))^2$ is also finite by assumption, so that consequently the integral on the left hand side is finite. Since the convergence of

$$\int_0^1 \frac{1}{\alpha(x)} \, dx$$

depends only on the behavior of this integral near zero, we obtain the desired conclusion. \qed
Remark 4.3. One could attempt to prove a converse to Theorem 1.2 by starting with the sequence \( \{A/i\}_{i>0} \) and attempting to find a convergent sequence of interval lengths \( \{\ell_i\}_{i\in\mathbb{Z}} \) such that

\[
\sup_i \frac{1}{\alpha(\ell_i)} \left( 1 - \frac{\ell_{i+1}}{\ell_i} \right) = K
\]

is finite, whereupon one can apply Corollary 3.5 in order to find the desired exceptional diffeomorphism. A natural choice of lengths would be

\[
\ell_i = \alpha^{-1} \left( \frac{A}{|i| + i_0} \right),
\]

where here \( i_0 \in \mathbb{N} \) is a suitable shift of indices.

With this choice, the finiteness of the supremum forces the following inequality to hold:

\[
\alpha^{-1} \left( \frac{A}{|i| + i_0 + 1} \right) \geq \alpha^{-1} \left( \frac{A}{|i| + i_0} \right) \left( 1 - \frac{KA}{|i| + i_0} \right),
\]

which is equivalent to

\[
\frac{A}{|i| + i_0 + 1} \geq \alpha \left( \alpha^{-1} \left( \frac{A}{|i| + i_0} \right) \left( 1 - \frac{KA}{|i| + i_0} \right) \right).
\]

One can then try and pull out the constant \( 1 - (KA)/(|i| + i_0) \) on the right hand side, but because this constant is less than one, the result will be less than or equal to the right hand side. In order to get around this difficulty, one can attempt a linear approximation of \( \alpha \) at \( \alpha^{-1}(A/(|i| + i_0)) \) in order to show that

\[
\frac{A}{|i| + i_0 + 1} \geq \left( 1 - \frac{KA}{|i| + i_0} \right) \left( \frac{A}{|i| + i_0} \right),
\]

possibly up to a nonzero multiplicative constant. After some straightforward manipulations, one quickly finds that the hypothesis

\[
\sup_{x>0} \frac{\alpha(x)}{x \alpha'(x)} < \infty
\]

is needed to make the linear estimate work, and this is precisely the other hypothesis of Theorem 1.4. This last supremum is easily seen not to be finite for arbitrary concave moduli and therefore its finiteness imposes a nontrivial hypothesis on \( \alpha \). As an example, one can consider \( \alpha(x) = 1/\log(1/x) \).

Acknowledgements

The authors are grateful to M. Triestino for helpful comments on an earlier draft of this paper. The first author is supported through the National Research Foundation funded by the government of Korea, by th Mid-Career Researcher Program
(2018R1A2B6004003). The second author is partially supported by an Alfred P. Sloan Foundation Research Fellowship, and by NSF Grant DMS-1711488. The second author is grateful to the Korea Institute for Advanced Study for its hospitality while this research was completed.

REFERENCES


School of Mathematics, Korea Institute for Advanced Study (KIAS), Seoul, 02455, Korea
Email address: skim.math@gmail.com
URL: http://cayley.kr

Department of Mathematics, University of Virginia, Charlottesville, VA 22904-4137, USA
Email address: thomas.koberda@gmail.com
URL: http://faculty.virginia.edu/Koberda