

Free products and the algebraic structure of diffeomorphism groups

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ABSTRACT. Let M be a compact one–manifold. We prove that the class of finitely generated subgroups of the group $\text{Diff}_+^{1+\text{bv}}(M)$, the group of C^1 orientation preserving diffeomorphisms of M whose first derivatives have bounded variation, is not closed under taking finite free products. As a corollary, we show that the class of finitely generated subgroups of $\text{Diff}^k(M)$, the group of C^k diffeomorphisms of M , is not closed under taking finite free products provided $k \geq 2$. By contrast, if the level of regularity is decreased to mere continuity, then the corresponding class of finitely generated groups of homeomorphisms is closed under taking free products. More precisely, let G be a group which is not virtually metabelian. We show that the group $(G \times \mathbb{Z}) * \mathbb{Z}$ is not a subgroup of $\text{Diff}_+^{1+\text{bv}}(M)$. As a corollary, many common groups of homeomorphisms cannot be realized as groups of $C^{1+\text{bv}}$ diffeomorphisms, for example $F * \mathbb{Z}$, where F denotes Thompson’s group F . We also complete the classification of right-angled Artin groups which can act smoothly on M and in particular, recover the main result of the joint work of the authors with Baik. Namely, a right-angled Artin group $A(\Gamma)$ either admits a faithful C^∞ action on M , or $A(\Gamma)$ admits no faithful $C^{1+\text{bv}}$ action on M . In the former case, $A(\Gamma) \cong A(\Gamma_1) \times \cdots \times A(\Gamma_k)$, where Γ_i is possibly empty for $i \geq 2$, and where each connected component of Γ_i is complete for $1 \leq i \leq k$. Finally, we develop a hierarchy of right-angled Artin groups, with the levels of the hierarchy corresponding to the number of semi-conjugacy classes of possible actions of these groups on the circle.

1. INTRODUCTION

Let M be a compact one–manifold. In this article, we study the algebraic structure of the group $\text{Diff}_+^{1+\text{bv}}(M)$, where here $\text{Diff}_+^{1+\text{bv}}(M)$ denotes the group of C^1 diffeomorphisms of M whose derivatives have bounded variation. Specifically, we consider the restrictions placed on subgroups of $\text{Diff}_+^{1+\text{bv}}(M)$ by the $C^{1+\text{bv}}$ regularity assumption. Our main result implies that the class of finitely generated subgroups of $\text{Diff}_+^{1+\text{bv}}(M)$ is not closed under taking finite free products. As a corollary, we complete a program initiated by Baik and the authors in [3, 2] to decide which

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right-angled Artin groups admit faithful C^∞ actions on a compact one-manifold, and exhibit many classes of finitely generated subgroups of $\text{Homeo}^+(M)$ which cannot be realized as subgroups of $\text{Diff}_+^{1+\text{bv}}(M)$.

1.1. Statement of results. Unless otherwise noted, M will denote a compact one-manifold. That is to say, M is a finite union of disjoint closed intervals $I = [0, 1]$ and circles $S^1 = \mathbb{R}/\mathbb{Z}$. An *action* on a one-manifold is always assumed to mean an orientation preserving action. For each $0 \leq r \leq \infty$ or $r = \omega$, we let $\text{Diff}_+^r(M)$ denote the group of orientation preserving C^r (analytic if $r = \omega$) diffeomorphisms. We write $\text{Homeo}^+(M) = \text{Diff}_+^0(M)$. Our main result is the following:

Theorem 1.1. *If G is a group which is not virtually metabelian, then the group $(G \times \mathbb{Z}) * \mathbb{Z}$ admits no faithful $C^{1+\text{bv}}$ action on M .*

Theorem 1.1 clearly implies that $(G \times \mathbb{Z}) * \mathbb{Z}$ does not embed into $\text{Diff}_+^r(M)$ for every level of regularity $r \geq 2$. A key step in our proof of Theorem 1.1 is the following result on C^1 -smoothability:

Theorem 1.2 (Theorem 3.1). *Let $X \in \{I, S^1\}$, and let $a, b, t \in \text{Diff}_+^1(X)$. If*

$$\text{supp } a \cap \text{supp } b = \emptyset,$$

*then the group $\langle a, b, t \rangle$ is not isomorphic to $\mathbb{Z}^2 * \mathbb{Z}$.*

We have stated Theorem 1.1 as above for clarity and concision, though several stronger statements can be deduced from the proof we give, especially for finitely generated groups:

Corollary 1.3. *If G is a finitely generated group which is not virtually abelian, then $(G \times \mathbb{Z}) * \mathbb{Z}$ admits no faithful $C^{1+\text{bv}}$ action on M .*

Corollary 1.4. *Let G be a group.*

- (1) *If G is nonabelian and if the center of G is nontrivial, then $G * \mathbb{Z}$ admits no faithful $C^{1+\text{bv}}$ action on I .*
- (2) *Suppose G is finitely generated. If G is not abelian-by-finite cyclic and if the center of G contains a copy of \mathbb{Z} , then $G * \mathbb{Z}$ admits no faithful $C^{1+\text{bv}}$ action on S^1 .*

Here, a group G is \mathcal{X} -by- \mathcal{Y} for group theoretic properties \mathcal{X} and \mathcal{Y} if there is an exact sequence

$$1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$$

such that K has property \mathcal{X} and Q has property \mathcal{Y} . We allow both K and Q to be trivial.

Theorem 1.1 has several other corollaries. Note that $G \times \mathbb{Z}$ often occurs as a subgroup of $\text{Diff}_+^{1+\text{bv}}(M)$, where G is not virtually metabelian. We have the following immediate consequence:

Corollary 1.5. *Let \mathcal{G} denote the class of finitely generated subgroups of $\text{Diff}_+^{1+\text{bv}}(M)$. The class \mathcal{G} is not closed under taking finite free products.*

The motivation for proving Theorem 1.1 came from investigating right-angled Artin subgroups of $\text{Diff}_+^{1+\text{bv}}(M)$ (cf. Corollary 1.8 below). The simplest right-angled Artin group to which Theorem 1.1 applies is $(F_2 \times \mathbb{Z}) * \mathbb{Z}$:

Corollary 1.6. *The group $(F_2 \times \mathbb{Z}) * \mathbb{Z}$ is not a subgroup of $\text{Diff}_+^{1+\text{bv}}(M)$.*

Since the hypotheses on Theorem 1.1 are relatively weak, there are many finitely generated subgroups of $\text{Homeo}^+(M)$ which can be thus shown to admit no faithful $C^{1+\text{bv}}$ actions on a compact manifold. Recall that *Thompson's group F* is a group of piecewise linear homeomorphisms of the interval, with dyadic break points and with all slopes being powers of two. *Thompson's group T* is the analogous group defined for the circle. A famous result of Ghys–Sergiescu [19] says that the standard actions of T and F are topologically conjugate into a group of C^∞ diffeomorphisms of the circle and of the interval, respectively. However, we have the following:

Corollary 1.7. *The groups $F * \mathbb{Z}$ and $T * \mathbb{Z}$ are not subgroups of $\text{Diff}_+^{1+\text{bv}}(M)$.*

Returning to the original motivation, Theorem 1.1 combined with results of Farb–Franks and Jorquera completes the classification of right-angled Artin groups admitting actions of various regularities on compact one-manifolds. Before stating the result, we define some terminology. We write Γ for a finite simplicial graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The *right-angled Artin group* $A(\Gamma)$ is defined by

$$A(\Gamma) = \langle V(\Gamma) \mid [v, w] = 1 \text{ if and only if } \{v, w\} \in E(\Gamma) \rangle.$$

Recall that a simplicial graph Γ is called *P_4 -free* if no induced (full) subgraph of Γ is isomorphic to a path P_4 on four vertices. Such graphs are often called *cographs* [11, 26]. We write \mathcal{K} for the class of cographs. It is well-known that cographs can be fit into a hierarchy which is defined as follows:

- (1) The class \mathcal{K}_0 consists of a single vertex;
- (2) If $n \geq 1$ is odd then \mathcal{K}_n is obtained by taking \mathcal{K}_{n-1} together with finite joins of elements in \mathcal{K}_{n-1} ;
- (3) If $n \geq 2$ is even then \mathcal{K}_n is obtained by taking \mathcal{K}_{n-1} together with finite disjoint unions of elements in \mathcal{K}_{n-1} .

We have that $\mathcal{K}_0 \subset \mathcal{K}_1 \subset \mathcal{K}_2 \subset \dots$ and

$$\mathcal{K} = \bigcup_i \mathcal{K}_i.$$

Note that join and disjoint union correspond to direct product and free product respectively, so that right-angled Artin groups on cographs are exactly the smallest

class of groups containing \mathbb{Z} , which is closed under finite direct products, and which is closed under finite free products.

Corollary 1.8 (cf. [2]). *Let $A(\Gamma)$ be a right-angled Artin group.*

- (1) (see [16, 24]) *There exists an injective homomorphism $A(\Gamma) \rightarrow \text{Diff}_+^1(M)$; thus, $A(\Gamma)$ admits a faithful C^1 action of M .*
- (2) *If there exists an injective homomorphism $A(\Gamma) \rightarrow \text{Diff}_+^{1+\text{bv}}(M)$ then $\Gamma \in \mathcal{H}_3$; conversely, if $\Gamma \in \mathcal{H}_3$ then $A(\Gamma) \leq \text{Diff}_+^\infty(M)$.*

Let S be an orientable surface of genus g and with n punctures or boundary components. We say that S is *sporadic* if

$$3g - 3 + n \leq 1.$$

We write $\text{Mod}(S)$ for the mapping class group of S , i.e. $\text{Mod}(S) = \pi_0(\text{Homeo}^+(S))$. Using the main result of [29], we immediately recover the following result as a corollary of Theorem 1.1:

Corollary 1.9 (cf. [2]). *Let M be a compact one-manifold. There exists a finite index subgroup $G \leq \text{Mod}(S)$ such that $G \leq \text{Diff}_+^{1+\text{bv}}(M)$ if and only if S is sporadic.*

In contrast to Corollary 1.5, we have the following, which is well-known from several contexts (see [32, 39] for instance):

Proposition 1.10. *The class of countable subgroups of $\text{Homeo}^+(I)$ is closed under finite free products.*

Theorem 1.1 allows us to build a hierarchy on right-angled Artin groups, whose levels correspond to right-angled Artin groups with more or fewer “dynamically different” actions on the circle. Roughly speaking, two group actions

$$\rho_1, \rho_2: G \rightarrow \text{Homeo}^+(S^1)$$

are *semi-conjugate* (or, *monotone-equivalent*) if there exists another action

$$\rho: G \rightarrow \text{Homeo}^+(S^1)$$

and monotone degree one maps $h_i: S^1 \rightarrow S^1$ such that

$$h_i \circ \rho = \rho_i \circ h_i$$

for each $i = 1, 2$. See [17, 9, 18, 36, 7, 28] for instance, and the many references therein. A *projective action* of a group G is a representation

$$\rho: G \rightarrow \text{PSL}_2(\mathbb{R}),$$

where $\text{PSL}_2(\mathbb{R})$ sits inside of $\text{Homeo}^+(S^1)$ as the group of projective analytic diffeomorphisms of S^1 .

Corollary 1.11. *Let $A(\Gamma)$ be a right-angled Artin group.*

- (1) If $\Gamma \in \mathcal{K}_2$, then $A(\Gamma)$ admits uncountably many distinct semi-conjugacy classes of faithful orientation preserving projective actions on S^1 ;
- (2) If $\Gamma \in \mathcal{K}_3 \setminus \mathcal{K}_2$ then any faithful orientation preserving $C^{1+\text{bv}}$ action of $A(\Gamma)$ on S^1 has a periodic point and no dense orbits and hence admits at most countably many distinct semi-conjugacy classes of $C^{1+\text{bv}}$ actions on S^1 ;
- (3) If $\Gamma \notin \mathcal{K}_3$ then $A(\Gamma)$ admits no faithful $C^{1+\text{bv}}$ action on S^1 .

In the case of analytic actions on a compact connected one-manifold M , Akhmedov and Cohen proved that $A(\Gamma)$ embeds into $\text{Diff}^\omega(M)$ if and only if $\Gamma \in \mathcal{K}_2$ [1].

1.2. Notes and references. This paper reveals some of the subtlety of the interplay between algebra and regularity in diffeomorphism groups of one-manifolds. Our paper arose during the effort to complete the classification of right-angled Artin subgroups of $\text{Diff}_+^\infty(S^1)$ in the spirit of [2]. The essential content of this paper is Lemma 3.11, which exhibits an explicit element of the kernel of any given $C^{1+\text{bv}}$ action of $(G \times \mathbb{Z}) * \mathbb{Z}$ action on a compact one-manifold. Since a group of the form $(G \times \mathbb{Z}) * \mathbb{Z}$ is generally simpler than the right-angled Artin groups considered in [2], it is more difficult to find elements in the kernel of a given action, and therefore we develop more sophisticated tools here. We note that our main result does subsume the main result of [2]. Indeed, the main result of [2] is that there is no injective homomorphism $A(P_4) \rightarrow \text{Diff}_+^{1+\text{bv}}(M)$, where here

$$A(P_4) = \langle a, b, c, d \mid [a, b] = [b, c] = [c, d] = 1 \rangle.$$

The group $A(P_4)$ contains a copy of $(F_2 \times \mathbb{Z}) * \mathbb{Z}$, which cannot embed in $\text{Diff}_+^{1+\text{bv}}(M)$ by Corollary 1.6. An explicit embedding of $(F_2 \times \mathbb{Z}) * \mathbb{Z}$ into $A(P_4)$ is given by $\langle a, b, c, a^d \rangle \leq A(P_4)$ (see [26] for a discussion of this fact).

The program completed by Corollary 1.8 fully answers a question raised in a paper of M. Kapovich (attributed to Kharlamov) as to which right-angled Artin groups admit faithful C^∞ actions on the circle [25].

Right-angled Artin subgroups of diffeomorphism groups of one-manifolds find an analogue in right-angled Artin subgroups of linear groups. It is well-known that right-angled Artin groups are always linear over \mathbb{Z} and hence admit injective homomorphisms into $\text{SL}_n(\mathbb{Z})$ [23, 22, 12]. For $\text{SL}_3(\mathbb{Z})$, it is still unclear which right-angled Artin groups appear as subgroups. Long-Reid [34] showed that $F_2 \times \mathbb{Z}$ is not a subgroup of $\text{SL}_3(\mathbb{Z})$, which implies that any right-angled Artin subgroup of $\text{SL}_3(\mathbb{Z})$ is a free product of free abelian groups of rank at most two. It is currently unknown whether or not $\mathbb{Z}^2 * \mathbb{Z}$ is a subgroup of $\text{SL}_3(\mathbb{Z})$.

A consequence of the technical work behind Theorem 1.1 is a certain criterion to prove that a group is not *virtually special* (see [20, 42]). More generally, we obtain a criterion to prove that a group is not even virtually or residually right-angled

Artin group, which is different from standard algebraic and geometric methods. See Lemma 3.5 and Theorem 3.12 for precise statements.

A crucial step in our proof of the main theorem is a C^1 -rigidity result, which is Theorem 3.1. We note that Thurston [41], Calegari [8], Navas [37] and Bonatti, Monteverde, Navas and Rivas [6] explored various remarkable C^1 -rigidity results. In particular, a C^1 -rigidity result on Baumslag–Solitar groups in [6] was employed in a very recent paper by Bonatti, Lodha and Triestino [5], to produce certain piecewise affine homeomorphism groups of \mathbb{R} which do not embed into $\text{Diff}_+^1(I)$.

As for other classes of groups of homeomorphisms which cannot be realized as groups of $C^{1+\text{bv}}$ diffeomorphisms, Corollary 1.7 is a complement to a result of the second author with Lodha [30], in which they show that certain “square roots” of Thompson’s group F may fail to act faithfully by $C^{1+\text{bv}}$ diffeomorphisms on a compact one–manifold, even though they are manifestly groups of homeomorphisms of these manifolds. Thus, the Ghys–Sergiescu Theorem appears to place F and T at the cusp of smoothability in the sense that even relatively minor algebraic variations on F and on T fail to be smoothable.

Finally, we remark on the differentiability hypothesis. The $C^{1+\text{bv}}$ hypothesis may look artificial at first glance, but just as explained in [2], it is actually an optimal hypothesis. Corollary 1.8 shows that every right-angled Artin group can act faithfully by C^1 diffeomorphisms, but nearly none of them can act by $C^{1+\text{bv}}$ diffeomorphisms. As for Corollary 1.5 and Proposition 1.10, we are left with the following threshold case which appears to remain open:

Question 1.12. *Let M be a compact one–manifold. Is the class of finitely generated subgroups of $\text{Diff}_+^1(M)$ closed under finite free products?*

Some of the difficulty of Question 1.12 stems from smoothing oscillatory behavior. By imitating the the proof of Proposition 1.10 below, it is not difficult to show that if $G \leq \text{Diff}_0^1(I)$ then $G * \mathbb{Z} \leq \text{Diff}^1[0, \infty)$, where here $\text{Diff}_0^1(I)$ denotes the group of diffeomorphisms of the interval which are tangent to the identity at $\{0, 1\}$. However, the resulting action of $G * \mathbb{Z}$ on $[0, \infty)$ will exhibit oscillatory behavior at infinity, and it is not clear how to conjugate this action back to $\text{Diff}_0^1(I)$.

2. BACKGROUND ON ONE–DIMENSIONAL SMOOTH DYNAMICS

We very briefly summarize the necessary background from one–dimensional dynamics. The reader may also consult [2], parts of which we repeat here, and where we direct the reader for proofs.

2.1. Poincaré's theory of rotation numbers. Let $f \in \text{Homeo}^+(S^1)$, and let $\tilde{f}: \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary lift of f . Then the *rotation number* of f is defined as

$$\text{rot } f = \lim_{n \rightarrow \infty} \frac{\tilde{f}^n(x)}{n} \in \mathbb{R}/\mathbb{Z} = S^1$$

where $x \in \mathbb{R}$. Then $\text{rot } f$ is well-defined, and independent of the choice of a lift \tilde{f} and a base point $x \in \mathbb{R}$; see [38] for instance. The set of periodic points of f is denoted as $\text{Per } f$. Let us record some elementary facts.

Lemma 2.1. *For $f \in \text{Homeo}^+(S^1)$, the following hold.*

- (1) $\text{rot}(f) = 0$ if and only if $\text{Fix } f \neq \emptyset$.
- (2) $\text{rot}(f) \in \mathbb{Q}$ if and only if $\text{Per } f \neq \emptyset$.
- (3) If $x \in S^1$ and $g \in \text{Homeo}^+(S^1)$ satisfy that

$$f^n(x) = g^n(x)$$

for all $n \in \mathbb{Z}$, then $\text{rot}(f) = \text{rot}(g)$.

The rotation number is a continuous class function (that is, constant on each conjugacy class)

$$\text{rot}: \text{Homeo}^+(S^1) \rightarrow S^1.$$

Moreover, the rotation number restricts to a group homomorphism on each amenable subgroup of $\text{Homeo}^+(S^1)$; see [18].

Let us recall the following classical result.

Theorem 2.2 (Hölder's Theorem [21]; see [38]). *A group acting freely on \mathbb{R} or on S^1 by orientation preserving homeomorphisms is abelian.*

We will use the following variation, which is similar to [15, Theorem 2.2].

Corollary 2.3 (cf. [15]). *Let X be a nonempty closed subset of S^1 , and let G be a group acting freely on X by orientation preserving homeomorphisms. Then the action of G extends to a free action $\rho: G \rightarrow \text{Homeo}^+(S^1)$ such that*

$$\text{rot} \circ \rho: G \rightarrow S^1$$

is an injective group homomorphism.

Proof. If (a, b) is a component of $S^1 \setminus X$ and $g \in G$, then (ga, gb) is also a component of $S^1 \setminus X$. So, G extends to some action ρ on S^1 in the affine manner. Put

$$S^1 \setminus X = \bigsqcup_{i \geq 1} I_i.$$

If $\rho(g)y = y$ for some $g \in G$ and for some $y \in S^1$, then $y \in I_i$ for some i . We have that $\rho(g)$ restricts to the identity on I_i by definition. This would imply $\rho(g)\partial I_i = g\partial I_i = \partial I_i$, and so, $g = 1$. That is, the action ρ of G on S^1 is free.

By Hölder's theorem, we see $\rho(G) \cong G$ is abelian. Since abelian groups are amenable, we have that $\text{rot} \circ \rho$ is a group homomorphism. The freeness of the action ρ implies that $\text{rot} \circ \rho(g) \neq 0$ for all nontrivial g . \square

2.2. Koppell–Denjoy theory. Let $M \in \{I, S^1\}$. We denote by $\text{var}(g; M)$ the total variation of a map $g: M \rightarrow \mathbb{R}$:

$$\text{var}(g; M) = \sup \left\{ \sum_{i=0}^{n-1} |g(a_{i+1}) - g(a_i)| : (a_i : 0 \leq i \leq n) \text{ is a partition of } M \right\}.$$

In the case $M = S^1$, we require $a_n = a_0$ in the above definition. Following [38], we say a C^1 diffeomorphism f on M is $C^{1+\text{bv}}$ if $\text{var}(f'; M) < \infty$. We let $\text{Diff}_+^{1+\text{bv}}(M)$ denote the group of orientation preserving $C^{1+\text{bv}}$ diffeomorphisms of M . The following two results play a fundamental role on the study of $C^{1+\text{bv}}$ diffeomorphisms.

Theorem 2.4 (Denjoy's Theorem [13], [38, Theorem 3.1.1]). *If $a \in \text{Diff}_+^{1+\text{bv}}(S^1)$ and $\text{Per } a = \emptyset$, then a is topologically conjugate to an irrational rotation.*

Theorem 2.5 (Kopell's Lemma [31], [38, Theorem 4.1.1]). *Suppose $a, b \in \text{Diff}_+^{1+\text{bv}}(I)$ and $[a, b] = 1$. If $\text{Fix } a \cap (0, 1) = \emptyset$ and $b \neq 1$, then $\text{Fix } b \cap (0, 1) = \emptyset$.*

Let X be a topological space. Then we define the *support* of $h \in \text{Homeo}(X)$ as

$$\text{supp } h = X \setminus \text{Fix } h.$$

For a subgroup $G \leq \text{Homeo}(X)$, we put

$$\text{supp } G = \bigcup_{g \in G} \text{supp } g.$$

We say that $f \in \text{Homeo}(X)$ is *grounded* if $\text{Fix } f \neq \emptyset$. We note that every $f \in \text{Homeo}^+(I)$ is grounded by definition.

The following important observations on commuting $C^{1+\text{bv}}$ diffeomorphisms essentially builds on Kopell's Lemma and Hölder's Theorem.

Lemma 2.6. *The following hold:*

- (1) (*Disjointness Condition*, [2]) *Let $M \in \{I, S^1\}$, and let $a, b \in \text{Diff}_+^{1+\text{bv}}(M)$ be commuting grounded diffeomorphisms. If A and B are components of $\text{supp } a$ and $\text{supp } b$ respectively, then either $A = B$ or $A \cap B = \emptyset$.*
- (2) (*Abelian Criterion*, [15]) *If $a, b, c \in \text{Diff}_+^{1+\text{bv}}(I)$ satisfy that $\text{Fix } a = \{0, 1\}$ and that $[a, b] = 1 = [a, c]$, then $[b, c] = 1$.*

Remark 2.7. Farb and Franks proved the *Abelian Criterion* in [15, Lemma 3.2] for C^2 diffeomorphisms, though their discussion applies to $C^{1+\text{bv}}$ diffeomorphisms as well. Actually, they stated the theorem in a stronger form for *fully supported* diffeomorphisms, which means that the closure of the support of each diffeomorphism is I .

The notation $f|_A$ for a function (or set of functions) and a set A means the restriction to A . We will need the following properties of centralizer groups.

Lemma 2.8. *Let $a \in \text{Diff}_+^{1+\text{bv}}(S^1)$ be an infinite order element, and let $Z(a)$ be the centralizer of a in $\text{Diff}_+^{1+\text{bv}}(S^1)$. Then the following hold.*

(1) *If a is grounded and if a group $H \leq Z(a)$ is generated by grounded elements, then every element in H is grounded and moreover,*

$$\text{supp } a \cap \text{supp}[H, H] = \emptyset.$$

(2) *If $\text{rot } a \notin \mathbb{Q}$, then $Z(a)$ is topologically conjugate to a subgroup of $\text{SO}(2, \mathbb{R})$.*

(3) *If $\text{rot } a \in \mathbb{Q}$, then $\text{rot } Z(a) \subseteq \mathbb{Q}$.*

(4) *(cf. [15, Lemma 3.4]) The rotation number restricts to a homomorphism on $Z(a)$; in particular, every element of $[Z(a), Z(a)]$ is grounded.*

Proof. (1) Let J be a component of $\text{supp } a$. By the Disjointness Condition, the group H acts on the open interval J . Since H fixes ∂J , every element of H is grounded. The Abelian Criterion implies that

$$[H, H]|_J = 1.$$

So, we have that $J \cap \text{supp}[H, H] = \emptyset$.

(2) By Denjoy's Theorem, the map a is topologically conjugate to an irrational rotation. The centralizer of an irrational rotation in $\text{Homeo}^+(S^1)$ is $\text{SO}(2, \mathbb{R})$; see [15, Proposition 2.10] or [38, Exercise 2.2.12].

(3) Suppose some element b in $Z(a)$ has an irrational rotation number. Since $a \in Z(b)$, the part (2) implies that a is conjugate to a rotation. This is a contradiction, for a rotation with a rational rotation number must have a finite order.

(4) If $\text{rot } a$ is irrational, then the conclusion follows from the part (2). So we may assume $\text{rot } a \in \mathbb{Q}$. Then a^p is grounded for some $p \neq 0$. Since $Z(a) \leq Z(a^p)$, it suffices to prove the lemma for a^p . In other words, we may further suppose that a is grounded.

Let us put

$$G = Z(a), \quad G_0 = \text{rot}^{-1}(0) \cap G.$$

The part (1) implies that

$$G_0 = \bigcup_{x \in S^1} \text{Stab}_G(x) = \langle G_0 \rangle$$

is a group. Since rot is a class function on $\text{Homeo}^+(S^1)$, we see that $G_0 \trianglelefteq G$.

For each $x \in \text{Fix } a$ and $g \in G$, we note

$$ag(x) = ga(x) = g(x).$$

So, $\text{Fix } a$ is G -invariant. We have a nonempty proper closed G -invariant set

$$X = \partial \text{Fix } a.$$

Claim 1. *We have the following:*

$$G_0 = \bigcup_{x \in X} \text{Stab}_G(x) = \bigcap_{x \in X} \text{Stab}_G(x).$$

It suffices to show that G_0 fixes an arbitrary $x \in X$. For each $J \in \pi_0 \text{supp } a$ and $g \in G_0$, we have seen in the part (1) that $\partial J \subseteq \text{Fix } g$. Since we can write

$$X = \overline{\bigcup \{\partial J \mid J \in \pi_0 \text{supp } a\}}$$

we see that $X \subseteq \text{Fix } g$. This proves the claim.

Let $p: G \rightarrow G/G_0$ denote the quotient map. By Claim 1, the natural action

$$G/G_0 \rightarrow \text{Homeo}^+(X), \quad p(g).x = g(x)$$

is well-defined and free. By Corollary 2.3, this free action extends to a free action

$$\rho: G/G_0 \rightarrow \text{Homeo}^+(S^1)$$

such that $\text{rot} \circ \rho$ is an injective homomorphism.

For each $g \in G$, $x \in X$ and $n \in \mathbb{Z}$, we have

$$\rho \circ p(g^n)(x) = g^n(x).$$

By Lemma 2.1 (3) we see that

$$\text{rot} \circ \rho \circ p = \text{rot} \upharpoonright_G,$$

and hence, that $\text{rot} \upharpoonright_G$ is a group homomorphism. As S^1 is abelian, we also obtain

$$[G, G] \leq \ker(\text{rot} \upharpoonright_G) = G_0. \quad \square$$

Note that a finitely generated subgroup of $\text{SO}(2, \mathbb{R})$ consisting of elements with rational rotation numbers is necessarily finite. So in Lemma 2.8 (3), if G is a finitely generated subgroup of $Z(a)$ then $\text{rot}(G)$ is a finite subgroup of $S^1 \cong \text{SO}(2, \mathbb{R})$.

Remark 2.9. The part (4) of Lemma 2.8 appears in the unpublished work of Farb and Franks [15], on which our argument is based. We included here a detailed, self-contained proof for readers' convenience. We also remark the necessity of the infinite-order hypothesis, which was omitted in [15]. For example, let us consider $a, b, c \in \text{Diff}_+^\infty(S^1)$ such that for each $x \in S^1 = \mathbb{R}/\mathbb{Z}$ we have

$$a(x) = x + 1/2, \quad b(x) = x + 1/4, \quad c(x + 1/2) = c(x) + 1/2$$

and such that

$$c(0) = 0, \quad c(1/8) = 1/4, \quad c(1/4) = 3/4.$$

Then $b, c \in Z(a)$ and $(bc)^3(0) = 0$. Hence we have

$$\text{rot}(b) + \text{rot}(c) = 1/4 + 0 \neq 1/3 = \text{rot}(bc).$$

Lemma 2.10. *Let $a \in \text{Diff}_+^{1+\text{bv}}(I)$, and let $Z(a)$ be the centralizer of a in $\text{Diff}_+^{1+\text{bv}}(I)$. Then we have*

$$\text{supp } a \cap \text{supp}[Z(a), Z(a)] = \emptyset.$$

The proof is almost identical to that of Lemma 2.8 (1).

2.3. The Two-jumps Lemma. The Two-jumps Lemma was developed by Baik and the authors in [2] and is the second essential analytic result needed to establish Theorem 1.1.

Lemma 2.11 (Two-jumps Lemma, [2]). *Let $M \in \{I, S^1\}$ and let $f, g: M \rightarrow M$ be continuous maps. Suppose $(s_i), (t_i)$ and (y_i) are infinite sequences of points in M such that for each $i \geq 1$, one of the following two conditions hold:*

- (i) $f(y_i) \leq s_i = g(s_i) < y_i < t_i = f(t_i) \leq g(y_i)$;
- (ii) $g(y_i) \leq t_i = f(t_i) < y_i < s_i = g(s_i) \leq f(y_i)$.

If $|g(y_i) - f(y_i)|$ converges to 0 as i goes to infinity, then f or g fails to be C^1 .

Figure 1 illustrates the case (i) of Lemma 2.11.

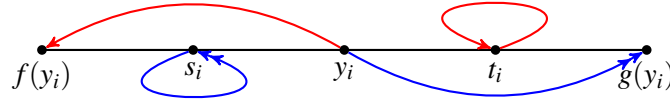


FIGURE 1. Two-jumps Lemma.

3. THE C^1 -SMOOTHABILITY OF $\mathbb{Z}^2 * \mathbb{Z}$

This section establishes the main technical result of the paper as below.

Theorem 3.1. *Let $M \in \{I, S^1\}$, and let $a, b, t \in \text{Diff}_+^1(M)$. If*

$$\text{supp } a \cap \text{supp } b = \emptyset,$$

*then the group $\langle a, b, t \rangle$ is not isomorphic to $\mathbb{Z}^2 * \mathbb{Z}$.*

Remark 3.2. (1) We emphasize that this theorem is about C^1 , rather than $C^{1+\text{bv}}$, diffeomorphisms.

(2) The regularity hypothesis of C^1 cannot be replaced by C^0 ; see Proposition 6.3.

Let us prove Theorem 3.1 through a sequence of lemmas in this section.

3.1. Finding a kernel element from a compact support. A crucial step in the proof of Theorem 3.1 is the following general construction.

Lemma 3.3 (Vanishing Commutator Lemma). *Let $H \leq \text{Homeo}^+(I)$, and let $1 \neq g_1 \in \text{Homeo}^+(I)$ such that the closure of $\text{supp } g_1$ is contained in $\text{supp } H$. Then there exists a positive integer k and elements $u_1, \dots, u_{k-1} \in H$ such that the recursively defined sequence*

$$g_{i+1} = [g_i, u_i g_i u_i^{-1}], \quad i = 1, 2, \dots, k-1,$$

satisfies that $g_i \neq u_i g_i u_i^{-1}$ for $1 \leq i < k$ and that $g_k = 1$.

Here and throughout this paper, when 1 refers to a homeomorphism or a group element then it means the identity, and otherwise it refers to the real number 1.

Proof of Lemma 3.3. Since $\overline{\text{supp } g_1}$ is a compact subset of the open set $\text{supp } H$, we can enumerate

$$I_1, I_2, \dots, I_N \in \pi_0(\text{supp } H)$$

such that $\text{supp } g_1 \cap I_i \neq \emptyset$ for each i and such that

$$\overline{\text{supp } g_1} \subseteq \bigcup_{i=1}^N I_i.$$

Note that each I_i is an open, H -invariant interval contained in $(0, 1)$.

Let us inductively construct the elements u_1, \dots, u_{k-1} satisfying the required properties. As a base case, we put $r(1) = 1 \in \mathbb{Z}$ and

$$K_1 := \overline{\text{supp } g_1} \cap I_1.$$

Since K_1 is a nonempty compact subset of $I_1 \subseteq \text{supp } H$, we have that

$$\sup\{u(\inf K_1) \mid u \in H\} = \sup I_1 > \sup K_1.$$

This implies that there exists $u_1 \in H$ such that $K_1 \cap u_1 K_1 = \emptyset$. We put

$$g_2 = [g_1, u_1 g_1 u_1^{-1}].$$

Let us first note

$$\overline{\text{supp } g_2} \subseteq \overline{\text{supp } g_1} \cup u_1 \overline{\text{supp } g_1} \subseteq \bigcup_{i=1}^N I_i.$$

Since $\text{supp } g_1 \upharpoonright_{I_1}$ and $\text{supp } u_1 g_1 u_1^{-1} \upharpoonright_{I_1}$ are nonempty and disjoint, we have $g_1 \neq u_1 g_1 u_1^{-1}$ and $g_2 \upharpoonright_{I_1} = 1$. If $N = 1$ or $g_2 \upharpoonright_{I_s} = 1$ for all $s > 1$, then $\{u_1\}$ is the desired sequence. Otherwise, we put

$$r(2) = \inf\{s > 1 \mid g_2 \upharpoonright_{I_s} \neq 1\} \geq 2.$$

Then we have

$$\overline{\text{supp } g_2} \subseteq \bigcup_{s=r(2)}^N I_s.$$

Moreover, the following set is nonempty and compact:

$$K_2 := \overline{\text{supp } g_2} \cap I_{r(2)}.$$

Suppose for some $i \geq 2$ we have constructed $u_1, \dots, u_{i-1}, g_2, \dots, g_i$ and some integer $r(i) \geq i$ such that the following hold:

(a) For each $1 \leq j \leq i-1$, we have

$$g_{j+1} = [g_j, u_j g_j u_j^{-1}], \quad g_j \neq u_j g_j u_j^{-1}.$$

(b) We have a compact and nonempty set:

$$K_i := \overline{\text{supp } g_i} \cap I_{r(i)}.$$

(c) We have the inclusion:

$$\overline{\text{supp } g_i} \subseteq \bigcup_{s=r(i)}^N I_s.$$

We proceed exactly as in the base case. That is, we find $u_i \in H$ such that $K_i \cap u_i K_i = \emptyset$ and define

$$g_{i+1} = [g_i, u_i g_i u_i^{-1}].$$

Then we again see that (a) holds for the index $i+1$ instead of i , and that $g_{i+1} \upharpoonright_{I_{r(i)}} = 1$. For each $s < r(i)$, the inductive hypothesis (b) implies that

$$g_{i+1} \upharpoonright_{I_s} = [1, u_i \cdot 1 \cdot u_i^{-1}] \upharpoonright_{I_s} = 1.$$

If $r(i) = N$ or $g_{i+1} \upharpoonright_{I_s} = 1$ for all $s > r(i)$, then the sequence u_1, \dots, u_i is the desired sequence of the lemma. Otherwise, we put

$$r(i+1) = \inf\{s > r(i) \mid g_{i+1} \upharpoonright_{I_s} \neq 1\}.$$

Then the conditions (a), (b), (c) above are satisfied with the index $i+1$. By induction, we obtain a desired sequence. \square

Lemma 3.4. *If g and u are elements of a right-angled Artin group such that g and ugu^{-1} commute, then $g = ugu^{-1}$.*

Proof. Two elements of a right-angled Artin group generate a free group or a free abelian group [4]; see also [27, Corollary 1.3]. If $g \neq ugu^{-1}$, then $\langle u, g \rangle$ is a rank-two free group; this would imply $[g, ugu^{-1}] \neq 1$. \square

Recall a group H is *residually RAAG* (*right-angled Artin group*) if for each non-trivial $h \in H$ there exists a right-angled Artin group A_h and a homomorphism $\phi_h: H \rightarrow A_h$ such that $\phi_h(h) \neq 1$. Each residually RAAG group admits an embedding into $\text{Diff}_+^\infty(\mathbb{R})$; see [3].

A group is *special* if it embeds into some right-angled Artin group [20]. A group *virtually* has some property P if a finite index subgroup of it has the property P . It is obvious that virtually special groups are virtually residually RAAG. Combining the previous two lemmas, we have the following:

Lemma 3.5 (No compact support for RAAG). *If $H \leq \text{Homeo}^+(I)$ is virtually residually RAAG, then for every $1 \neq g \in H$, we have*

$$\overline{\text{supp } g} \not\subseteq \text{supp } H.$$

Proof. Assume for a contradiction that

$$\overline{\text{supp } g} \subseteq \text{supp } H.$$

Let H_0 be a finite index normal subgroup of H that is residually RAAG, and let $R = [H : H_0]$. Then for each $h \in H$ we have that $h^R \in H_0$ and that $\text{supp } h^R = \text{supp } h$. In particular, we have

$$\overline{\text{supp } g^R} = \overline{\text{supp } g} \subseteq \text{supp } H = \text{supp } H_0.$$

We now apply Lemma 3.3 to

$$1 \neq g^R \in H_0.$$

Then we find some $h = g_{k-1} \in H_0$ and $u = u_{k-1} \in H_0$ such that $[h, u] \neq 1$ and such that $[h, uhu^{-1}] = 1$.

We have some right-angled Artin group A and a homomorphism $\phi: H_0 \rightarrow A$ such that $\phi[h, u] \neq 1$. Then Lemma 3.4 implies that $\phi[h, uhu^{-1}] \neq 1$, which is a contradiction. \square

Remark 3.6. In the above lemma, one can replace the virtually residually RAAG condition by the “virtually residually braid group” condition. This is because Lemma 3.4 also holds for pure braid groups [33]. Actually, every RAAG is a subgroup of some pure braid group [27].

Let us note a consequence of Lemma 3.5 for circle homeomorphisms.

Lemma 3.7. *Let $a, b, c, d \in \text{Homeo}^+(S^1)$ be nontrivial elements such that*

$$\text{supp } a \cap \text{supp } b = \emptyset \quad \text{and} \quad \text{supp } c \cap \text{supp } d = \emptyset.$$

If $G = \langle a, b, c, d \rangle$ is virtually residually RAAG, then $\text{supp } G \neq S^1$.

Proof. Suppose $\text{supp } G = S^1$. For simplicity, let us abbreviate $\mathcal{A} = \pi_0 \text{supp } a$, and similarly define \mathcal{B} , \mathcal{C} and \mathcal{D} . Since S^1 is compact, there exists a finite open covering \mathcal{V} of S^1 such that

$$\mathcal{V} \subseteq \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}.$$

By minimizing the cardinality, we can require that \mathcal{V} forms a *chain of intervals*. More precisely, this means that $\mathcal{V} = \{V_1, \dots, V_k\}$ for some $k \geq 1$ and that

$$\inf V_i < \sup V_{i-1} \leq \inf V_{i+1}$$

for $i = 1, 2, \dots, k$, where here the indices are taken cyclically.

Without loss of generality, let us assume $V_1 \in \mathcal{A}$. Then we have $V_{2i-1} \in \mathcal{A} \cup \mathcal{B}$ and $V_{2i} \in \mathcal{C} \cup \mathcal{D}$ for each i . Note that k is an even number and that $x = \inf V_2$ is a global fixed point of $H = \langle b, c, d \rangle$. In particular, we can regard H as acting on I , which is a two-point compactification of $S^1 \setminus \{x\}$. Note that

$$\emptyset \neq \overline{\text{supp } b} \subseteq S^1 \setminus \bigcup (\mathcal{V} \cap \mathcal{A}) \subseteq \bigcup (\mathcal{V} \cap (\mathcal{B} \cup \mathcal{C} \cup \mathcal{D})) \subseteq \text{supp } H.$$

This contradicts Lemma 3.5. \square

3.2. Supports of commutators. The following lemma illustrates how one can use commutators to find elements with compact supports.

Lemma 3.8. *If f and g are homeomorphisms of a topological space X , then*

$$\overline{\text{supp}[f, g]} \subseteq \text{supp } f \cup \text{supp } g \cup \overline{\text{supp } f \cap \text{supp } g}.$$

Proof. Suppose

$$x \notin \text{supp } f \cup \text{supp } g \cup \overline{\text{supp } f \cap \text{supp } g}.$$

Then $f(x) = x = g(x)$. Moreover, for some open neighborhood U of x we have

$$U \cap \text{supp } f \cap \text{supp } g = \emptyset.$$

We can find an open neighborhood $V \subseteq U$ of x such that

$$f^{\pm 1}(V) \cup g^{\pm 1}(V) \subseteq U.$$

Let $y \in V$. We see $[f, g](y) = y$, by considering the following three cases separately:

$$y \in V \cap \text{supp } f, \quad y \in V \cap \text{supp } g, \quad y \in V \cap \text{Fix } f \cap \text{Fix } g.$$

So we obtain that

$$[f, g]|_V = 1.$$

This implies

$$x \notin \overline{\text{supp}[f, g]}.$$

\square

Lemma 3.9. *Let X be a topological space. If $b, c, d \in \text{Homeo}(X)$ satisfy*

$$\text{supp } c \cap \text{supp } d = \emptyset,$$

then for $\phi = [c, bdb^{-1}]$ we have that

$$\text{supp } \phi \subseteq \text{supp } b \cup cb(\text{supp } b \cap \text{supp } d) \cup db^{-1}(\text{supp } b \cap \text{supp } c).$$

Proof. For brevity, let us write

$$\tilde{b} = \text{supp } b, \quad \tilde{c} = \text{supp } c, \quad \tilde{d} = \text{supp } d.$$

Let us consider three equivalent expressions for ϕ :

$$[c, bdb^{-1}] = cbd(cb)^{-1} \cdot bd^{-1}b^{-1} = c \cdot b \cdot db^{-1}c(db^{-1})^{-1} \cdot b^{-1}.$$

It follows that

$$\begin{aligned} \text{supp } \phi &\subseteq (\tilde{c} \cup b\tilde{d}) \cap (cb\tilde{d} \cup b\tilde{d}) \cap (\tilde{c} \cup \tilde{b} \cup db^{-1}\tilde{c}) \\ &\subseteq ((\tilde{c} \cap cb\tilde{d}) \cup b\tilde{d}) \cap (\tilde{c} \cup \tilde{b} \cup db^{-1}\tilde{c}) \\ &\subseteq (\tilde{c} \cap cb\tilde{d}) \cup \tilde{b} \cup (b\tilde{d} \cap (\tilde{c} \cup db^{-1}\tilde{c}) \setminus \tilde{b}) \\ &\subseteq (\tilde{c} \cap cb\tilde{d}) \cup \tilde{b} \cup (b(\tilde{d} \setminus \tilde{b}) \cap (\tilde{c} \cup db^{-1}\tilde{c})) \\ &\subseteq (\tilde{c} \cap cb\tilde{d}) \cup \tilde{b} \cup (\tilde{d} \cap db^{-1}\tilde{c}). \end{aligned}$$

Hence, it suffices for us to prove the claim below.

Claim. *We have the following:*

$$\begin{aligned} \tilde{c} \cap cb\tilde{d} &\subseteq cb(\tilde{b} \cap \tilde{d}), \\ \tilde{d} \cap db^{-1}\tilde{c} &\subseteq db^{-1}(\tilde{b} \cap \tilde{c}). \end{aligned}$$

To see the first part of the claim, let us consider $x \in X$ satisfying

$$cb(x) \in \tilde{c} \cap cb\tilde{d}.$$

Then we have $x \in \tilde{d}$ and $cb(x) \in \tilde{c}$. Since $\tilde{c} \cap \tilde{d} = \emptyset$ and $b(x) \in c^{-1}\tilde{c} = \tilde{c}$, we see $x \neq b(x)$. In particular, we have $x \in \tilde{b}$ and $cb(x) \in cb(\tilde{b} \cap \tilde{d})$. This proves the first part of the claim. The second part follows by symmetry. \square

Lemma 3.10. *If $b, c, d \in \text{Diff}_+^1(I)$ are given such that*

$$\text{supp } c \cap \text{supp } d = \emptyset,$$

then for $\phi = [c, bdb^{-1}]$ we have that

$$\overline{\text{supp } \phi \setminus \text{supp } b} \subseteq \text{supp } c \cup \text{supp } d.$$

Proof. For each $B \in \pi_0 \text{supp } b$, we define

$$J_B = B \cup cb(B \cap \text{supp } d) \cup db^{-1}(B \cap \text{supp } c).$$

By Lemma 3.9, we have that

$$\text{supp } \phi \subseteq \bigcup \{J_B \mid B \in \pi_0 \text{supp } b\} = \bigcup \{J_B \setminus B \mid B \in \pi_0 \text{supp } b\} \cup \text{supp } b.$$

Moreover, for each $B \in \pi_0 \text{supp } b$ we note that

$$\overline{J_B \setminus B} \subseteq \overline{c(B) \setminus B} \cup \overline{d(B) \setminus B} \subseteq \text{supp } c \cup \text{supp } d.$$

Claim. *The following set is a finite collection of intervals:*

$$\mathcal{B}_0 = \{B \in \pi_0 \text{supp } b \mid J_B \neq B\}.$$

We will employ the C^1 -hypothesis for this claim. Assume for a contradiction that \mathcal{B}_0 is infinite. Without loss of generality, we may suppose there are infinitely many distinct $B_1, B_2, \dots \in \mathcal{B}$ and $x_i \in B_i \cap \text{supp } d$ such that $cb(x_i) \notin B_i$. Then we have $C_1, C_2, \dots \in \mathcal{C}$ such that $b(x_i), cb(x_i) \in C_i$ and such that $x_i \notin C_i$. Let us consider the interval $J_i = [x_i, cb(x_i)]$ which contains $b(x_i)$ in the interior, up to switching the endpoints of this interval; see Figure 2. Then we have

$$(b(x_i), cb(x_i)) \cap \partial B_i \neq \emptyset, \quad [x_i, b(x_i)) \cap \partial C_i \neq \emptyset.$$

We can apply Two-jumps Lemma (Lemma 2.11) to the intervals $\{J_i\}_{i \geq 1}$ to conclude that b or c is not C^1 . This is a contradiction and so, the claim is proved.

From the claim above, we deduce the conclusion as follows.

$$\overline{\text{supp } \phi \setminus \text{supp } b} \subseteq \overline{\bigcup \{J_B \setminus B \mid B \in \mathcal{B}_0\}} = \bigcup \{\overline{J_B \setminus B} \mid B \in \mathcal{B}_0\} \subseteq \text{supp } c \cup \text{supp } d. \quad \square$$

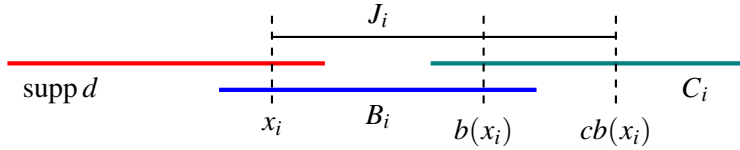


FIGURE 2. Lemma 3.10.

3.3. Finding compact supports. We will deduce Theorem 3.1 from the following, seemingly weaker result.

Lemma 3.11. *Let $M \in \{I, S^1\}$ and let $a, b, c, d \in \text{Diff}_+^1(M)$. If*

$$\text{supp } a \cap \text{supp } b = \emptyset, \quad \text{supp } c \cap \text{supp } d = \emptyset,$$

*then the group $\langle a, b, c, d \rangle$ is not isomorphic to $\mathbb{Z}^2 * \mathbb{Z}^2$.*

Proof of Theorem 3.1 from Lemma 3.11. Assume $\langle a, b, t \rangle \cong \mathbb{Z}^2 * \mathbb{Z}$. Right-angled Artin groups are Hopfian, since they are finitely generated and residually finite. So the natural surjection between groups

$$\langle A, B, T \mid [A, B] = 1 \rangle \rightarrow \langle a, b, t \rangle$$

is actually an isomorphism. It follows that

$$\langle a, b, a', b' \rangle \cong \langle A, B, A^T, B^T \rangle \cong \mathbb{Z}^2 * \mathbb{Z}^2.$$

This contradicts Lemma 3.11, since the four diffeomorphism a, b, a', b' satisfy the conditions of the lemma. \square

Proof of Lemma 3.11. We put $G = \langle a, b, c, d \rangle$ and consider an abstract group

$$G_0 = \langle a_0, b_0, c_0, d_0 \mid [a_0, b_0] = 1 = [c_0, d_0] \rangle \cong \mathbb{Z}^2 * \mathbb{Z}^2.$$

There is a natural surjection $p: G_0 \rightarrow G$ defined by

$$(a_0, b_0, c_0, d_0) \mapsto (a, b, c, d).$$

It suffices for us to show that p is not injective. For, the Hopficity of G_0 will then imply that $G \not\cong G_0$.

Assume for a contradiction that p is injective and $G \cong G_0$. By Lemma 3.7, we may assume that G has a global fixed point. In other words, we can assume $M = I$.

Let us define $\phi = [c, bdb^{-1}]$ and $\psi = [\phi, a]$. Lemma 3.8 implies that

$$\overline{\text{supp } \psi} \subseteq \text{supp } \phi \cup \text{supp } a \cup \overline{\text{supp } \phi \cap \text{supp } a}.$$

We see from Lemma 3.10 that

$$\overline{\text{supp } \phi \cap \text{supp } a} \subseteq \overline{\text{supp } \phi \setminus \text{supp } b} \subseteq \text{supp } c \cup \text{supp } d.$$

So, it follows that

$$\overline{\text{supp } \psi} \subseteq \text{supp } G.$$

As we are assuming p is injective, we have

$$\psi = [\phi, a] = [[c, bdb^{-1}], a] \neq 1.$$

Lemma 3.5 implies that G is not virtually residually RAAG, which is a contradiction. This completes the proof. \square

Our proof actually shows the following.

Theorem 3.12. *Let $M \in \{I, S^1\}$ and let $a, b, c, d \in \text{Diff}_+^1(M)$ such that*

$$\text{supp } a \cap \text{supp } b = \emptyset, \quad \text{supp } c \cap \text{supp } d = \emptyset.$$

If the element

$$[[c, bdb^{-1}], a]$$

is a nontrivial diffeomorphism, then $\langle a, b, c, d \rangle$ is not virtually residually RAAG. In particular, $\langle a, b, c, d \rangle$ is not virtually special.

4. PROOF OF THEOREM 1.1

In this section, we apply the facts we have gathered to complete the proof of the main result.

4.1. Reducing to the connected case. We first show that without loss of generality, $M \in \{I, S^1\}$. The following lemmas are straightforward, and we include proofs for the convenience of the reader.

Lemma 4.1. *Suppose A, B, C, D are groups, and suppose that $A \times B$ is a normal subgroup of $C * D$. Then at least one of these four groups is trivial.*

Proof. If $A \times B$ is a subgroup of $C * D$ then the Kurosh Subgroup Theorem implies that there is a free product decomposition

$$A \times B \cong F * \ast_i H_i,$$

where F is a free group (possibly of infinite rank) and where each H_i is conjugate into C or into D . By analyzing centralizers of elements, it is easy to show that a nontrivial free product is never isomorphic to a nontrivial direct product (cf. [35, p.177]). It follows that $A \times B$ is conjugate into C or D , which contradicts the normality of $A \times B$. \square

An alternative proof of Lemma 4.1 can be given using Bass–Serre theory (see [40]).

Lemma 4.2. *Suppose A, B, C, D are nontrivial groups, and that $A * B \leq C \times D$. Then there is an injective homomorphism from $A * B$ into either C or D .*

Proof. Suppose the contrary, so that K_C and K_D are the (nontrivial) kernels of the inclusion of $A * B$ into $C \times D$ composed with the projections onto C and D . Then $K_C \cap K_D = 1$ and K_C and K_D normalize each other, so that $K_C K_D \cong K_C \times K_D \leq A * B$. This contradicts Lemma 4.1. \square

Lemma 4.3. *Suppose*

$$M = \prod_{i=1}^n M_i$$

*is a compact one-manifold, and suppose that $A * B$ embeds into $\text{Diff}_+^{1+\text{bv}}(M)$. Then for some finite index subgroups $A_0 \leq A$ and $B_0 \leq B$, and for some i , we have an embedding of $A_0 * B_0$ into $\text{Diff}_+^{1+\text{bv}}(M_i)$.*

Proof. This follows immediately from Lemma 4.2, using the fact that $\text{Diff}_+^{1+\text{bv}}(M)$ is commensurable with

$$\prod_{i=1}^n \text{Diff}_+^{1+\text{bv}}(M_i).$$

Note that passage to finite index subgroups is necessary, since M may consist of a union of diffeomorphic manifolds which are permuted by the action of $A * B$. \square

4.2. Taming supports. Let us denote the center of a group G by Z_G . If M is a one-manifold and if $s \in \text{Diff}_+^{1+\text{bv}}(M)$, then we denote by $Z(s)$ the centralizer of s in $\text{Diff}_+^{1+\text{bv}}(M)$. The following lemma is crucial for applying Lemma 3.11:

Lemma 4.4. *Assume one of the following:*

- (i) $M = I$ and G is a nonabelian group such that $Z_G \neq 1$.
- (ii) $M = S^1$ and G is a non-metablian group such that $\mathbb{Z} \leq Z_G$.
- (iii) $M = S^1$ and G is a finitely generated group such that $\mathbb{Z} \leq Z_G$ and such that G is not abelian-by-finite cyclic.

In each of the cases, if $G \leq \text{Diff}_+^{1+\text{bv}}(M)$, then there is a subgroup $\mathbb{Z}^2 \leq G$ generated by diffeomorphisms a and b such that $\text{supp } a \cap \text{supp } b = \emptyset$.

Proof. *Case (i).* Let us pick $s \in Z_G \setminus 1$ and $b \in [G, G] \setminus 1$. Since $G \leq Z(s)$, Lemma 2.10 implies that

$$\text{supp } s \cap \text{supp } b = \emptyset.$$

Since $\text{Homeo}^+(I)$ is torsion-free, we have $\langle b, s \rangle \cong \mathbb{Z}^2$ as desired.

Case (ii). We are given with some $s \in Z_G$ such that $\langle s \rangle \cong \mathbb{Z}$. As G is nonabelian, Lemma 2.8 (2) implies that $\text{rot}(s) \in \mathbb{Q}$; in particular, s^n is grounded for some $n \geq 1$. From the part (4) of the same lemma and from that $G \leq Z(s)$, we see every element of $[G, G]$ is grounded. Since $[G, G] \leq Z(s^n)$, we note from Lemma 2.8 (1) that

$$\text{supp}(s^n) \cap \text{supp } G'' = \emptyset.$$

From the metablian hypothesis, we can find $b \in G'' \setminus 1$. As s^n and b are grounded, they have infinite orders. It follows that $\langle s^n, b \rangle \cong \mathbb{Z}^2$.

Case (iii). Let us proceed similarly to the case (ii). Namely, pick $s \in Z_G$ such that $\langle s \rangle \cong \mathbb{Z}$. By Lemma 2.8, we have a homomorphism

$$\text{rot} \upharpoonright_G: G \rightarrow \mathbb{Q}.$$

We fix $n \geq 1$ such that s^n is grounded. As G is finitely generated, we see $\text{rot}(G)$ is finite cyclic. The hypothesis implies that $G_0 = \ker(\text{rot} \upharpoonright_G)$ is not abelian. Since every element of G_0 is grounded, we can apply Lemma 2.8 (1) and deduce

$$\text{supp } s^n \cap \text{supp}[G_0, G_0] = \emptyset.$$

Each $b \in [G_0, G_0] \setminus 1$ then yields the desired subgroup $\langle b, s^n \rangle \cong \mathbb{Z}^2$. □

4.3. Main result.

Proof of Theorem 1.1. Suppose $(G \times \mathbb{Z}) * \mathbb{Z} \leq \text{Diff}_+^{1+\text{bv}}(M)$ for some compact one-manifold M . Replacing G by a finite index subgroup if necessary, we may assume that M is connected, by Lemma 4.3. By applying the cases (i) and (ii) of Lemma 4.4

to the group $G \times \mathbb{Z}$, we can find a subgroup $\langle a, b \rangle \cong \mathbb{Z}^2 \leq G \times \mathbb{Z}$ such that $\text{supp } a \cap \text{supp } b = \emptyset$. If we write the \mathbb{Z} -free factor of $(G \times \mathbb{Z}) * \mathbb{Z}$ as $\langle t \rangle$, then

$$\langle a, b, t \rangle \cong \langle a, b \rangle * \langle t \rangle \cong \mathbb{Z}^2 * \mathbb{Z}.$$

This contradicts Theorem 3.1. \square

One can also deduce Corollaries 1.3 and 1.4 from Lemma 4.4, in the exact same fashion as Theorem 1.1.

5. SMOOTH RIGHT-ANGLED ARTIN GROUP ACTIONS ON COMPACT ONE-MANIFOLDS

In this and the remaining sections, we deduce several corollaries from Theorem 1.1. We first complete the classification of right-angled Artin groups which admit faithful C^∞ actions on a compact one-manifold (Corollary 1.8).

Lemma 5.1. *Let $A(\Gamma)$ be a right-angled Artin group. Then one of the following mutually exclusive conclusions holds:*

- (1) *We have $(F_2 \times \mathbb{Z}) * \mathbb{Z} \leq A(\Gamma)$;*
- (2) *The graph Γ lies in \mathcal{K}_3 .*

Proof. Let us consider a stratification of graph classes:

$$\mathcal{K}_2 \subseteq \mathcal{K}_3 \subseteq \mathcal{K}.$$

Suppose $\Gamma \in \mathcal{K}_2$. Then $A(\Gamma)$ is the free product of free abelian groups, and hence contains no copy of $(F_2 \times \mathbb{Z}) * \mathbb{Z}$.

Let $\Gamma \in \mathcal{K}_3 \setminus \mathcal{K}_2$. Then Γ is the join of at least two graphs Γ_1, Γ_2 in \mathcal{K}_2 . We write

$$A(\Gamma) = A(\Gamma_1) \times A(\Gamma_2).$$

If $A(\Gamma)$ contains a copy of $(F_2 \times \mathbb{Z}) * \mathbb{Z}$, then so does $A(\Gamma_1)$ or $A(\Gamma_2)$ by Lemma 4.2; this would contradict the previous paragraph.

Assume $\Gamma \in \mathcal{K} \setminus \mathcal{K}_3$. First consider the case that $\Gamma \in \mathcal{K}_{2i} \setminus \mathcal{K}_{2i-1}$ for some $i \geq 2$. We can write

$$\Gamma = \prod_{j=1}^k \Gamma_j$$

for some $k \geq 2$ and for some nonempty connected graphs $\Gamma_j \in \mathcal{K}_{2i-1}$. These graphs Γ_j cannot all be complete graphs, for otherwise $\Gamma \in \mathcal{K}_2$. So at least one graph Γ_j contains P_3 , the path on three vertices, as an induced subgraph. This implies that $A(\Gamma)$ contains a copy of $(F_2 \times \mathbb{Z}) * \mathbb{Z}$.

We then consider the case that $\Gamma \in \mathcal{K}_{2i+1} \setminus \mathcal{K}_{2i}$ for some $i \geq 2$. Note Γ is the join of some graphs $\Gamma_1, \dots, \Gamma_k$ in \mathcal{K}_{2i} . By the previous graph, each $A(\Gamma_j)$ contains $(F_2 \times \mathbb{Z}) * \mathbb{Z}$.

Finally assume $\Gamma \notin \mathcal{K}$, so that Γ is not a cograph. Then we have that P_4 is an induced subgraph of Γ , so that $A(P_4) \leq A(\Gamma)$. The group $A(P_4)$ contains every

right-angled Artin group $A(F)$, where F is a finite forest (see [26]). Since the defining graph of $(F_2 \times \mathbb{Z}) * \mathbb{Z}$ is a copy of a path P_3 on three vertices together with an isolated vertex, its defining graph is a finite forest. We see that $(F_2 \times \mathbb{Z}) * \mathbb{Z} \leq A(\Gamma)$. \square

We complete the proof of Corollary 1.8 with the following proposition:

Proposition 5.2. *Let $\Gamma \in \mathcal{K}_3$ and let M be a compact one-manifold. Then $A(\Gamma) \leq \text{Diff}_+^\infty(M)$.*

In the case when $M = S^1$, we will prove more precise facts in Section 7.

Proof of Proposition 5.2. Let $\text{Diff}_0^\infty(I)$ denote the group of C^∞ diffeomorphisms of the interval which are infinitely tangent to the identity at $\{0, 1\}$. It suffices to prove that $A(\Gamma) \leq \text{Diff}_0^\infty(I)$, since I is a submanifold of every compact one-manifold M and every element of $\text{Diff}_0^\infty(I)$ can, by definition, be extended to all of M . If $\Gamma \in \mathcal{K}_2$ then $A(\Gamma)$ is a (possibly trivial) free product of free abelian groups. By the Kurosh Subgroup Theorem, it suffices to prove the proposition for $\mathbb{Z} * \mathbb{Z}^k$.

To do this, we first find a copy of $\mathbb{Z}^k \leq \text{Diff}_0^\infty(I)$ such that the support of each nontrivial element of \mathbb{Z}^k is all of $(0, 1)$. The existence of such a copy of \mathbb{Z}^k follows from choosing a C^∞ vector field on I which vanishes only at $\{0, 1\}$ and integrating it to get a flow, which gives an \mathbb{R} -worth of commuting elements of $\text{Diff}_0^\infty(I)$. Now, choosing a generic (in the sense of Baire) element ψ of $\text{Diff}_0^\infty(I)$, we have that ψ and this copy of \mathbb{Z}^k generate a copy of $\mathbb{Z} * \mathbb{Z}^k \leq \text{Diff}_0^\infty(I)$ (cf. [28]).

Finally, if $\Gamma \in \mathcal{K}_3 \setminus \mathcal{K}_2$ then $A(\Gamma)$ is a finite direct product of right-angled Artin groups with defining graphs in \mathcal{K}_2 . Write N for the maximal rank of an abelian subgroup of $A(\Gamma)$. We choose a finite collection of disjoint intervals $\{J_1, \dots, J_k\}$ with nonempty interior inside of I , and realize a copy of $\mathbb{Z} * \mathbb{Z}^N$ on J_i , extending by the identity outside of J_i . It is clear that $A(\Gamma)$ is thus realized as a subgroup of $\text{Diff}_0^\infty(I)$. \square

6. LOWER REGULARITY

In this section, we prove Proposition 1.10. We recall that a group G is *orderable* (or *left orderable*) if there exists a total order $<$ on which is left invariant, i.e. for all triples $a, b, g \in G$ we have $a < b$ if and only if $ga < gb$. The following is a standard result in orderable groups:

Theorem 6.1 (See [38, 14], for instance). *Let G be a countable group. The following are equivalent:*

- (1) *The group G is orderable;*
- (2) *The group G admits a faithful homomorphism into $\text{Homeo}^+(\mathbb{R})$;*
- (3) *The group G admits a faithful homomorphism into $\text{Homeo}^+(I)$.*

If $G \leq \text{Homeo}^+(\mathbb{R})$, one can construct an ordering on G as follows: let $\{a_n\}_{n \in \mathbb{N}}$ be an enumeration of \mathbb{Q} . If $g \neq h \in G$, we set $g < h$ if $g(a_n) < h(a_n)$ at the first index n where g and h differ on \mathbb{Q} . Observe that with this ordering, if $1 < g$ then $a_n < g(a_n)$ for some n . Conversely, if $1 < g$ for some given order $<$ on a countable group G , then one can construct an injective homomorphism $G \rightarrow \text{Homeo}^+(\mathbb{R})$ such that $x < g(x)$ for some $x \in \mathbb{R}$.

Note that if $<$ is an order on G then G also admits the opposite order $<^{opp}$, where $g < h$ if and only if $h <^{opp} g$. It follows that if $1 \neq g \in G$, there is always a faithful action of G on \mathbb{R} such that $x < g(x)$ for some $x \in \mathbb{R}$, provided G is countable.

We now establish Proposition 1.10:

Proposition 6.2. *Let \mathcal{G} denote the class of countable subgroups of $\text{Homeo}^+(I)$. Then \mathcal{G} is closed under countable free products.*

Proposition 6.2 also follows from the general fact that if G and H are countable left orderable groups then so is $G * H$, as was shown in [32, 39]. We give a proof here which is purely dynamical.

Proof. By induction, it suffices to show that if $G_1, G_2, \dots \in \mathcal{G}$ then $*_{i \geq 1} G_i \in \mathcal{G}$. Since $K = \langle \{G_i \mid i \geq 1\} \rangle \in \mathcal{G}$ and since $*_{i \geq 1} K \leq \mathbb{Z} * K$ by the normal form theorem for free products, it suffices to show that if $G \in \mathcal{G}$ then $\mathbb{Z} * G \in \mathcal{G}$.

We enumerate the countably many elements of $\mathbb{Z} * G$, writing a typical element as

$$h = g_k t^{n_k} \cdots g_1 t^{n_1},$$

where t is a generator of \mathbb{Z} , where each g_i is an element of G , and where possibly n_1 and g_k are trivial.

Choose countably many disjoint subintervals $\{I_h\}_{1 \neq h \in \mathbb{Z} * G}$ of I , each with nonempty interior. It is straightforward to see that

$$\prod_{1 \neq h \in \mathbb{Z} * G} \text{Homeo}^+(I_h) \leq \text{Homeo}^+(I).$$

We now focus on I_h , finding an action of $\mathbb{Z} * G$ on I_h where h does not lie in the kernel. We choose k disjoint intervals

$$\{J_1, \dots, J_k\} \subset I_h$$

with nonempty interior, $J_i = [a_i, b_i]$ and $a_i < b_i < a_j < b_j$ for $1 \leq i < j \leq k$. We choose an action of G on J_i for each $1 \leq i \leq k$ for which $x_i < g_i(x_i)$ for some $x_i \in J_i$.

We now define an action of t on I_h . We choose k disjoint intervals

$$\{L_1, \dots, L_k\} \subset I_h$$

of the form $L_i = [c_i, d_i]$. We choose $c_1 < x_0 < a_1$ and $x_1 < d_1 < g_1(x_1)$. For $i > 1$, we choose points $d_{i-1} < c_i < g_{i-1}(x_{i-1})$ and $x_i < d_i < g_i(x_i)$. We now define t on

$$\bigcup_{i=1}^k L_i$$

so that $t^{n_1}(x_0) = x_1$ and so that $t^{n_i}(g_{i-1}(x_{i-1})) = x_i$. Since the $\{L_1, \dots, L_k\}$ are disjoint, such a choice for the definition of t is possible. It is straightforward to check that $x_0 < h(x_0) = g_k(x_k)$. It follows that h is not in the kernel of this action of $\mathbb{Z} * G$. \square

We note that the C^1 hypothesis in Theorem 3.1 cannot be lowered to C^0 :

Proposition 6.3. *There exists an embedding*

$$\rho: \langle a, b, t \mid [a, b] = 1 \rangle \cong \mathbb{Z}^2 * \mathbb{Z} \rightarrow \text{Homeo}^+(I)$$

such that $\text{supp } \rho(a)$ and $\text{supp } \rho(b)$ are disjoint.

Proof. The proof is based on the idea of Proposition 6.2 and [3]. Let us put $H = \langle a, b, t \rangle$ and pick an arbitrary $h \in H \setminus 1$. Up to conjugacy, we may write

$$h = t^{r_\ell} a^{p_\ell} b^{q_\ell} \dots t^{r_1} a^{p_1} b^{q_1} \cdot t^{r_0} a^{p_0} b^{q_0}.$$

For brevity, we only consider the case $r_i \neq 0$ and $(p_i, q_i) \neq (0, 0)$ for each i . As in Proposition 6.2, one can easily find

$$\rho_h: H \rightarrow \text{Homeo}^+(\mathbb{R})$$

such that $\rho_h(a^{p_i} b^{q_i})(2i) = 2i + 1$ and $\rho_h(t^{r_i})(2i + 1) = 2i + 2$, and furthermore,

$$\text{supp } \rho_h(a) \cap \text{supp } \rho_h(b) = \emptyset.$$

Note that $\rho_h(h) \neq 1$. By concatenating ρ_h , we obtain the desired embedding

$$\rho = \prod_{h \in H \setminus 1} \rho_h: H \rightarrow \prod_{h \in H \setminus 1} \text{Homeo}^+(\mathbb{R}) \leq \text{Homeo}^+(I)$$

such that $\text{supp } \rho(a)$ and $\text{supp } \rho(b)$ are disjoint. \square

7. COMPLEXITY OF RIGHT-ANGLED ARTIN GROUPS VERSUS DIVERSITY OF CIRCLE ACTIONS

In this section, we prove Corollary 1.11. The proof follows easily from the results in [28], together with Corollary 1.8.

In a joint work with Mj, the authors defined a class of finitely generated groups \mathcal{F} and called each group in the class as *liftable-flexible*. We will extract the necessary facts which are demonstrated in [28] without recalling all the technical details necessary for parsing the most general statements found therein. Recall from the introduction that a *projective* action of a group is a representation into $\text{PSL}(2, \mathbb{R})$.

Theorem 7.1 (cf. Theorem 1.1 of [28]). *There exists a class of finitely generated groups \mathcal{F} satisfying the following.*

- (1) *For each $G \in \mathcal{F}$, there exist uncountably many distinct semiconjugacy classes of faithful projective actions on S^1 .*
- (2) *Every limit group is in \mathcal{F} ; in particular, every free group and every free abelian group is in \mathcal{F} .*
- (3) *If $G, H \in \mathcal{F}$, then $G * H \in \mathcal{F}$.*

Lemma 7.2. *Let A and B be nontrivial torsion-free finitely generated subgroups such that B is nonabelian. We assume that $A \times B \leq \text{Diff}_+^{1+\text{bv}}(S^1)$. Then there exist finite index normal subgroups $A_1 \trianglelefteq A$ and $B_1 \trianglelefteq B$ such that $A_1 \times B_1$ has a global fixed point on S^1 . Furthermore, the closure of $\text{supp } A_1$ is a proper subset of S^1 .*

Proof. Choose an arbitrary element $a \in A \setminus 1$. Since the centralizer $Z(a)$ is nonabelian and A is torsion-free, Lemma 2.8 implies that $\text{rot } Z(a) \subseteq \mathbb{Q}$. Since B and $a \times B$ are subsets of $Z(a)$, we see that $\text{rot}(B), \text{rot}(a \times B) \subseteq \mathbb{Q}$. This shows that $\text{rot}(A \times B) \subseteq \mathbb{Q}$.

For each $a \in A$ and $b \in B$, note that

$$a, b, ab \in Z(a).$$

By Lemma 2.8 (4), it follows that $\text{rot}(a) + \text{rot}(b) = \text{rot}(ab)$. In particular, we have a homomorphism

$$\text{rot}: A \times B \rightarrow \mathbb{Q}.$$

Since $A \times B$ is finitely generated, the image of the above map is finite. So we can find finite index normal subgroups $A_1 \trianglelefteq A$ and $B_1 \trianglelefteq B$ such that every element of $A_1 \times B_1$ is grounded.

Let $a \in A_1 \setminus 1$. Since B_1 centralizes a , it acts on each component of $\text{supp } a$ as an abelian group; see Lemma 2.6. It follows that B_1 preserves the set $\text{supp } A_1$ and

$$\text{supp } A_1 \cap \text{supp}[B_1, B_1] = \emptyset.$$

Since B_1 is nonabelian, we see $\overline{\text{supp } A_1} \neq S^1$. It follows that $\partial \text{supp } A_1$ is a global fixed point of $A_1 \times B_1$. \square

Proof of Corollary 1.11. (1) If $\Gamma \in \mathcal{K}_2$, then $A(\Gamma)$ is a finite free product of free abelian groups. Combining the parts (2) and (3) of Theorem 7.1, we see $A(\Gamma)$ is in the class \mathcal{F} . The first part of the same theorem implies the desired conclusion.

(2) Suppose that $\Gamma \in \mathcal{K}_3 \setminus \mathcal{K}_2$, so that $A(\Gamma)$ decomposes as a nontrivial direct product $A(\Gamma_1) \times A(\Gamma_2)$, where at least one of $A(\Gamma_1)$ and $A(\Gamma_2)$ is nonabelian. Say $A(\Gamma_2)$ is nonabelian. By Lemma 7.2, there exist finite index normal subgroups $G \trianglelefteq A(\Gamma_1)$ and $H \trianglelefteq A(\Gamma_2)$ such that $G \times H$ has a global fixed point; in particular, $A(\Gamma)$ has a finite orbit. We put

$$X = \overline{\text{supp } G}, \quad Y = S^1 \setminus X.$$

Since G is normal in $A(\Gamma)$ we see that X and Y are $A(\Gamma)$ -invariant sets of S^1 , which are proper by the same lemma. This implies that X and Y have nonempty interiors, and hence $A(\Gamma)$ does not have a dense orbit.

(3) This follows immediately from Corollary 1.8. \square

We briefly remark that if a group acts on S^1 with a global fixed point then it is semi-conjugate to a trivial action, and if it acts with a periodic point then the action is semi-conjugate to a rational rotation group. Corollary 1.11 implies that if $\Gamma \in \mathcal{K}_3 \setminus \mathcal{K}_2$ then $A(\Gamma)$ admits only countably many semi-conjugacy classes of faithful actions, and it is not difficult to realize one semi-conjugacy class for each rational rotation.

Observe that since every right-angled Artin group surjects to \mathbb{Z} , every right-angled Artin group admits uncountably many distinct semi-conjugacy classes of non-faithful actions on S^1 , so only faithful actions are interesting for our purposes.

8. THOMPSON'S GROUPS

Let F and T be the Thompson's groups acting on the closed interval and on the circle, respectively [10]. Recall that these are the groups of piecewise linear homeomorphisms of the interval and circle respectively, with dyadic breakpoints and all slopes given by powers of two. It is known that the standard action of T is conjugate to a C^∞ action [19]. The restriction of such a smooth action yields a smooth action of F on a closed interval. On the other hand, the groups $F * \mathbb{Z}$ and $T * \mathbb{Z}$ do not admit any smooth actions on the circle, or indeed on any compact one-manifold (cf. Corollary 1.7):

Corollary 8.1. *If M is a compact one-manifold, then $F * \mathbb{Z}$ and $T * \mathbb{Z}$ are not subgroups of $\text{Diff}^{1+\text{bv}}(M)$.*

Proof. Since $F \leq T$, it suffices to prove the corollary for $F * \mathbb{Z}$ only. In order to apply Theorem 1.1, it suffices to show that $F \times \mathbb{Z} \leq F$ and that F is not virtually metabelian, whence the conclusion will be immediate. These claims follow immediately from the well-known facts that F is not virtually solvable and that $F \times F \leq F$, and we make these details explicit below for the reader's convenience.

Since F is a group of homeomorphisms of the interval, it is immediate that it is torsion-free. Moreover, conjugating F by the homeomorphism of \mathbb{R} given by $x \mapsto x/2$ scales F to be the group of piecewise linear homeomorphisms with dyadic breakpoints and slopes given by powers of two, only scaled to act on the interval $[0, 1/2]$. It follows that we may realize $F \times F \leq F$, since we can realize one copy of F on $[0, 1/2]$ and a second one on the interval $[1/2, 1]$, with the points $\{0, 1/2, 1\}$ globally invariant. It follows that $\mathbb{Z} \times F \leq F$, since F is torsion-free.

To see that F is not virtually metabelian, we use the standard fact that $[F, F]$ is an infinite simple group and that $Z(F) = \{1\}$ (cf. [10]). It follows that if $H \leq F$ is

a finite index subgroup then $[F, F] \leq H$. Since H contains an infinite simple group, it cannot be solvable, much less metabelian. \square

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