Flexibility of Projective Representations
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Motivation I : Space of Faithful Representations
L : fg group, G : topological group

Theme Describe
\[ X_{\text{faithful}}(L) \subseteq X(L) = \text{Out}(L) \setminus \text{Hom}(L,G) / \text{Inn}(G) \]

\textbf{e.g.} nonempty? how big?

Today G = PSL(2,R).

\textbf{e.g.} \( X_f(\pi_1(S)) \supseteq \text{Mod}(S) \setminus \text{Hom}(\pi_1(S),\text{PSL}(2,R)) / \text{PSL}(2,R) \)

Discrete repns : Moduli space (up to ±)

\textbf{e.g.} Triangle group
\[ \mathcal{A}(p,q,r) = \langle a, b, c \mid a^p = b^q = c^r = 1 = abc \rangle \subseteq \text{PSL}(2,R) \]
\[ \forall f : \mathcal{A}(p,q,r) \to \text{PSL}(2,R) \]
f(a), f(b), f(c) are rotations
\[ \Rightarrow X_f(\mathcal{A}(p,q,r)) \text{ is finite.} \]

Motivation II : Exotic Circle Actions

\textbf{Question} L \subseteq \text{Homeo}(S^1), \text{ say \textasciitilde classical.}
Are there dynamically different faith. actions L on S^1 ?

\textbf{e.g.} L : Fuchsian. \( F_n, \text{ Mod}(S), \text{ limit groups...} \)

\textbf{Rotation Spectrum}
\[ f \in \text{Homeo}(S^1) \quad \text{rot} f = \lim_{n \to \infty} f^n(0) / n \in \mathbb{R}/\mathbb{Z} = S^1 \]
\[ \text{rot} f^n = n \cdot \text{rot} f \quad \text{rot} f^g = \text{rot} f : \text{cont. class fn.} \]
\( \rho : L \to \text{Homeo}(S^1), \Sigma(\rho) = \text{rot} \cdot \rho(L) : \text{(unmarked) rotation spectrum} \)

**Trace**

tr: PSL(2,\(R\)) \(\to\) \(R / (x \sim -x)\), conjugacy invariant

g : elliptic \(\text{tr}(g) \leftrightarrow \text{rot}(g)\)

g : hyperbolic \(\text{tr}(g) \leftrightarrow \text{length}(g)\)

**Equivalence of actions**

1. \(\rho_1, \rho_2 : L \to \text{Homeo}(S^1)\) are conjugate: 
   \(\exists\) homeo \(h\) s.t. \(\forall g \in L: \)

2. \(\rho_1 \sim_s \rho_2 \) (semi-conjugate)
   if their "minimalizations" are conjugate.

3. \(\rho_1\) is equivalent to \(\rho_2\) if \(\rho_1 \sim_s \alpha \circ \rho_2 \circ \beta\) 
   for \(\exists \alpha \in \text{Inn}(\text{Homeo}(S^1)), \beta \in \text{Aut}(L)\).

(1) > (2) > (3)

• \(\rho_1 \sim_s \rho_2 \implies \text{rot} \circ \rho_1 = \text{rot} \circ \rho_2\), as maps.

• \(\rho_1\) equiv to \(\rho_2 \implies \text{rot} \circ \rho_1(L) = \text{rot} \circ \rho_2(L)\), as sets.

**Discrete vs dense**

(Poincaré) Classified all discrete \(L \leq \text{PSL}(2,\(R\)).

**Thm** (Goldman 1988)

• \(\text{Hom}(\pi_1(S_g), \text{PSL}(2,\(R\)))\) has \(2(2g-2)+1\) components.

• Two (maximal) compo corr. to discrete faithful repns.

**Thm** (DuBlois–Kent 2008)

{faithful repns} is dense in \(\text{Hom}(\pi_1(S_g), \text{PSL}(2,\(R\)))\).

**Lemma** \(L\) : f.g. non-virt. solvable \(\leq \text{PSL}(2,\(R\)).

Then, \(L\) : dense \(\iff\) \(L\) : indiscrete \(\iff\) rot(\(L\)) is infinite.

**Flexible groups**

\(1 \to Z \to \text{PSL}^\sim(2,\(R\)) \to \text{PSL}(2,\(R\)) \to 1\)
1 \to \mathbb{Z} \to \text{Homeo}_\mathbb{Z}(\mathbb{R}) \to \text{Homeo}(S^1) \to 1

**Def** (K.-Koberda-Mj) $L : f.g., TL := \{\text{torsions}\}$.

(1) **L is flexible with an anchor** $g_0 \in L$, if $\forall$ countable $W \subseteq \mathbb{R}/\mathbb{Z}$, $\exists \rho : L \to \text{PSL}(2,\mathbb{R})$ s.t. $\rho(g_0)$ is elliptic and $\text{tr} \circ \rho\ (L \setminus TL) \cap W = \emptyset$.

(2) **L is liftable-flexible**, if $L$ satisfies (1) and $\rho$ lifts to $\rho : L \to \text{PSL}^\sim(2,\mathbb{R})$.

**Q** (1) Which groups are (lift-)flexible?
(2) Why do we care (lift-)flexible groups?

**Strong inequivalence**

**Thm** (KKM)

$L : \text{flexible} \implies \exists$ uncount. set $\Lambda \subseteq X_F(L)$ of indiscrete parabolic-free repns $L \to \text{PSL}(2,\mathbb{R})$ s.t. $\text{rot } \rho(L \setminus TL) \cap \text{rot } \rho'(L \setminus TL) = \emptyset$ for $\forall$ distinct $\rho, \rho' \in \Lambda$.

In particular, $\rho(g)$ is not conj to $\rho'(h)$ $\forall g, h \in L$.

**Quasi-morphisms**

**Def** $f : \mathbb{Z} \to \mathbb{Z}$ is a quasi-morphism if $\partial f(a,b) = f(a) + f(b) - f(a+b)$ is bounded. defect $D_f = \sup |\partial f|$

Recall $1 \to \text{QM}(L;\mathbb{Z})/\text{Hom+Bdd} \to H^2_b(L;\mathbb{Z}) \to H^2(L;\mathbb{Z})$ exact.

**Thm** (KKM) $L : \text{liftable-flexible} \implies \exists$ uncount. set of $\mathbb{Z}$-valued subadditive defect one quasimorphisms, which are all linearly independent in $H^2_b(L;\mathbb{Z})$.

Use $\text{Hom}(L,\text{Homeo}(S^1)) / \text{semi-conj} \to H^2_{\{0,1\}}(L;\mathbb{Z}) \subseteq H^2_b(L;\mathbb{Z})$.

**Examples of flexible groups**

**Def** $F_n : \text{fixed. } L$ is a limit group if for each finite $S \subseteq L \setminus 1$, there is some $f : L \to F_n$ such that $1 \not\in f(S)$.

(Bestvina-Feighn; Barlev-Gelander 2010 for density)

$\forall$ limit group $L$, $\exists$ dense embedding $L \to \text{PSL}(2,\mathbb{R})$, coming from dense $F_2 \to \text{PSL}(2,\mathbb{R})$.

**Thm** (KKM)
(1) Limit groups are liftable-flexible.
(2) Torsion-free Fuchsian groups are liftable-flexible.
(3) Non-uniform Fuchsian groups are flexible.

**Thm (KKM)**

Every non-sporadic (e.g. $g \geq 3$ or $g=2$ w/ $\geq 3$ cone points) Fuchsian group admits uncount. many inequiv indisc faith repns into PSL(2,R).

**Thm (Mann 2015)** If $\rho_0 : \pi_1(S_g) \to \text{Homeo}(S^1)$ is a "geometric", then the connected component of $\text{Hom}(\pi_1(S_g), \text{Homeo}(S^1))$ containing $\rho_0$ consists of a single semi-conj class.

**Cor (KKM)** If $L$ is flexible, then $\text{Hom}(L, \text{PSL}(2,R))$ has a component containing uncountably many distinct equiv classes of faithful repns.

**Topological Baumslag Lemma**

**Lemma (Baumslag 1962)**

$F : \text{free}$, $b_1, b_2, \ldots, b_k, u \in F$ s.t. $[b_i,u] \neq 1$, $m_1, m_2, \ldots, m_k \in \mathbb{Z}\setminus 0$

Then $f(n) = b_1 u^{m_1 n} b_2 u^{m_2 n} \ldots b_k u^{m_k n} \neq 1$ for $\forall |t| >> 0$.

**Lemma (KKM)**

$X : \text{connected Hausdorff}$, $\mu : \mathbb{R} \to \text{Homeo}(X)$ gp hom s.t.

(i) $\text{Fix } \mu$ is cpt;

(ii) $\forall$ open $U \supset \text{Fix } \mu$, $\exists N>0$ s.t. $\mu(t)(X\setminus U) \subseteq U$ whenever $|t| \geq N.$

If $b_1, b_2, \ldots, b_k, \in \text{Homeo}(X)$ s.t. $\text{Fix } \mu \cap b_i \text{Fix } \mu = \emptyset$, then

$f(t) = b_1 \mu(m_1 t) b_2 \mu(m_2 t) \ldots b_k \mu(m_k t) \neq 1$ for $\forall |t| >> 0$.

**Idea**

Fix $\mu \subseteq U$, $b_i \text{Fix } \mu \subseteq V$ s.t. $U \cap V = \emptyset$

$\forall |t| >> 0 \implies f(t)$ plays a ping-pong between $U$ and $V.$
Combination lemmas

Lemma (KKM)

(1) A, B ≤ PSL(2,R). Then for a very general ν ∈ PSL(2,R),
\[ < A, B^\nu > \cong A \ast B. \]

(2) A, B ≤ PSL(2,R), μ : R → PSL(2,R). Suppose C = N(μ) ∩ A = N(μ) ∩ B ⊆ μ. Then ρ_ν is faithful for a very general ν ∈ μ:

Combination Theorems

C ≤ A is malnormal if C ∩ C^g = \{1\} for all g ∈ A\C.

Thm (KKM)

Let H = \{flexibles\} or \{liftable-flexibles\}.

(1) A, B are flexible ⇒ A \ast B is flexible

(1') A is flexible ⇒ A \ast Z_m is flexible

(2) \langle A, B \rangle is flexible. C ≤ A, C ≤ B : maximal abel, maln
⇒ A \ast_C B is flexible.

(3) \langle L \rangle is flexible. C ≤ L : maximal abel and maln
⇒ A \ast_C is flexible.

The same hold for liftable-flexible groups except for (3)'.