

Finite subgraphs of an extension graph

SANG-HYUN KIM, THOMAS KOBERDA, AND JUYOUNG LEE

ABSTRACT. Let Γ be a finite graph and let Γ^e be its extension graph. We inductively define a sequence $\{\Gamma_i\}$ of finite induced subgraphs of Γ^e through successive applications of an operation called “doubling along a star”. Then we show that every finite induced subgraph of Γ^e is isomorphic to an induced subgraph of some Γ_i . This result strengthens [9, Lemma 3.1].

1. STATEMENT OF THE RESULT

1.1. **Notations.** *Throughout this note,* let us fix a finite graph Γ and its vertex set

$$V = V(\Gamma) = \{a_0, \dots, a_{n-1}\}.$$

We will mostly follow the terminology and notations in [9].

For $U, W \subseteq A(\Gamma)$, we define

$$U^W = \{u^w : u \in U \text{ and } w \in W\} \subseteq A(\Gamma).$$

We put

$$V^e = V^{A(\Gamma)}.$$

Recall that the *extension graph* Γ^e is defined as the commutation graph of V^e in $A(\Gamma)$; see Definition 2.1.

It will be convenient for us to denote

$$V_{\mathbb{Z}}^e = \{(v^k)^g : v \in V, k \in \mathbb{Z} \setminus 0, g \in A(\Gamma)\} \subseteq A(\Gamma).$$

We have a map $(\cdot)^* : V_{\mathbb{Z}}^e \rightarrow V^e$ defined by the formula

$$((v^k)^g)^* = v^g.$$

For example, if $a, b, c \in V$ then we have $(a^{-2})^{bc})^* = a^{bc}$.

For each $w \in A(\Gamma)$, we let $\|w\|$ denote the *word length* of w . In other words, $\|w\|$ is the smallest nonnegative integer ℓ such that we can write

$$w = s_1^{e_1} \cdots s_\ell^{e_\ell}$$

Date: August 7, 2017.

Key words and phrases. right-angled Artin group; extension graph; acylindrically hyperbolic group.

for some $s_i \in V$ and $e_i \in \{-1, 1\}$. In this case, we define the *support* of w as

$$\text{supp } w = \{s_1, s_2, \dots, s_\ell\} \subseteq V.$$

1.2. Double of a graph along a star. Let X be a graph. For $S \subseteq V(X)$, we denote by $X(S)$ or by XS the subgraph of X induced by S . The *star* of a vertex v in X is the set of vertices in X that are either equal or adjacent to v . We denote the star of v as $\text{St}_X(v)$ or $\text{St}(v)$. We define the *link* of v as

$$\text{Lk}_X(v) = \text{St}_X(v) \setminus \{v\}.$$

Fix a vertex v of X . Let S_0, S_1 be sets with some fixed bijections

$$\rho_i: S_i \rightarrow V(X) \setminus \text{St}(v).$$

Then we can define a new graph Y by requiring that

$$V(Y) = S_0 \coprod S_1 \coprod \text{St}(v)$$

and that $\{a, b\} \in E(Y)$ if and only if one of the following holds:

- (i) $a, b \in S_i$ and $\{\rho_i(a), \rho_i(b)\} \in E(X)$ for some $i = 0$ or 1 .
- (ii) $a \in S_i, b \in \text{St}(v)$ and $\{\rho_i(a), b\} \in E(X)$ for some $i = 0$ or 1 .
- (iii) $a, b \in \text{St}(v)$ and $\{a, b\} \in E(X)$.

The graph Y thus obtained is called the *double of X along the star of v* .

The main result of this paper is the following.

Theorem 1.1 (compare with [9, Lemma 3.1]). *There exists an infinite sequence of finite induced subgraphs $\{\Gamma_i\}_{i \geq 0}$ of Γ^e such that Γ_{i+1} is the double of Γ_i along a star and such that every finite induced subgraph of Γ^e admits an embedding into some Γ_i as an induced subgraph.*

This theorem strengthens Lemma 3.1 in [9], where the proof of the lemma is omitted. As there has been much interest recently concerning on the combinatorial structures of extension graphs [4, 3, 10, 5, 6, 7], we decided to write down a very detailed construction of such a sequence $\{\Gamma_i\}_{i \geq 0}$.

2. UNIVERSAL SEQUENCE

We denote the symmetric difference of two sets A and B as $A \Delta B$.

Definition 2.1. Let X be a subset of a group G . The *commutation graph* of X , denoted as $\text{CG}(X)$, is the simplicial graph whose vertex set is X and in which two distinct vertices $x, y \in X$ are adjacent if and only if $[x, y] = 1$.

Note that we have the *natural homomorphism*

$$A(\text{CG}(X)) \rightarrow \langle X \rangle$$

defined by the unique extension of Id_X .

Lemma 2.2 ([2, 9], cf. [8, 1]). *Let G be a group and $X_0 \subseteq G$ be a subset such that the natural homomorphism*

$$A(\text{CG}(X_0)) \rightarrow \langle X_0 \rangle$$

is an isomorphism. Suppose $u \in X_0$, and define

$$X_1 = (X_0 \cup X_0^u) \Delta \{u, u^{-2}\}.$$

We define $\phi: \langle X_0 \rangle \rightarrow \mathbb{Z}_2$ by $\phi(u) = 1$ and $\phi(v) = 0$ for all $v \in X_0 \setminus \{u\}$. Then $\text{CG}(X_1)$ is the double of $\text{CG}(X_0)$ along the star of $u \in X_0$. Moreover, we have the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & A(\text{CG}(X_1)) & \longrightarrow & A(\text{CG}(X_0)) & \longrightarrow & \mathbb{Z}_2 \longrightarrow 1 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \text{Id} \\ 1 & \longrightarrow & \langle X_1 \rangle & \longrightarrow & \langle X_0 \rangle & \xrightarrow{\phi} & \mathbb{Z}_2 \longrightarrow 1 \end{array}$$

where the left two vertical maps are natural isomorphisms.

Let us now define an infinite sequence $\{u_i\}_{i \geq 0} \subseteq V_{\mathbb{Z}}^e$ as follows:

$$\begin{aligned} u_0 &= a_0, u_1 = a_0^{-2}, u_2 = a_1, u_3 = a_1^{-2}, \dots, u_{2n-1} = a_{n-1}^{-2}, \\ u_{2n} &= a_0^4, u_{2n+1} = a_0^{-8}, u_{2n+2} = a_1^4, \dots, u_{4n-1} = a_{n-1}^{-8}, \\ u_{4n} &= a_0^{16}, u_{4n+1} = a_0^{-32}, \dots \text{ and so forth.} \end{aligned}$$

For each $i \geq 0$ and $0 \leq j < 2n$, one can more succinctly write

$$u_{2ni+j} = a_{\lfloor j/2 \rfloor}^{r_{i,j}}, \quad r_{i,j} = (-2)^{2i+j-2\lfloor j/2 \rfloor}.$$

We then have an infinite sequence $\{U_i\}_{i \geq 0}$ of subsets of $V_{\mathbb{Z}}^e$ defined as

$$\begin{aligned} U_0 &= V, \\ U_{i+1} &= (U_i \cup U_i^{u_i}) \Delta \{u_i, u_i^{-2}\}. \end{aligned}$$

We will call the sequence $\{(u_i, U_i)\}_{i \geq 0}$ as a *universal sequence* in Γ^e . Note that a universal sequence depends on the choice of the enumeration $V = \{a_0, \dots, a_{n-1}\}$.

The following lemma is the basic building block for our construction.

Lemma 2.3. *The following hold for each $i \geq 0$.*

- (1) $u_i \in U_i$.
- (2) *The natural homomorphism $A(\text{CG}(U_i)) \rightarrow \langle U_i \rangle$ is an isomorphism, and moreover, $\text{CG}(U_{i+1})$ is the double of $\text{CG}(U_i)$ along the star of u_i .*
- (3) *The map $x \mapsto x^*$ is injective on U_i .*
- (4) *The two graphs $\text{CG}(U_i)$ and $\Gamma^e U_i^*$ are isomorphic by the isomorphism $x \mapsto x^*$.*

Proof. For $i \geq 0$ and $0 \leq j < n$ let us note

$$\begin{aligned} u_{2ni+2j} &= a_j^{4^i}, \quad \{a_0, \dots, a_{j-1}\}^{4^{i+1}} \cup \{a_j, \dots, a_{n-1}\}^{4^i} \subseteq U_{2ni+2j}. \\ u_{2ni+2j+1} &= a_j^{-2 \cdot 4^i}, \quad \{a_0, \dots, a_{j-1}\}^{4^{i+1}} \cup \{a_j\}^{-2 \cdot 4^i} \cup \{a_{j+1}, \dots, a_{n-1}\}^{4^i} \subseteq U_{2ni+2j+1}. \end{aligned}$$

So, part (1) is obvious. Part (2) follows from an induction combined with Lemma 2.2.

For part (3), assume $x, y \in U_i$ satisfy $x^* = y^*$. Then we can write $x = (u^g)^k$ and $y = (u^h)^m$ for some $u \in V$, $g, h \in A(\Gamma)$ and $k, m \in \mathbb{Z} \setminus 0$. Since $x^m = y^k$ in $A(\Gamma)$, part (2) implies that $x = y$.

Consider part (4). The map $(\cdot)^*$ defines a natural bijection

$$\text{CG}(U_i) \rightarrow \Gamma^e U_i^*.$$

Note that for $x, y \in V_{\mathbb{Z}}^e$, we have

$$[x, y] = 1 \Leftrightarrow [x^*, y^*] = 1.$$

Hence $\text{CG}(U_i)$ and $\Gamma^e U_i^* = \text{CG}(U_i^*)$ are isomorphic. \square

The following lemma shows that the sequence $\{\text{CG}(U_i)\}_{i \geq 0}$ eventually contains copies of all the finite induced subgraphs of Γ^e .

Lemma 2.4. *For each finite set $W \subseteq A(\Gamma)$, there exists $K > 0$ and a map*

$$\sigma: W \rightarrow A(\Gamma)$$

such that the following hold:

- (i) $\Gamma^e(V^W) \cong \Gamma^e(V^{\sigma W})$.
- (ii) $V^{\sigma W} \subseteq U_K^*$.

The proof of this lemma is postponed until the next section. Let us first deduce the main theorem of this paper.

Proof of Theorem 1.1 assuming Lemma 2.4. We can find a finite set $W \subseteq A(\Gamma)$ such that $V(\Lambda) \subseteq V^W$. By the conditions (i) and (ii) of the lemma, the graph Λ is an induced subgraph of $\Gamma^e U_K^*$. Lemma 2.3 implies that $\Gamma^e U_K^*$ is obtained from Γ by successive applications of doubling along stars, as desired. \square

3. INFLATING POWERS OF LETTERS

In this section, we find a map σ satisfying the conditions of Lemma 2.4.

3.1. Canonical expression. Consider an arbitrary $w \in A(\Gamma)$. We can write

$$(*) \quad w = s_1^{e_1} s_2^{e_2} \cdots s_\ell^{e_\ell}$$

where $\ell = \|w\|$, $s_i \in V$ and $e_i \in \{-1, 1\}$. For each $i = 1, 2, \dots, \ell$, we put

$$f_i = \min_y \|y\|$$

where y varies among the words in $A(\Gamma)$ such that we can write

$$s_i^{e_i} s_{i+1}^{e_{i+1}} \cdots s_\ell^{e_\ell} = x \cdot s_i^{e_i} \cdot y$$

for some word $x \in \langle \text{Lk}_\Gamma(s_i) \rangle$. Roughly speaking, we minimize the length of a word y that “remains” on the right of $s_i^{e_i}$. We call (f_1, \dots, f_ℓ) as the *right-counting vector* corresponding to the word in (*).

Example 3.1. Let $V = \{a_0 = a, a_1 = b, a_2 = c\}$ and

$$A(\Gamma) = \langle a, b, c \mid [a, b] = 1 \rangle, \quad w = a^2 b^{-3} c b.$$

The right-counting vector for this word is

$$(f_1, \dots, f_7) = (3, 2, 4, 3, 2, 1, 0).$$

Let us consider a different word representing the same element:

$$w = b^{-1} a b^{-1} a b^{-1} c b.$$

Then the corresponding right-counting vector becomes

$$(4, 3, 3, 2, 2, 1, 0).$$

A word in (*) is called a *canonical expression for w* if the corresponding right-counting vector satisfies the following two conditions:

- (A) $f_1 \geq f_2 \geq \cdots \geq f_\ell$;
- (B) If $f_i = f_j$ for some $i < j$ and if we write $s_i = a_p, s_j = a_q$ for some $0 \leq p, q < n$, then we have that $p < q$ and that $[a_p, a_q] = 1$.

Lemma 3.2. *Each $w \in A(\Gamma)$ admits a unique canonical expression.*

Proof. Let (f_1, \dots) be the right-counting vector for a word representing w . Put

$$A = (\# \text{ of } (i, j) \text{ where } i < j \text{ and } f_i < f_j),$$

$$B = (\# \text{ of } (i, j) \text{ where } i < j, f_i = f_j \text{ and } s_i = a_p, s_j = a_q \text{ for some } p > q).$$

A canonical expression for w is then obtained by minimizing the lexicographical order of the tuple (A, B) , resulting in $(0, 0)$. \square

3.2. Proof of Lemma 2.4. Recall our notation $\{(u_i, U_i)\}_{i \geq 0}$, which is defined in the previous section. In order to prove Lemma 2.4, it suffices for us to consider the case when

$$W = B(M) = \{w \in A(\Gamma) : \|w\| \leq M\}$$

for some positive integer M .

Let $w \in B(M)$ be written as (*), which is not necessarily canonical. Denote by (f_1, \dots, f_ℓ) the corresponding right-counting vector. We define

$$N_i = \frac{3e_i - 1}{2} \cdot 4^{M-1-f_i} \in \{\pm 2^m : m \geq 0\},$$

$$(**) \quad \sigma(s_1^{e_1} s_2^{e_2} \cdots s_\ell^{e_\ell}) = s_1^{N_1} s_2^{N_2} \cdots s_\ell^{N_\ell}.$$

Lemma 3.3. *The following hold.*

- (1) *The map $\sigma : B(M) \rightarrow A(\Gamma)$ is well-defined. That is, if two words x_1 and x_2 represent the same element w in $A(\Gamma)$, then $\sigma(x_1) = \sigma(x_2)$ in $A(\Gamma)$.*
- (2) *If (*) is a canonical expression for $w \in B(M)$ and if $\sigma(w)$ is written as (**), then $\{s_i^{N_i}\}_{1 \leq i \leq \ell}$ is a subsequence of $\{u_i\}_{i \geq 0}$.*

Proof. (1) One goes through the definition of σ for a different expression

$$w = s_1^{e_1} \cdots s_{i-1}^{e_{i-1}} s_{i+1}^{e_{i+1}} s_i^{e_i} s_{i+2}^{e_{i+2}} \cdots s_\ell^{e_\ell}$$

when $[s_i, s_{i+1}] = 1$ and verifies that the resulting element coincides with $\sigma(w)$.

(2) Immediate from the definition of a right-counting vector and a canonical expression. \square

Example 3.4. Continuing Example 3.1 and setting $M = 7$, we have

$$\sigma(w) = \sigma(b^{-1}ab^{-1}ab^{-1}cb) = b^{-2 \cdot 4^2} a^{4^3} b^{-2 \cdot 4^3} a^{4^4} b^{-2 \cdot 4^4} c^{4^5} b^{4^6}.$$

We claim the map $\sigma : B(M) \rightarrow A(\Gamma)$ thus defined satisfies the conditions of Lemma 2.4. The condition (ii) is implied by Lemma 3.3 (2), so it remains to show the condition (i). For $v \in V$, we let $Z(v) = \langle \text{St}_\Gamma(v) \rangle$, which is the centralizer group of v in $A(\Gamma)$.

Lemma 3.5. *For $u, v \in V$ and $x, y \in B(M)$, we have the following.*

- (1) *$xy^{-1} \in Z(u)$ iff $\sigma(x)\sigma(y)^{-1} \in Z(u)$.*
- (2) *$xy^{-1} \in Z(v)Z(u)$ iff $\sigma(x)\sigma(y)^{-1} \in Z(v)Z(u)$.*

Proof. Let us consider reduced expressions

$$x = x_0 \cdot p, \quad y = y_0 \cdot p$$

such that $x_0 \cdot y_0^{-1}$ is reduced. Then we have reduced expressions

$$\sigma(x) = x_1 \cdot \sigma(p), \quad \sigma(y) = y_1 \cdot \sigma(p)$$

for some words x_1 and y_1 . We can write

$$x_0 y_0^{-1} = \prod_{i=1}^k t_i^{g_i},$$

$$x_1 y_1^{-1} = \prod_{i=1}^k t_i^{h_i}.$$

for some $k \geq 0$, $t_i \in V$ and $g_i, h_i \in \mathbb{Z} \setminus \{0\}$; furthermore, g_i and h_i have the same sign for each i . In particular, $x_1 \cdot y_1^{-1}$ is reduced and so, the conclusion follows. \square

For each $u, v \in V$ and $x, y \in B(M)$, Lemma 3.5 (1) implies the following equivalences:

$$u^x = v^y \Leftrightarrow u = v \text{ and } xy^{-1} \in Z(u)$$

$$\Leftrightarrow u = v \text{ and } \sigma(x)\sigma(y)^{-1} \in Z(u) \Leftrightarrow u^{\sigma x} = v^{\sigma y}.$$

$$\{u^x, v^y\} \in E(\Gamma^e) \Leftrightarrow u \neq v \text{ and } [u^x, v^y] = 1$$

$$\Leftrightarrow u \neq v, [u, v] = 1, u^x = u^g, v^y = v^g \text{ for some } g \in A(\Gamma)$$

$$\Leftrightarrow u \neq v, [u, v] = 1, xy^{-1} \in Z(u)Z(v)$$

$$\Leftrightarrow u \neq v, [u, v] = 1, \sigma(x)\sigma(y)^{-1} \in Z(u)Z(v)$$

$$\Leftrightarrow \{u^{\sigma x}, v^{\sigma y}\} \in E(\Gamma^e).$$

So the map σ satisfies the condition (i) of Lemma 2.4, as desired.

ACKNOWLEDGEMENTS

The authors thank Ilya Kapovich, Sang-jin Lee and Bertold Wiest for helpful discussions.

REFERENCES

1. Robert W Bell, *Combinatorial Methods for Detecting Surface Subgroups in Right-Angled Artin Groups*, ISRN Algebra **2011** (2011), , Article ID 102029.
2. M. Bestvina, B. Kleiner, and M. Sageev, *The asymptotic geometry of right-angled Artin groups. I*, *Geom. Topol.* **12** (2008), no. 3, 1653–1699. MR2421136 (2009f:20061)
3. Montserrat Casals-Ruiz, *Embeddability and universal theory of partially commutative groups*, *Int. Math. Res. Not. IMRN* (2015), no. 24, 13575–13622. MR3436157
4. Montserrat Casals-Ruiz, Andrew Duncan, and Ilya Kazachkov, *Embeddings between partially commutative groups: two counterexamples*, *J. Algebra* **390** (2013), 87–99. MR3072113
5. J. Huang, *Quasi-isometric classification of right-angled Artin groups I: the finite out case*, *ArXiv e-prints* (2014).
6. Jingyin Huang, *Quasi-isometry classification of right-angled artin groups ii: several infinite out cases*, (2016).
7. Jingyin Huang and Bruce Kleiner, *Groups quasi-isometric to raag's*, (2016).

8. Sang-hyun Kim, *Co-contractions of graphs and right-angled Artin groups*, *Algebr. Geom. Topol.* **8** (2008), no. 2, 849–868. MR2443098 (2010h:20093)
9. Sang-hyun Kim and Thomas Koberda, *Embedability between right-angled Artin groups*, *Geom. Topol.* **17** (2013), no. 1, 493–530. MR3039768
10. E.-K. Lee and S.-J. Lee, *Embeddability of right-angled Artin groups on complements of trees*, *ArXiv e-prints* (2017).

DEPARTMENT OF MATHEMATICAL SCIENCES, SEOUL NATIONAL UNIVERSITY, SEOUL, KOREA

E-mail address: s.kim@snu.ac.kr

URL: <http://cayley.kr>

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VA 22904-4137, USA

E-mail address: thomas.koberda@gmail.com

URL: <http://faculty.virginia.edu/Koberda>

DEPARTMENT OF MATHEMATICAL SCIENCES, SEOUL NATIONAL UNIVERSITY, SEOUL, KOREA

E-mail address: ljy219@snu.ac.kr