UNSMOOTHABLE GROUP ACTIONS ON COMPACT ONE–MANIFOLDS

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Abstract. We show that no finite index subgroup of a sufficiently complicated mapping class group or braid group can act faithfully by $C^{1+\mathrm{bv}}$ diffeomorphisms on the circle, which generalizes a result of Farb–Franks, and which parallels a result of Ghys and Burger–Monod concerning differentiable actions of higher rank lattices on the circle. This answers a question of Farb, which has its roots in the work of Nielsen. We prove this result by showing that if a right-angled Artin group acts faithfully by $C^{1+\mathrm{bv}}$ diffeomorphisms on a compact one–manifold, then its defining graph has no subpath of length three. As a corollary, we also show that no finite index subgroup of $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ for $n \geqslant 3$, the Torelli group for genus at least 3, and of each term of the Johnson filtration for genus at least 5, can act faithfully by $C^{1+\mathrm{bv}}$ diffeomorphisms on a compact one–manifold.

1. Introduction

1.1. **Main results.** Let $S = S_{g,n,b}$ be an orientable surface of genus g, with n marked points, and with b boundary components, and let $\mathrm{Mod}(S)$ denote its mapping class group. We will write

$$c(S) = 3g - 3 + n + b$$

for the *complexity* of S. Throughout this paper, we let M be a compact (possibly disconnected) one–manifold, and we let $\mathrm{Diff}_+^{1+\mathrm{bv}}(M)$ be the group of C^1 orientation–preserving diffeomorphisms of M whose first derivatives have bounded variation; see Section 4 for a more detailed discussion. The following is a classical result of Nielsen:

Theorem 1.1 (Nielsen [13])**.** *If $S = S_{g,1,0}$, the group $\mathrm{Mod}(S)$ admits a faithful continuous action on S^1 without global fixed points.*

In this article we prove the following result which shows that the action in Theorem 1.1, and in fact any action at all even after taking a finite index subgroup, is not smoothable to C^1 with bounded variation.

Theorem 1.2. *There exists a finite index subgroup $G < \mathrm{Mod}(S)$ and an injective homomorphism from G to $\mathrm{Diff}_+^{1+\mathrm{bv}}(M)$ if and only if $c(S) \leqslant 1$.*

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Theorem 1.2 generalizes a result of Farb–Franks [11], wherein it is shown that if $S = S_{g,n,0}$, where $g \geqslant 3$ and where $n \in \{0, 1\}$, then there is no nontrivial C^2 action of $\mathrm{Mod}(S)$ on the circle or the closed interval. As Farb and Franks remark, Ghys established this fact independently in the case of the circle.

Note that a C^2 action or a C^1 action with Lipschitz derivatives on M is always $C^{1+\mathrm{bv}}$. For a lower regularity, Parwani [31] proves that every C^1 mapping class group action on the circle is trivial if the genus of S is at least six.

We say a group H *embeds* into another group G if there exists an injective homomorphism from H into G, and H *virtually embeds* into G if a finite index subgroup of H embeds into G. Theorem 1.2 has the following immediate corollary:

Corollary 1.3. *The n–strand braid group B_n virtually embeds into $\mathrm{Diff}_+^{1+\mathrm{bv}}(M)$ if and only if $n \leqslant 3$.*

Previous results about the non–existence of smooth mapping class group actions on one–manifolds used particular relations which hold in mapping class groups, especially braid relations. The difficulty with generalizing such arguments is that most relations other than commutation/non–commutation of elements do not persist in finite index subgroups.

We are able to establish Theorem 1.2 by exhibiting a right-angled Artin group which cannot act faithfully by $C^{1+\mathrm{bv}}$ diffeomorphisms on a compact one–manifold. Recall that if Γ is a finite simplicial graph with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$, the right-angled Artin group $A(\Gamma)$ is the finitely presented group

$$A(\Gamma) = \langle V(\Gamma) \mid [v_i, v_j] = 1 \text{ if and only if } \{v_i, v_j\} \in E(\Gamma)\rangle.$$

We let P_n denote the path on n vertices (see Figure 2).

The core of the paper is in proving the following result:

Theorem 1.4. *The group $A(P_4)$ does not virtually embed into $\mathrm{Diff}_+^{1+\mathrm{bv}}(M)$.*

Note that Theorem 1.4 provides a (relatively simple) purely algebraic obstruction for a group to act faithfully on a compact one-manifold by $C^{1+\mathrm{bv}}$ diffeomorphisms. Whereas the regularity assumption in Theorem 1.4 may appear artificial at first glance, it is actually optimal; in particular, it cannot be weakened to C^1. See Theorem 1.9 below.

Recall that a graph Λ is an *induced subgraph* of a graph Γ if $V(\Lambda) \subset V(\Gamma)$, and whenever $v_1, v_2 \in V(\Lambda)$ span an edge of Γ, they also span an edge of Λ. A graph Γ is called *P_4–free* if no induced subgraph of Γ is isomorphic to P_4.

Corollary 1.5. *Suppose that $A(\Gamma)$ embeds in $\mathrm{Diff}_+^{1+\mathrm{bv}}(M)$. Then Γ is P_4–free.*

There are two main steps in the proof of Theorem 1.4. The first step is finding a certain configuration of infinitely many intervals in the supports of the generators of $A(P_4)$. This step constitutes most of the original content of this paper, and builds on the classical results in one–dimensional dynamics described in Section 4. The results of Section 5 give a detailed description of right-angled Artin group actions on one–dimensional manifolds. Labeling the vertices of P_4 as shown in Figure 2, we prove the following proposition in Section 5, which is crucial for the proof of Theorem 1.4:

Proposition 1.6. *Suppose M is connected. If there is a faithful representation $\phi\colon A(P_4) \to \mathrm{Diff}_+^{1+\mathrm{bv}}(M)$ such that $\mathrm{Fix}\,\phi(v) \neq \emptyset$ for each $v \in V(P_4)$, then there exist collections of infinitely many intervals*

$$\{I_a^j\colon j \geqslant 1\} \subseteq \pi_0(\mathrm{supp}\,\phi(a)) \quad \textit{and} \quad \{I_d^j\colon j \geqslant 1\} \subseteq \pi_0(\mathrm{supp}\,\phi(d))$$

such that the following two conditions hold:

(i) $I_v^j \neq I_v^k$ whenever $v \in \{a, d\}$ and $j \neq k$.
(ii) For each j, we have $I_a^j \neq I_d^j$ and $\phi(cb)I_a^j \cap I_d^j \neq \emptyset$.

In Proposition 1.6 above and in what follows, π_0 denotes the set of connected components.

The second step of the proof of Theorem 1.4 consists of derivative estimates of the diffeomorphisms $\{\phi(v) \mid v \in V(P_4)\}$; see Section 6.

In the last section, we will give a generalization of Theorem 1.2 to finitely generated groups quasi–isometric to mapping class groups.

1.2. Notes and references.

1.2.1. *Higher rank phenomena for mapping class groups.* Theorem 1.2 is an example of a higher rank phenomenon for mapping class groups. The reader may compare Theorem 1.2 to the following result of Ghys [14] and Burger–Monod [4], which built on work of Witte [34]:

Theorem 1.7. *Let G be a lattice in a simple Lie group of higher rank. Then every continuous action of G on the circle has a finite orbit, and every C^1 action of G on the circle factors through a finite group.*

Lattices in higher rank simple Lie groups have Kazhdan's property (T). Navas [27] proved the following non-smoothability result for all countable groups with property (T). Here, we use the terminology $C^{1+\tau}$ *diffeomorphism* for a C^1 diffeomorphism whose first derivative is Hölder continuous of exponent $\tau > 0$:

Theorem 1.8. *Let G be a countable group with property (T) and $\tau > 1/2$. Then every $C^{1+\tau}$ action of G on the circle factors through a finite group.*

Navas refined Theorem 1.8 to groups with relative property (T) in [28], and produced a finitely generated, locally indicable group with no faithful differentiable action on the interval in [29]. In those papers, as in the present one, the main subtleties arise from differentiability.

Braid groups and mapping class groups of genus two surfaces (virtually) admit nontrivial homomorphisms to $\mathbb{Z}$, so that these groups do admit (non–faithful) actions on the circle without a global fixed point, which can even be taken to be analytic.

In this context, Theorem 1.2 can be viewed as a generalization of known results to “lattices in mapping class groups”, i.e. finite index subgroups of mapping class groups. The reader may note that Theorem 1.2 resolves a question posed by Labourie in his 2002 ICM talk, as well as by Farb (see page 47 of [10]).

For C^1 actions, we can contrast Theorem 1.4 with the following result of Farb–Franks [12, Theorem 1.6]:

Theorem 1.9. *If X be a connected one–manifold, then every finitely generated residually torsion–free nilpotent group embeds into the orientation–preserving C^1 diffeomorphism group of X.*

In particular, right-angled Artin groups, pure braid groups, and Torelli groups of surfaces all admit faithful C^1 actions on the circle. In particular, Theorem 1.4 cannot be weakened to cover C^1 actions of right-angled Artin groups. In light of Parwani’s Theorem [31], one may ask the following:

Question 1. *Let S be a surface with genus at least two. Does* $\mathrm{Mod}(S)$ *virtually admit a faithful C^1 action on a compact one–manifold?*

Ivanov conjectured that every finite index subgroup of a higher genus mapping class group has finite abelianization [20]. Parwani [31, Conjecture 1.6] noted that Ivanov’s conjecture anticipates a negative answer to Question 1 when the genus is at least four.

1.2.2. *Other group actions.* Theorem 1.4 shows that a group which contains $A(P_4)$ cannot act by $C^{1+\mathrm{bv}}$ diffeomorphisms on the circle. There are many groups which contain $A(P_4)$, including many subgroups of mapping class groups which do not have finite indices.

We recall the following: let $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ denote the automorphism and the outer automorphism groups of the free group F_n on n generators. The *Torelli group* $\mathscr{I}(S)$ of S is defined to be the kernel of the standard representation $\mathrm{Mod}(S) \to \mathrm{Aut}(H_1(S,\mathbb{Z}))$. When S has a distinguished marked point, one can write $\mathfrak{J}_k(S)$ for the k^{th} term of the *Johnson filtration*, i.e. $\mathfrak{J}_k(S)$ is the kernel of the natural action of $\mathrm{Mod}(S)$ on the k^{th} universal nilpotent quotient G_k of $\pi_1(S)$. In other words, we set $\gamma_1(\pi_1(S)) = \pi_1(S)$ and $\gamma_{k+1}(\pi_1(S)) = [\pi_1(S), \gamma_k(\pi_1(S))]$. Then, $G_k = \pi_1(S)/\gamma_k(\pi_1(S))$. Note that with this definition, $\mathfrak{J}_1(S) = \mathrm{Mod}(S)$ and $\mathfrak{J}_2(S) = \mathscr{I}(S)$.

Corollary 1.10. *No finite index subgroup of the following groups acts faithfully by $C^{1+\mathrm{bv}}$ diffeomorphisms on M:*

(1) $\mathrm{Aut}(F_n)$ *and* $\mathrm{Out}(F_n)$ *for* $n \geqslant 3$*;*
(2) $\mathscr{I}(S)$ *and* $\mathfrak{J}_3(S)$*, where* S *has genus at least* 3*;*
(3) $\mathfrak{J}_k(S)$*, where* S *has genus at least* 5 *if* $k > 3$*.*

It was proved in [3] that $\mathrm{Aut}(F_n)$ does not admit a nontrivial action by circle homeomorphisms, though the Bridson–Vogtmann proof relies essentially on torsion inside $\mathrm{Aut}(F_n)$ and thus does not generalize to finite index subgroups of $\mathrm{Aut}(F_n)$. We will give a proof of Corollary 1.10 in Section 3 below.

1.2.3. *Compactness and smooth actions.* Theorem 1.4 also underlines a fundamental difference between compact and noncompact one–manifolds, as far as their diffeomorphism groups are concerned. We see in this paper that many right-angled Artin groups do not embed in $\mathrm{Diff}_+^{1+\mathrm{bv}}(M)$ when M is compact, though no such restrictions hold when M is noncompact, as is illustrated by the following result of the authors [1]:

Theorem 1.11. *Every right-angled Artin group embeds into the orientation–preserving* C^∞ *diffeomorphism group of* $\mathbb{R}$*.*

Actions on noncompact manifolds have applications to mapping class groups of surfaces in their own right:

Question 2. *Let* $n \geqslant 4$*. Does* B_n *virtually admit a faithful* C^∞ *action on* $\mathbb{R}$*?*

A negative answer would prove that a braid group on at least four strands does not virtually embed into a right-angled Artin group, by Theorem 1.11. It has recently been proved that such braid groups do not virtually admit *cocompact* actions on CAT(0) cube complexes [17], [19].

1.2.4. *Structure of finitely generated subgroups of the diffeomorphism group.* Theorems 1.2 and 1.4 raise natural questions about the structure of finitely generated subgroups of $\mathrm{Diff}_+^{1+\mathrm{bv}}(M)$, which are known to be quite complicated [15]. Already, we see that the possible right-angled Artin subgroups of $\mathrm{Diff}_+^{1+\mathrm{bv}}(M)$ are rather restricted whenever M is a compact one–manifold. Indeed, Corollary 1.5 states that if Γ is a graph and $A(\Gamma)$ acts faithfully by $C^{1+\mathrm{bv}}$ diffeomorphisms on M then Γ must be P_4–free. The class of P_4–free graphs is well–understood: it is the smallest class of finite simplicial graphs which contains a single vertex graph, and which is closed under finite disjoint unions and finite joins. See [7] and also [21] for a more detailed discussion of P_4–free graphs and the right-angled Artin groups defined by such graphs.

Let us denote by $\mathrm{Diff}_+^{1+\mathrm{bv}}(I, \partial I)$ the group of $C^{1+\mathrm{bv}}$ diffeomorphisms of I which are the identity near the boundary of I. By concatenating intervals, we see that if G and H are finitely generated subgroups of $\mathrm{Diff}_+^{1+\mathrm{bv}}(I, \partial I)$, then $G \times H$ occurs as a subgroup of $\mathrm{Diff}_+^{1+\mathrm{bv}}(I, \partial I)$. A much more subtle question is the following:

Question 3. *Let $G, H < \mathrm{Diff}_+^{1+\mathrm{bv}}(M)$ be finitely generated subgroups. Does $G * H$ occur as a subgroup of $\mathrm{Diff}_+^{1+\mathrm{bv}}(M)$?*

A positive answer to this question would characterize the right-angled Artin subgroups of $\mathrm{Diff}_+^{1+\mathrm{bv}}(M)$ as exactly the ones with P_4–free defining graphs. Recall that a diffeomorphism of a space is *fully supported* if the fixed point set has empty interior [11]. The referee has pointed out the following partial answer to Question 3: If G and H consist of fully supported diffeomorphisms, then one can embed $G * H$ into $\mathrm{Diff}_+^2(M)$ by conjugating one of G and H by some diffeomorphism f, and the choice of such an f is dense in $\mathrm{Diff}_+^2(M)$. Similar genericity arguments are described in [15, Proposition 4.5] and in [33, Theorem 2.1].

1.2.5. *Sections for the mapping class group.* Let

$$\pi\colon \mathrm{Homeo}_+(S) \to \mathrm{Mod}(S)$$

be the projection that takes a homeomorphism of S its mapping class. A well–known result of Markovic [26], which is closely related to mapping class group actions on the circle, says that π does not admit a section whenever S is a closed surface with genus at least six.

Question 4. *Let $G < \mathrm{Mod}(S)$ be a finite index subgroup. Does there exist a section $s\colon G \to \mathrm{Diff}_+^1(S)$ which splits π? How about to $\mathrm{Homeo}_+(S)$?*

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3. Right-angled Artin groups and finite index subgroups of mapping class groups

For our purposes, right-angled Artin groups are an essential tool for understanding all finite index subgroups of a given mapping class group. In this section, we summarize the relevant facts and reduce the results of this paper to Theorem 1.4.

Lemma 3.1. *If a right-angled Artin group A embeds into a group G, then A embeds into each finite index subgroup of G.*

Proof. Let H be a finite index subgroup of G. By taking a smaller finite index subgroup if necessary, we may assume H is normal in G. Let $\phi\colon A \to G$ be the given injective homomorphism and $N = [G : H]$. Write $A = A(\Gamma)$. Then $K = \langle v^N \mid v \in V(\Gamma)\rangle \leqslant A$ is isomorphic to A and $\phi(K) \leqslant H$. □

Let S be a surface. Recall that the *curve graph* $\mathscr{C}(S)$ is a graph whose vertices are isotopy classes of non–peripheral simple closed curves on S, and the edge relation is given by disjoint realization on S. A graph Γ is called an *induced subgraph* of $\mathscr{C}(S)$ if there exists an injective map of graphs $\iota\colon \Gamma \to \mathscr{C}(S)$ which preserves adjacency and non–adjacency.

Theorem 3.2 (See [23])**.** *Let $\Gamma < \mathscr{C}(S)$ be a finite induced subgraph. Then there exists an injective homomorphism $A(\Gamma) \to \mathrm{Mod}(S)$.*

Explicitly, let us suppose a graph map $\iota\colon \Gamma \to \mathscr{C}(S)$ preserves adjacency and non–adjacency. For each $N \in \mathbb{N}$, we get an induced map

$$\iota_{*,N}\colon A(\Gamma) \to \mathrm{Mod}(S)$$

by sending $v \mapsto T^N_{\iota(v)}$, where $T_{\iota(v)}$ denotes the Dehn twist about the curve $\iota(v)$. A more detailed version of Theorem 3.2 as proved in [23] is that for $N \gg 0$, the homomorphism $\iota_{*,N}$ is injective.

Corollary 3.3. *The group $A(P_4)$ embeds into* $\mathrm{Mod}(S)$ *whenever $c(S) \geqslant 2$.*

Proof of Theorem 1.2. For surfaces S with $c(S) \geqslant 2$, the conclusion of the theorem follows from Theorem 1.4, combined with Lemma 3.1 and Corollary 3.3. For surfaces with $c(S) < 2$, we have that $\mathrm{Mod}(S)$ is virtually a product of a free group and a cyclic group [5] and thus virtually admits a $C^{1+\mathrm{bv}}$ action on M; for example, see [16]. □

Proof of Corollary 1.10. By Theorem 1.4, it suffices to find a copy of $A(P_4)$ in each of these groups.

(1) Note that the surface $S_{1,0,2}$ of genus one with two boundary components contains a *chain* of four simple closed curves, by which we mean a collection of four pairwise non–isotopic essential simple closed curves $\{\gamma_1, \ldots, \gamma_4\}$ in minimal position with the property that $\gamma_i \cap \gamma_{i+1} \neq \varnothing$, but $\gamma_i \cap \gamma_j = \varnothing$ otherwise. See Figure 1 (a)

below. By Theorem 3.2, this realizes $A(P_4)$ in $\mathrm{Mod}(S_{1,0,2}) < \mathrm{Out}(F_3)$. Equipping $S_{1,0,2}$ with a marked point realizes $\mathrm{Mod}(S_{1,1,2})$ as a subgroup of $\mathrm{Aut}(F_3)$.

(2) A closed surface S of genus at least 3 contains a chain of four separating closed curves as shown in Figure 1 (b). The curves are labeled by a, b, c, d so that the intersection graph coincides with the graph shown in Figure 2 which is a copy of P_4. By Theorem 3.2 this furnishes a copy of $A(P_4)$ in $\mathscr{I}(S)$. The reader may note that this actually furnishes $A(P_4)$ in $\mathfrak{J}_3(S)$.

(3) (Sketch) If S has genus 5 or more then there exists a chain of four subsurfaces $\{S_1, \ldots, S_4\}$ in S, where each S_i is homeomorphic to a genus two surface with one boundary component. A chain of subsurfaces is defined analogously to a chain of simple closed curves, and a chain of two subsurfaces is illustrated in Figure 1 (c) below. Let $k > 3$.

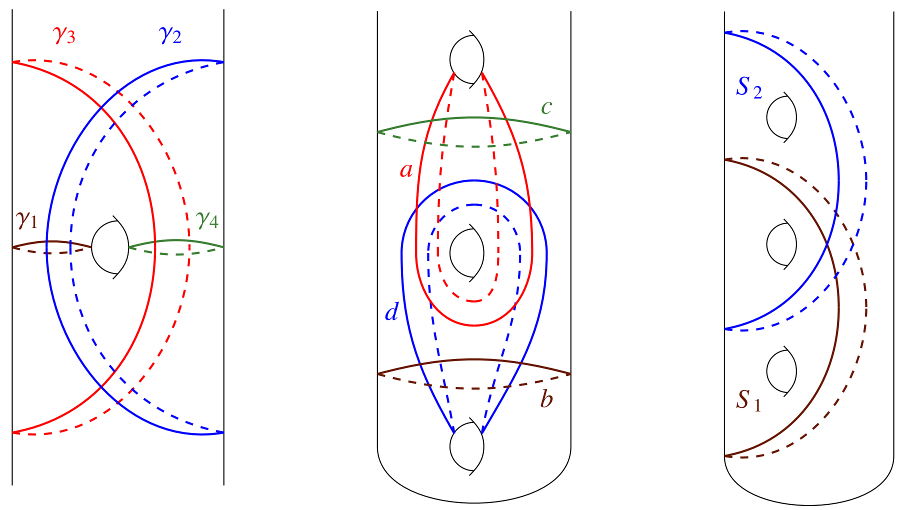
For each $i = 1, 2, 3, 4$, we claim the surface S_i supports a pseudo-Anosov mapping classes ψ_i which lie in the Johnson kernel $\mathfrak{J}_k(S)$. This follows from a well–known fact that every non–central normal subgroup of a mapping class group contains pseudo-Anosov elements (see [20]). One way to see this fact is to appeal to a result of Dahmani–Guirardel–Osin in [8], which shows that the mapping class group contains an infinitely generated, purely pseudo-Anosov, normal subgroup $\mathfrak{N}$. If $1 \neq \psi \in \mathfrak{J}_k(S)$ is pseudo-Anosov then there is nothing to show. If ψ fails to be pseudo-Anosov then we take $1 \neq n \in \mathfrak{N}$, and consider $[n, \psi]$. Since ψ and n will not commute as the centralizer of a pseudo-Anosov mapping class is virtually cyclic, we have that $\phi = [n, \psi]$ is nontrivial. Since $\mathfrak{J}_k(S)$ is normal, $\phi \in \mathfrak{J}_k(S)$. Since $\mathfrak{N}$ is normal, we also have $\phi \in \mathfrak{N}$ and is therefore pseudo-Anosov.

Finally, sufficiently large powers of $\{\psi_1, \ldots, \psi_4\}$ will generate a copy of $A(P_4) < \mathfrak{J}_k(S)$, by [6, Theorem 1.1] or [23, Theorem 1.1]. □

We also record the following fact, which allows us to reduce to the case where M is a connected manifold:

Lemma 3.4. *Let $\phi\colon A(P_4) \to G \times H$ be an injective homomorphism, and let ϕ_G and ϕ_H be the composition of ϕ with the projections to G and H respectively. Then one of ϕ_G or ϕ_H is injective.*

Proof. Let K_G and K_H be the kernels of ϕ_G and ϕ_H respectively, which we assume are nontrivial. On the one hand, note that $K_G K_H \cong K_G \times K_H$, so that if $g \in K_G$ and $h \in K_H$ are nontrivial, then $\mathbb{Z}^2 \cong \langle g, h \rangle < A(P_4)$. On the other hand, we have that K_G and K_H both contain loxodromic elements, each of which has a cyclic centralizer in $A(P_4)$ (see Lemma 52 of [22] and Theorem 2.4 of [24]). This is a contradiction, so that at least one of K_G and K_H is trivial. □



(a) A chain of curves

(b) A chain of four separating curves

(c) A chain of subsurfaces

FIGURE 1. Corollary 1.10.

4. CLASSICAL ONE–DIMENSIONAL DYNAMICS

From Sections 4 through 6, we will assume M is connected. In other words, we let $M = I = [0, 1]$ or $M = S^1 = \mathbb{R}/\mathbb{Z}$. For $x, y \in S^1$, the open interval $(x, y) = \{z \in M : x < z < y\}$ is naturally defined as a subset of the image of $(x, x + 1) \subseteq \mathbb{R}$. We let $\mathrm{Homeo}_+(M)$ denote the group of orientation–preserving homeomorphisms of M. For $f \in \mathrm{Homeo}_+(M)$, we use the notations $\mathrm{Fix}\, f = \{x \in M : fx = x\}$ and $\mathrm{supp}\, f = M \setminus \mathrm{Fix}\, f$. The set $\mathrm{supp}\, f$ is called the *support* (or, *open support*) of f. We will broadly appeal to the following equality, which is trivial to prove: $\mathrm{supp}\, f = \mathrm{supp}\, f^{-1}$ for an arbitrary automorphism of an arbitrary set.

We define the set of *periodic points*:

$$\mathrm{Per}\, f = \{x \in M : f^p x = x \text{ for some } 0 \neq p \in \mathbb{Z}\} = \bigcup_{p=1}^{\infty} \mathrm{Fix}(f^p).$$

For $Y \subseteq M$, the set of the connected components of Y is denoted as $\pi_0(Y)$. For two group elements a and b, we let $[a, b] = a^{-1}b^{-1}ab$.

Lemma 4.1. *For $f, g \in \mathrm{Homeo}_+(M)$, we have the following.*

(1) $\mathrm{Fix}(gfg^{-1}) = g\,\mathrm{Fix}\, f$, $\mathrm{supp}(gfg^{-1}) = g\,\mathrm{supp}\, f$ *and* $\mathrm{Per}(gfg^{-1}) = g\,\mathrm{Per}\, f$.

(2) Let $Y \subseteq Z$ be intervals in M such that each component of $\operatorname{supp} f$ *is either contained in Y or disjoint from Z. If $g(Y) \subseteq Y$ and $g(Z) = Z$, then $gf^{\pm 1}g^{-1}(Y) = Y$.*

(3) If f and g commute, then f permutes the elements of $\pi_0(\operatorname{supp} g)$.

Proof. (1) For $x \in M$, we have

$$x \in \operatorname{Fix}(gfg^{-1}) \Leftrightarrow gfg^{-1}x = x \Leftrightarrow f(g^{-1}x) = g^{-1}x \Leftrightarrow g^{-1}x \in \operatorname{Fix} f.$$

And the other two assertions follow similarly.

(2) The assertion is immediate from

$$\operatorname{supp}(gf^{\pm 1}g^{-1}) \cap Z = g \operatorname{supp} f \cap Z \subseteq g(Y) \subseteq Y.$$

(3) Let $Y \in \pi_0(\operatorname{supp} g)$. Since fY is connected and contained in $f \operatorname{supp} g = \operatorname{supp} g$, there exists $Z \in \pi_0(\operatorname{supp} g)$ containing fY. Since $f^{-1}Z$ is connected and contained in $f^{-1}(\operatorname{supp} g) = \operatorname{supp} g$, we have $Y = f^{-1}Z$ and $fY = Z$. □

Let us say f is *grounded* if $\operatorname{Fix} f \neq \varnothing$. For example, every map in $\operatorname{Homeo}_+(I)$ is grounded. We say a group action of G on M is *free* (or, G acts *freely* on M) if $\operatorname{Fix} g = \varnothing$ for each $g \in G \setminus \{1\}$.

For $f \in \operatorname{Homeo}_+(S^1)$, choose an arbitrary lift $\tilde{f} \colon \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$. Then we define the *rotation number* of f as $\operatorname{rot} f = \lim_{n\to\infty} \left(\tilde{f}^n(x) - x\right)/n$, which is uniquely defined in $\mathbb{R}/\mathbb{Z}$ regardless of the choice of the lift $\tilde{f}$ and x. We have $\operatorname{rot}(f^n) = n \operatorname{rot}(f)$ for each $n \in \mathbb{Z}$. It is a standard fact that $\operatorname{rot}(f) = 0$ if and only if f is grounded.

Theorem 4.2 (Hölder's Theorem [30])**.** *If G is a subgroup of* $\operatorname{Homeo}_+(M)$ *such that G acts freely on $M \setminus \partial M$, then G is abelian. In this case, if $M = S^1$ then the rotation number is an embedding from G to the multiplicative group S^1.*

Theorem 4.3 ([11, Proposition 2.10])**.** *The centralizer of an irrational rotation in* $\operatorname{Homeo}_+(S^1)$ *is* $\operatorname{SO}(2, \mathbb{R})$.

We denote by $\operatorname{var}(g; M)$ the total variation of a map $g \colon M \to \mathbb{R}$:

$$\operatorname{var}(g; M) = \sup \left\{ \sum_{i=0}^{n-1} |g(a_{i+1}) - g(a_i)| : (a_i \colon 0 \leqslant i \leqslant n) \text{ is a partition of } M \right\}.$$

In the case $M = S^1$, we require $a_n = a_0$ in the above definition. Following [30], we say a C^1 diffeomorphism f on M is $C^{1+\mathrm{bv}}$ if $\operatorname{var}(f'; M) < \infty$. We let $\operatorname{Diff}_+^{1+\mathrm{bv}}(M)$ denote the group of orientation–preserving $C^{1+\mathrm{bv}}$ diffeomorphisms of M. Our approach essentially builds on the following two fundamental results on $C^{1+\mathrm{bv}}$ diffeomorphisms.

Theorem 4.4 (Denjoy's Theorem [9], [30, Theorem 3.1.1])**.** *If $f \in$* $\operatorname{Diff}_+^{1+\mathrm{bv}}(S^1)$ *and* $\operatorname{Per} f = \varnothing$*, then f is topologically conjugate to an irrational rotation.*

Theorem 4.5 (Kopell's Lemma [25], [30, Theorem 4.1.1])**.** *Suppose* $f, g \in \mathrm{Diff}_+^{1+\mathrm{bv}}(I)$ *and* $[f, g] = 1$. *If* $\mathrm{Fix}\, f \cap (0, 1) = \varnothing$ *and* $g \neq 1$, *then* $\mathrm{Fix}\, g \cap (0, 1) = \varnothing$.

Farb and Franks proved the following *Abelian Criterion* in [11, Lemma 3.2] for C^2 diffeomorphisms. Actually, they stated the theorem in a stronger form for *fully supported* diffeomorphisms, which means that the closure of the support of each diffeomorphism is I. Our proof given here follows [11] closely.

Theorem 4.6 (Abelian Criterion)**.** *If* $f, g, h \in \mathrm{Diff}_+^{1+\mathrm{bv}}(I)$ *satisfy that* $\mathrm{Fix}\, g = \{0, 1\}$ *and that* $[f, g] = 1 = [g, h]$, *then* $[f, h] = 1$.

Proof. Suppose $w \in \langle f, h\rangle$ fixes a point in $(0, 1)$. Since $[w, g] = 1$, Kopell's Lemma implies that $w = 1$. Hence $\langle f, h\rangle$ acts freely on $(0, 1)$ and is abelian by Hölder's theorem. □

The *commutation graph* of a subset S of a group G is a graph on the vertex set S such that two vertices u and v are joined if and only if they are commuting in G. A *complete graph* is a finite graph in which every pair of vertices are joined.

Lemma 4.7. *Suppose a nonabelian subgroup* $G \leqslant \mathrm{Diff}_+^{1+\mathrm{bv}}(S^1)$ *is generated by a finite set* $V \subseteq G$ *such that the commutation graph of* V *is connected. If* V *consists of infinite order elements, then the rotation number of each* $v \in V$ *is rational.*

Proof. Suppose $v_1 \in V$ has an irrational rotation number, and choose a path

$$(v_1, \ldots, v_m)$$

in the commutation graph of V such that all the vertices are visited at least once. By Denjoy's Theorem, there exists $f \in \mathrm{Homeo}_+(S^1)$ such that $f v_1 f^{-1}$ is an irrational rotation. From Theorem 4.3 and the assumption that v_2 is of infinite order, we see that $f v_2 f^{-1}$ is also an irrational rotation. An inductive argument shows that $f v f^{-1}$ is an irrational rotation for each $v \in V$. This would imply that G is abelian. □

Remark 4.8. In Lemma 4.7, one cannot drop the condition that V consists of infinite order elements. For instance, consider the three maps $a, b, c \in \mathrm{Diff}_+^{\infty}(S^1)$:

$$a(x) = x + \frac{\pi}{3}, \quad b(x) = x + \frac{1}{2}, \quad c(x) = x + \frac{\sin 4\pi x}{8\pi} \mod \mathbb{Z}.$$

It is easy to see that $[a, b] = [b, c] = 1$, $[a, c] \neq 1$ and that the rotation number of a is irrational.

We write $A \pitchfork B$ if two sets A and B intersect nontrivially.

Lemma 4.9 (Disjointness Condition)**.** *Components of the supports of two commuting grounded* $C^{1+\mathrm{bv}}$ *diffeomorphisms on* M *are either pairwise equal or pairwise disjoint.*

Proof. Suppose $f, g \in \mathrm{Diff}_+^{1+\mathrm{bv}}(M)$ satisfy $[f, g] = 1$, and $A \pitchfork B$ for some $A = (p, q) \in \pi_0(\operatorname{supp} f)$ and $B = (x, y) \in \pi_0(\operatorname{supp} g)$. Assume further that $A \neq B$. By symmetry, we only need to consider the following two cases.

First, assume $p < x < q \leqslant y$. By replacing f by f^{-1} if necessary, we may assume $gx = x < fx < q$. Then $fx \in \operatorname{supp} g$ and so, $gfx \neq fx = fgx$. This contradicts the commutativity of f and g. Note that we have also covered the case when we have a circular ordering $p < y \leqslant x < q \leqslant p$.

Second, we assume $p \leqslant x < y \leqslant q$. Again, we may assume $f(t) > t$ for $t \in A$. For each $b \in B$, we have $\lim_{k\to-\infty} f^k(b) = p$ and $\lim_{k\to\infty} f^k(b) = q$. Since $gb \in B$, we have

$$gp = \lim_{k\to-\infty} gf^k b = \lim_{k\to-\infty} f^k(gb) = p,\ gq = \lim_{k\to\infty} gf^k b = \lim_{k\to\infty} f^k(gb) = q.$$

Kopell's Lemma for the closure of A yields a contradiction. □

In the following two lemmas, we explain how conjugation can be used to control the supports of homeomorphisms.

Lemma 4.10. *Let* $f, g \in \mathrm{Homeo}_+(M)$. *Suppose* $Y \subseteq Z$ *are open intervals in* M *such that each component of* $\operatorname{supp} f$ *is either contained in* Y *or disjoint from* Z. *If* $g(Z) = Z$, *then there exist* $s, t \in \{-1, 1\}$ *such that for* $u = gf^sg^{-1}$ *and* $w = uf^tu^{-1}$ *we have* $w(Y) \subseteq Y$.

Proof. Write $Y = (p, q)$ and $Z = (P, Q)$. Let $u = gfg^{-1}$. We may assume $p \leqslant up$ by replacing f by f^{-1} if necessary.

Case 1. $p \leqslant up < uq \leqslant q$. Let $w = ufu^{-1}$. Since $u(Y) \subseteq Y$, Lemma 4.1 (2) implies that $w(Y) \subseteq Y$.

Case 2. $p \leqslant up \leqslant q < uq \leqslant Q$. Since $P \leqslant u^{-1}p \leqslant p$, we have $uf^{\pm1}u^{-1}p = uu^{-1}p = p$. Hence, if $ufu^{-1}q \leqslant q$, we are done: we choose $w = ufu^{-1}$. Otherwise, we have $ufu^{-1}q \geqslant q$, hence $uf^{-1}u^{-1}q \leqslant q$. In this case, we choose $w = uf^{-1}u^{-1}$. In either case, we have $wY \subseteq Y$.

Case 3. $p < q \leqslant up < uq < Q$. Let $w = ufu^{-1}$. Since $P \leqslant u^{-1}p < u^{-1}q \leqslant p$, we have $wp = uu^{-1}p = p$ and $wq = uu^{-1}q = q$. Hence, $w(Y) = Y$. □

Let $F(x, y)$ be the free group on $\{x, y\}$ and $g \in F(x, y)$. We say a word $w \in F(x, y)$ is a *successive conjugation of* x *by* g if there exist $n \geqslant 1$ and $s_1, s_2, \ldots, s_n \in \{1, -1\}$ such that for the sequence of words in $F(x, y)$ defined by

$$w_1 = gx^{s_1}g^{-1}, w_{i+1} = w_i x^{s_{i+1}} w_i^{-1} \text{ for } 1 \leqslant i \leqslant n-1$$

we have $w = w_n$.

Lemma 4.11. *Let* $\mathscr{Y} = \{Y_1, Y_2, \ldots\}$ *and* $\mathscr{Z} = \{Z_1, Z_2, \ldots\}$ *be two (possibly infinite) collections of open intervals in* M, *such that* $Y_i \subseteq Z_i$ *for each* i *and* $Z_i \cap Z_j = \varnothing$ *whenever* $i \neq j$. *If* $m \geqslant 0$ *and* $f, g \in \mathrm{Homeo}_+(M)$ *satisfy* $\operatorname{supp} f \subseteq \bigcup \mathscr{Y}$ *and*

$\operatorname{supp} g \subseteq \bigcup \mathscr{Z}$, then there exists a successive conjugation $w(x,y)$ of x by y in $F(x,y)$ such that for $w = w(f,g)$ and $j \leqslant m$, we have $wY_j \subseteq Y_j$ and moreover, $\operatorname{supp}(wfw^{-1}) \subseteq \bigcup_{j\leqslant m} Y_i \cup \bigcup_{j>m} Z_i$.

Proof. For each $Y = Y_i \in \mathscr{Y}$, we define $Z(Y) = Z_i$ so that the group $\langle f, g\rangle$ acts on $Z(Y)$. Moreover, each component of $\operatorname{supp} f$ is either contained in Y or disjoint from $Z(Y)$. By Lemma 4.10, we can choose $s_1, t_1 \in \{-1, 1\}$ such that for $u_1 = u_1(x,y) = yx^{s_1}y^{-1}$ and for $w_1(x,y) = u_1 x^{t_1} u_1^{-1}$, we have $w_1(f,g)(Y_1) \subseteq Y_1$.

Inductively, suppose we have chosen $w_i \in F(x,y)\backslash\langle x\rangle$ such that w_i is a successive conjugation of x by y and $w_i(f,g)(Y_j) \subseteq Y_j$ for $1 \leqslant j \leqslant i < m$. Since $w_i(f,g)(Z_{i+1}) = Z_{i+1}$, we can apply Lemma 4.10 again. That is, for some $s_{i+1}, t_{i+1} \in \{-1, 1\}$, if we let

$$u_{i+1}(x,y) = w_i(x,y)x^{s_{i+1}}w_i(x,y)^{-1},\ w_{i+1}(x,y) = u_i(x,y)x^{t_{i+1}}u_i(x,y)^{-1},$$

then we have $w_{i+1}(f,g)(Y_{i+1}) \subseteq Y_{i+1}$. By Lemma 4.1 (2), we see $u_{i+1}(f,g)(Y_j) \subseteq Y_j$ and $w_{i+1}(f,g)(Y_j) \subseteq Y_j$ for $1 \leqslant j \leqslant i$. Inductively, we have $w_m \in \langle x,y\rangle\backslash\langle x\rangle$ such that $w_m(f,g)Y_j \subseteq Y_j$ for each $1 \leqslant j \leqslant m$. For $j > m$, we have $w_m(f,g)Y_j \subseteq w_m(f,g)Z_j \subseteq Z_j$. In conclusion, if we put $w = w_m(f,g)$, then

$$\begin{aligned}\operatorname{supp} wfw^{-1} = w \operatorname{supp} f &= \bigcup_{j\leqslant m} w(\operatorname{supp} f \cap Y_j) \cup \bigcup_{j>m} w(\operatorname{supp} f \cap Y_j)\\ &\subseteq \bigcup_{j\leqslant m} Y_j \cup \bigcup_{j>m} Z_j.\end{aligned}$$

□

5. The right-angled Artin group on a path

The main objective of this section is to prove Proposition 1.6. Throughout this section, we assume to have a faithful representation $\phi\colon A(P_4) \to \operatorname{Diff}_+^{1+\mathrm{bv}}(M)$ such that $\phi(v)$ is grounded for each $v \in V(P_4)$. We will continue to assume M is connected. For brevity, we let each $v \in V(P_4)$ also denote the diffeomorphism $\phi(v)$. We label the vertices of P_4 as in Figure 2, and use the notations $V = V(P_4) = \{a,b,c,d\}$ and $\mathscr{J}_v = \pi_0(\operatorname{supp} v)$ for each $v \in V$. We choose this particular labeling of the vertices so that none of $[a,b]$, $[b,c]$ or $[c,d]$ is trivial.

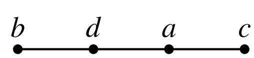


Figure 2. The graph P_4.

Lemma 5.1. *If $J \in \mathscr{J}_a \cap \mathscr{J}_d$, then $\langle a,b,c,d\rangle$ acts on J as an abelian group.*

Proof. For each $v \in \{b,c\}$ and $J' \in \mathscr{J}_v$, the Disjointness Condition (Lemma 4.9) implies that either $J \cap J' = \varnothing$ or $J = J'$ holds. By the Abelian Criterion (Theorem 4.6), we see $\langle a,b,c,d\rangle$ acts as an abelian group on J. □

For $f, g \in \mathrm{Homeo}_+(M)$, we define

$$J(f,g) = \bigcup\{I_f \cup I_g \colon I_f \in \pi_0(\operatorname{supp} f), I_g \in \pi_0(\operatorname{supp} g), I_f \pitchfork I_g, I_f \neq I_g\}$$
$$\cup \bigcup\{J \colon J \in \pi_0(\operatorname{supp} f) \cap \pi_0(\operatorname{supp} g) \text{ and } \langle f, g\rangle\restriction_J \text{ is nonabelian}\}.$$

Lemma 5.2. *For $f, g \in \mathrm{Homeo}_+(M)$, we have $\operatorname{supp}[f,g] \subseteq J(f,g)$.*

Proof. If $x \in M \backslash J(f,g)$, then one of the following holds.

(i) $x \notin \operatorname{supp} f \cup \operatorname{supp} g$;
(ii) $x \in J \in \pi_0(\operatorname{supp} f)$ such that $J \cap \operatorname{supp} g = \varnothing$;
(iii) $x \in J \in \pi_0(\operatorname{supp} g)$ such that $J \cap \operatorname{supp} f = \varnothing$;
(iv) $x \in J \in \pi_0(\operatorname{supp} f) \cap \pi_0(\operatorname{supp} g)$ such that $\langle f, g\rangle$ acts on J as an abelian group.

In each of the cases, we have that $[f,g]x = x$. □

Lemma 5.3. *The following statements hold:*

(1) For each $J \in \mathcal{J}_c \cup \mathcal{J}_d$, we have either $J \subseteq J(c,d)$ or $J \cap J(c,d) = \varnothing$.
(2) $\operatorname{supp} a \cap J(c,d) = \varnothing$.
(3) An interval in $\mathcal{J}_b$ cannot properly contain a component of $J(c,d)$.

Proof. (1) is immediate. For (2), suppose $I_a \in \mathcal{J}_a$ nontrivially intersects $J \in \mathcal{J}_v$ for some $v \in \{c,d\}$ such that $J \subseteq J(c,d)$. By the Disjointness Condition and the Abelian Criterion, we have $I_a = J$ and $\langle a, c, d\rangle$ acts on J as an abelian group. This contradicts the hypothesis that $J \subseteq J(c,d)$.

(3) Each component of $J(c,d)$ contains an interval J in $\mathcal{J}_d$. The Disjointness Condition implies that J cannot be properly contained in $\operatorname{supp} b$. □

Remark 5.4. If $\operatorname{Fix} a$ has an empty interior, then $\operatorname{supp}[c,d] \subseteq J(c,d) \subseteq M \backslash \overline{\operatorname{supp} a} = \varnothing$ and $[c,d] = 1$ by Lemmas 5.2 and 5.3. This recovers the *Abelian Criterion* in the form given in [11, Lemma 3.2].

We say a sequence of four intervals (I_1, I_2, I_3, I_4) forms a *chain* when $I_i \pitchfork I_j$ if and only if $|i - j| = 1$. Figure 3 illustrates examples where sequences of four intervals (I_a, I_b, I_c, I_d) form chains.

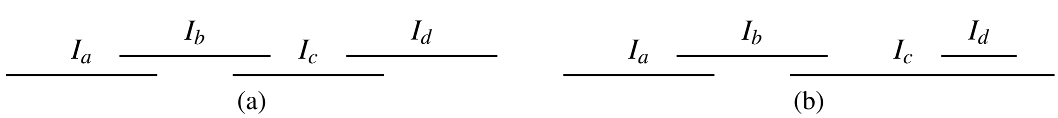


FIGURE 3. Chains of intervals.

Lemma 5.5. *If $I_a \in \mathcal{J}_a$ and $I_d \in \mathcal{J}_d$ satisfy $cbI_a \pitchfork I_d$, then one of the following holds.*

(1) $I_a = I_d = cbI_a$ *and* $\langle a,b,c,d\rangle$ *acts on* I_a *as an abelian group.*
(2) $I_a \cap I_d = \varnothing$ *and there exist* $I_b \in \mathscr{J}_b$ *and* $I_c \in \mathscr{J}_c$ *such that* $bI_a \pitchfork I_c$ *and that* (I_a, I_b, I_c, I_d) *forms a chain; in particular, we have* $I_d \subseteq J(c,d)$.

Proof. The case $I_a = I_d$ is obvious from Lemma 5.1, so let us assume $I_a \cap I_d = \varnothing$. Choose $x \in I_a$ such that $cbx \in I_d$. Let us first assume $bx \in I_a$. The Disjointness Condition implies that each interval belonging to the family $\mathscr{J}_c \cup \mathscr{J}_d$ is either equal to or disjoint from I_a. So we have $cbx \in I_a$. But this is a contradiction, for I_a and I_d are disjoint. It follows that $x \in I_a \cap I_b$ and $bx \in I_b \backslash I_a$ for some $I_b \in \mathscr{J}_b$. We have $I_b \cap I_d = \varnothing$ and so, $cbx \notin I_b$. This means that there exist $I_c \in \mathscr{J}_c$ such that $bx \in I_b \cap I_c$ and $cbx \in I_c \cap I_d$. Since $I_c \pitchfork I_d$ and $I_c \neq I_d$, we see $I_d \subseteq J(c,d)$. ◻

We set $\mathscr{Y} = \pi_0\left(M \backslash \overline{J(c,d)}\right)$.

Lemma 5.6. *If* $I_a \in \mathscr{J}_a$ *and* $I_a \subseteq Y$ *for some* $Y \in \mathscr{Y}$*, then* cbI_a *is contained in the component of* $M \backslash \left(\bigcup \mathscr{Y} \backslash \{Y\}\right)$ *which contains* Y.

Proof. We have the following:

Claim. For each $Y' \in \mathscr{Y} \backslash \{Y\}$, the intervals cbI_a and Y' are disjoint.

Suppose the contrary and choose $x \in I_a$ and $Y' \in \mathscr{Y} \backslash \{Y\}$ such that $cbx \in Y'$. If $bx \in I_a$ then $cbx \in I_a \subseteq Y$ and we have a contradiction. So $bx \notin I_a$, and $\{x, bx\} \subseteq I_b$ for some $I_b \in \mathscr{J}_b$. Since $I_a \pitchfork I_b$ and $I_a \neq I_b$, we see $I_b \notin \mathscr{J}_d$. So $I_b \cap \operatorname{supp} d = \varnothing$. If $cbx \in I_b$, then $I_b \pitchfork Y'$. From $I_b \pitchfork Y$, we see that I_b would properly contain a component of $J(c,d)$. This contradicts Lemma 5.3. It follows that $cbx \notin I_b$, and that we can find $I_c \in \mathscr{J}_c$ such that $\{bx, cbx\} \subseteq I_c$. In particular, $cbx \in I_c \cap Y'$. Since the interval $I_b \cup I_c$ intersects both Y and Y', a component J of $J(c,d)$ is contained in $I_b \cup I_c$. By Lemma 5.3, the interval I_b cannot properly contain J, and so, $I_c \subseteq J(c,d)$. Since $cbx \in Y' \subseteq M \backslash J(c,d)$, this is a contradiction.

To complete the proof, let us assume that cbI_a is contained in some component Z of $M \backslash \left(\bigcup \mathscr{Y} \backslash \{Y\}\right)$ which does not contain Y. Fix $x \in I_a$ so that we have $cbx \in Z$. Each interval joining x and cbx contains an interval in $\mathscr{Y} \backslash \{Y\}$. As in the previous paragraph, one can see that there exist $I_v \in \mathscr{J}_v$ for $v \in \{b, c\}$ such that $bx \in I_b \cap I_c \backslash I_a$ and $cbx \in I_c \backslash I_b$, and moreover, $I_b \cup I_c$ is an interval containing both x and cbx. By assumption, $I_b \cup I_c$ contains an interval $Y' \in \mathscr{Y} \backslash \{Y\}$. This implies that $I_b \cup I_c$ contains at least one components J of $J(c,d)$ such that $J \cap I_c = \varnothing$. This implies that I_b would properly contain J, and we have a contradiction. ◻

By Lemma 5.6, for each $Y \in \mathscr{Y}$ there uniquely exists a minimal open interval $Z(Y)$ contained in

$$M \backslash \left(\bigcup \mathscr{Y} \backslash \{Y\}\right)$$

such that $Y \subseteq Z(Y)$ and such that for each $I_a \in \mathscr{J}_a$ with $I_a \subseteq Y$, we have $cbI_a \subseteq Z(Y)$.

Lemma 5.7. *If an open interval J intersects $Z(Y)$ for some $Y \in \mathscr{Y}$, then either $J \subseteq Z(Y)$ or $cbI_a \pitchfork J$ for some $I_a \in \mathscr{J}_a$ contained in Y.*

Proof. Suppose $J \nsubseteq Z(Y)$ so that one of the endpoints of $Z(Y)$ belongs to J. Then a component of $J \cap Z(Y)$ must intersect cbI_a for some $I_a \in \mathscr{J}_a$ satisfying $I_a \subseteq Y$, by the minimality of $Z(Y)$. □

Lemma 5.8. *For two distinct intervals Y and Y' in $\mathscr{Y}$, we have $Z(Y) \cap Z(Y') = \varnothing$.*

Each open interval $J \subseteq M$ can be written as $J = (\inf J, \sup J)$ for some points $\inf J, \sup J \in M$.

Proof of Lemma 5.8. Suppose $Z(Y) \pitchfork Z(Y')$. By definition, we have $Z(Y) \cap Y' = \varnothing$ and so, $Z(Y') \nsubseteq Z(Y)$. By Lemma 5.7, we can choose $I_a \in \mathscr{J}_a$ with $I_a \subseteq Y$ such that $cbI_a \pitchfork Z(Y')$. If $cbI_a \nsubseteq Z(Y')$, then Lemma 5.7 again implies that there is $I'_a \in \mathscr{J}_a$ with $I'_a \subseteq Y$ such that $cbI_a \pitchfork cbI'_a$. But this is impossible since $I_a \cap I'_a = \varnothing$.

It follows that $cbI_a \subseteq Z(Y')$. By minimality of $Z(Y')$, we can find $I'_a \in \mathscr{J}_a$ such that $I'_a \subseteq Y$ and such that, up to switching the roles of Y and Y', we have a (possibly circular) ordering of points as following:

$$\inf I_a < \sup I_a \leqslant \inf(cbI'_a) < \sup(cbI'_a) \leqslant \inf(cbI_a) < \sup(cbI_a) \leqslant \inf I'_a.$$

This inequality is absurd since cb is order–preserving. □

We let $g = cbab^{-1}c^{-1}$ so that $\operatorname{supp} g = cb \operatorname{supp} a$.

Lemma 5.9. *For each $Y \in \mathscr{Y}$, we have $aY = Y, dY = Y$ and $gZ(Y) = Z(Y)$. Furthermore, we have $\operatorname{supp} g \subseteq \bigcup\{Z(Y) \colon Y \in \mathscr{Y}\}$.*

Proof. From $\operatorname{supp} a \cap Y = \bigcup\{J \in \mathscr{J}_a \colon J \pitchfork Y\}$, we see $aY \subseteq Y$. If $I_d \in \mathscr{J}_d$ satisfies $I_d \pitchfork Y$ but $I_d \nsubseteq Y$, then $I_d \pitchfork J(c,d)$, and hence by Lemma 5.3 we have $I_d \subseteq J(c,d)$. This would be a contradiction, and so, we have $dY = Y$. For the last two assertions, consider an arbitrary component cbI_a of $\operatorname{supp} g = cb \operatorname{supp} a$. If $I_a \subseteq Y' \in \mathscr{Y}$ for some $Y' \neq Y$, then $cbI_a \subseteq Z(Y') \in M \setminus Z(Y)$ by Lemma 5.8. If $I_a \subseteq Y$, then $cbI_a \subseteq Z(Y)$ by definition. This shows $gZ(Y) \subseteq Z(Y)$ and $\operatorname{supp} g \subseteq \bigcup\{Z(Y) \colon Y \in \mathscr{Y}\}$. □

Let $\mathscr{Y}_0$ be the collection of intervals $Y \in \mathscr{Y}$ such that for some $I_a \in \mathscr{J}_a$ contained in Y and some $I_d \in \mathscr{J}_d$, we have $cbI_a \cap I_d \cap (M \setminus Y) \neq \varnothing$; in this case, Lemma 5.5 and the fact $\operatorname{supp} a \cap J(c,d) = \varnothing$ together imply that $I_a \neq I_d$ and $I_d \subseteq J(c,d)$.

Lemma 5.10. *Let g be as above. Suppose $Y \in \mathscr{Y}$ and $I_d \in \mathscr{J}_d$ satisfies $I_d \pitchfork Z(Y)$. Assume either*

(1) $I_d \subseteq Y$, or
(2) $Y \notin \mathscr{Y}_0$.

Then either g acts on I_d as the identity or $\langle a,b,c,d\rangle$ acts on I_d as an abelian group.

Proof. Suppose g is not the identity on I_d, so that $cb\operatorname{supp} a \pitchfork I_d$. We can find some $I_a \in \mathscr{J}_a$ such that $cbI_a \pitchfork I_d$.

(1) Suppose $I_d \subseteq Y$. Since $I_d \subseteq M \backslash J(c,d)$, the case (2) of Lemma 5.5 does not occur. It follows that $I_a = I_d = cbI_a$ and that $\langle a,b,c,d\rangle$ is abelian on I_d.

(2) Suppose $Y \notin \mathscr{Y}_0$. If $I_d \subseteq Z(Y)$, then we have $I_a \subseteq Y$ by the definition of $Z(Y)$ and Lemma 5.8. If $I_d \nsubseteq Z(Y)$, then Lemma 5.7 implies that we can find $I'_a \in \mathscr{J}_a$ such that $I'_a \subseteq Y$ and $cbI'_a \pitchfork I_d$. So we may simply assume $I_a \subseteq Y$. If $I_d \neq cbI_a$, then Lemma 5.5 implies that $I_d \subseteq J(c,d)$ and $Y \in \mathscr{Y}_0$, and this would violate the assumption. So, we have $I_d = cbI_a$. By Lemma 5.5, the group $\langle a,b,c,d\rangle$ acts on I_d as an abelian group. □

Lemma 5.11. *$\mathscr{Y}_0$ is an infinite collection of intervals.*

Proof. Suppose $\mathscr{Y} = \{Y_1, Y_2, \ldots, Y_m, \ldots\}$ such that $\mathscr{Y}_0 = \{Y_1, Y_2, \ldots, Y_m\}$. By Lemma 4.11, there is a successive conjugation $w(x,y) \in F(x,y)$ of x by y such that $w = w(a,g)$ acts on Y_j for $j \leqslant m$ and on $Z(Y_j)$ for $j > m$. Moreover for waw^{-1}, we can require that $\operatorname{supp} waw^{-1} \subseteq \bigcup_{j\leqslant m} Y_j \cup \bigcup_{j>m} Z(Y_j)$.

Claim. If some $wI_a \pitchfork \operatorname{supp} d$ for some $I_a \in \mathscr{J}_a$, then $wI_a \in \mathscr{J}_d$ and $\langle a,b,c,d\rangle$ acts on wI_a as an abelian group.

Assume $wI_a \pitchfork I_d$ for some $I_d \in \mathscr{J}_d$. Let us first consider the case $I_a \subseteq Y_j$ for some $j \leqslant m$. We have $wI_a \subseteq Y_j$ and $I_d \pitchfork Y_j$. By Lemma 5.3 (1), we see that $I_d \subseteq Y_j$. Then Lemma 5.10 (1) implies that g is the identity on I_d or $\langle a,b,c,d\rangle$ is abelian on I_d. If g is the identity on I_d, then $wI_d = I_d$ and hence, $I_d = I_a = wI_a$ and $\langle a,b,c,d\rangle$ is abelian on $I_d = wI_a$ by Lemma 5.1. If $\langle a,b,c,d\rangle$ is abelian on I_d, then we also have $wI_d = I_d$ and so, $I_d = I_a = wI_a$.

Let us now suppose $I_a \subseteq Y_j$ for some $j > m$, so that $I_d \pitchfork Z(Y_j)$. By Lemma 5.10 (2), we see that $I_a \cap \langle a,g\rangle I_d = I_a \cap I_d$. If $I_a \cap \operatorname{supp} d = \varnothing$, then $I_a \cap I_d = \varnothing$ and $wI_a \cap I_d = w(I_d \cap w^{-1}I_d) = \varnothing$. So $I_a \in \mathscr{J}_d$, and Lemma 5.1 implies that $I_a = wI_a = I_d$ and that $\langle a,b,c,d\rangle$ acts on I_a as an abelian group.

From the claim above and Lemma 5.2, we see that

$$\operatorname{supp}[waw^{-1}, d] \subseteq J(waw^{-1}, d) = \varnothing,$$

and so, $[waw^{-1}, d] = 1$. On the other hand, there exists a successive conjugation u of a by b such that $[waw^{-1}, d] = [cuc^{-1}, d]$. Since $[cuc^{-1}, d]$ is a nontrivial reduced word in $A(P_4)$, we have a contradiction. □

Now Lemma 5.11 completes the proof of Proposition 1.6.

6. The Two Jumps Lemma

In this section we prove some quantitative estimates on first derivatives which naturally arise in our setup. We let $M = I$ or $M = S^1$.

Lemma 6.1. *If $f\colon M \to M$ is a C^1 map and x is an accumulation point of* $\operatorname{Fix} f$*, then $f'(x) = 1$.*

Proof. Choose a sequence $(x_n)_{n\geqslant 1}$ converging to x such that $x_n \in \operatorname{Fix} f$ and $x_n \neq x$ for each n. Then there exists y_n between x_n and x satisfying $f'(y_n) = (f(x_n) - f(x))/(x_n - x) = 1$. We have $1 = \lim_{n\to\infty} f'(y_n) = f'(\lim_{n\to\infty} y_n) = f'(x)$. □

The length of an interval J is denoted by $|J|$.

Lemma 6.2 (Two Jumps Lemma)**.** *Let $f, g\colon M \to M$ be continuous maps and $(y_j)_{j\geqslant 1}$ be an infinite sequence of points in M. For each $j \geqslant 1$, suppose I_j is a closed interval bounded by $f(y_j)$ and $g(y_j)$ such that y_j belongs to the interior of I_j and such that* $\lim_{j\to\infty} |I_j| = 0$*. For each $j \geqslant 1$, let A_j and B_j be the closed intervals determined by the following conditions:*

$$I_j = A_j \cup B_j,\ A_j \cap B_j = y_j,\ f(y_j) \in A_j,\ g(y_j) \in B_j.$$

If $A_j \pitchfork \operatorname{Fix} g$ and $B_j \pitchfork \operatorname{Fix} f$ for each $j \geqslant 1$, then f or g fails to be C^1.

Figure 4 (a) illustrates an example of the hypotheses in Lemma 6.2.

Proof of Lemma 6.2. Suppose f and g are C^1, and choose $s_j \in A_j \cap \operatorname{Fix} g$ and $t_j \in B_j \cap \operatorname{Fix} f$ for each $j \geqslant 1$. We have a configuration of points as shown in Figure 4 (a), up to choosing a subsequence and reversing the orientation of M. So,

$$f(y_j) \leqslant s_j < y_j < t_j \leqslant g(y_j).$$

By further passing to a subsequence, we can also assume that $\lim_{j\to\infty} y_j = y$ for some $y \in M$ so that $f(y) = y = g(y)$. Since (s_j) and (t_j) both accumulate at y, we see $f'(y) = 1 = g'(y)$ from Lemma 6.1.

Using the Mean Value Theorem, we can find $u_j \in (s_j, y_j)$ and $v_j \in (y_j, t_j)$ satisfying

$$g'(u_j) = \frac{g(y_j) - s_j}{y_j - s_j} = 1 + \frac{|B_j|}{y_j - s_j} \geqslant 1 + \frac{|B_j|}{|A_j|}$$

$$f'(v_j) = \frac{t_j - f(y_j)}{t_j - y_j} = 1 + \frac{|A_j|}{t_j - y_j} \geqslant 1 + \frac{|A_j|}{|B_j|}$$

and hence, $f'(v_j)g'(u_j) \geqslant 4$. We have a contradiction, since (u_j) and (v_j) both converge to y. □

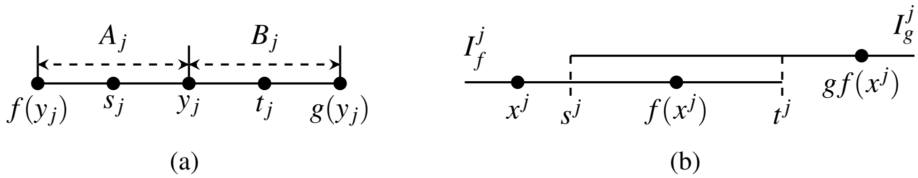


FIGURE 4. Lemmas 6.2 and 6.3.

Lemma 6.3. *Let f, g be orientation–preserving homeomorphisms of M. For each $v \in \{f, g\}$, suppose we have an infinite collection of disjoint open intervals $\{I_v^j \colon j \geqslant 1\}$ such that $v(I_v^j) = I_v^j$ for $j \geqslant 1$. Suppose furthermore that for each $j \geqslant 1$, there exists some $x^j \in I_f^j \setminus I_g^j$ with the property that $f(x^j) \in I_g^j$ and $gf(x^j) \notin I_f^j$. Then one of f or g fails to be a C^1 diffeomorphism.*

Proof. Immediate by applying Lemma 6.2 to f^{-1} and g on the sequence $\left(f(x^j)\right)_{j \geqslant 1}$. See Figure 4 (b). □

7. PROOF OF THE MAIN THEOREM

Proof of Theorem 1.4. Suppose $\phi \colon A(P_4) \to \mathrm{Diff}_+^{1+\mathrm{bv}}(M)$ is an injective homomorphism, and we let the vertex set $V = V(P_4) = \{a, b, c, d\}$ as shown in Figure 2. By raising the generators to suitable powers, we may assume that $\phi(A(P_4))$ preserves each component of M. From Lemma 3.4, we may further assume that M is connected. By raising to higher powers if necessary, we can also require that $\phi(v)$ is grounded for each v; see Lemma 4.7. By Proposition 1.6 and Lemma 5.5, we have an infinite family of distinct intervals $\{I_v^j \colon j \geqslant 1\} \subseteq \pi_0(\operatorname{supp} v)$ for each v such that the hypotheses of Lemma 6.3 are satisfied after setting:

$$f = b, g = c.$$

We have a contradiction by Lemma 6.3. □

8. QUASI–ISOMETRIC RIGIDITY

In this section, we prove the following generalization of Theorem 1.2:

Theorem 8.1. *Let G be a finitely generated group which is quasi–isometric to the mapping class group of a finite type surface with complexity at least two. Then there is no injective homomorphism $G \to \mathrm{Diff}_+^{1+\mathrm{bv}}(M)$.*

Recall that a *finite type surface* is a closed surface minus finitely many (possibly zero) punctures. Let us denote the center $Z(S) = Z(\mathrm{Mod}(S))$. The proof of Theorem 8.1 relies on the following result:

Theorem 8.2 (QI rigidity of mapping class groups [2], [18])**.** *Let S be a finite type surface with $c(S) \geqslant 2$, and let G be a finitely generated group which is quasi–isometric to* $\mathrm{Mod}(S)$*. Then after passing to a finite index subgroup of G if necessary, there is a homomorphism $G \to \mathrm{Mod}(S)/Z(S)$ with finite index image and finite kernel.*

To prove Theorem 8.1, let us start with a homomorphism $\phi\colon H_0 \to L_0$ such that $[G : H_0] < \infty$, $[\mathrm{Mod}(S)/Z(S) : L_0] < \infty$ and such that $K_0 = \ker\phi$ is finite. Let L_1 be the preimage of L_0 with respect to the projection $\mathrm{Mod}(S) \to \mathrm{Mod}(S)/Z(S)$. See the diagram below.

$$\begin{array}{ccccccccc}
 & & & & G & & \mathrm{Mod}(S)/Z(S) & \twoheadleftarrow & \mathrm{Mod}(S) \\
 & & & & \Big|{\scriptstyle <\infty} & & \Big|{\scriptstyle <\infty} & & \Big|{\scriptstyle <\infty} \\
1 & \longrightarrow & K_0 & \longrightarrow & H_0 & \xrightarrow{\ \phi\ } & L_0 & \twoheadleftarrow & L_1 \\
 & & \Big| & & \Big| & & \Big| & & \Big| \\
1 & \longrightarrow & K & \longrightarrow & H = \phi^{-1}(L) & \longrightarrow & L & \xleftarrow{\ \cong\ } & L \cong A(P_4)
\end{array}$$

Since $c(S) \geqslant 2$, there exists $L \cong A(P_4)$ inside L_1. Since L is torsion-free and $|Z(S)| < \infty$, we have that $\mathrm{Mod}(S)/Z(S)$ also contains a copy of L. We denote this copy again as L. Put $H = \phi^{-1}(L)$ and $K = H \cap K_0$. We obtain an extension

$$1 \to K \to H \to A(P_4) \to 1.$$

We wish to show that H cannot act faithfully by $C^{1+\mathrm{bv}}$ diffeomorphisms on M.

Choosing arbitrary lifts of generators of $A(P_4)$ to H, we have that the conjugation action acts by automorphisms of K. Since K is finite and acts faithfully on M, Hölder's Theorem implies that K is abelian. Moreover, some positive power of each generator of H acts by the identity on K, so that we may assume K is central. Theorem 8.1 now follows immediately from Theorem 1.4 and the following lemma:

Lemma 8.3. *Let H be a group and K be a finite subgroup of H such that we have a central extension*

$$1 \to K \to H \to A(P_4) \to 1.$$

Then $A(P_4)$ embeds into H.

Proof. Label P_4 as in Figure 2, and choose lifts $\{\alpha, \beta, \gamma, \delta\}$ of $\{a, b, c, d\}$ to H. We have that $[b, d] = 1$ in $A(P_4)$, so that $[\beta, \delta] = q \in K$ in H. Suppose q has order n. We then compute $[\beta^n, \delta] = q^n = 1$ in H. Thus, we may replace $\{\alpha, \beta, \gamma, \delta\}$ by appropriate positive powers so that pairs which commute in the projection to $A(P_4)$ will commute in H. It follows then that there is a homomorphism $A(P_4) \to H$ which splits the surjection $H \to A(P_4)$, whence it follows that H contains a subgroup isomorphic to $A(P_4)$. □

Remark 8.4. In the proof of Lemma 8.3, it is immediate that the group $\langle \beta, \delta \rangle$ is nilpotent. The Plante–Thurston Theorem [32], [30, Theorem 4.2.3] states that every nilpotent subgroup of $\mathrm{Diff}_+^{1+\mathrm{bv}}(M)$ is abelian. Hence with an additional hypothesis that $H \leqslant \mathrm{Diff}_+^{1+\mathrm{bv}}(M)$, we deduce that $[\beta, \delta] = [\delta, \alpha] = [\alpha, \gamma] = 1$ and that $\langle \alpha, \beta, \gamma, \delta \rangle \cong A(P_4)$.

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