RESIDUAL PROPERTIES OF FIBERED AND HYPERBOLIC 3–MANIFOLDS

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Abstract. We study the residual properties of geometric 3–manifold groups. In particular, we study conditions under which geometric 3–manifold groups are virtually residually p for a prime p, and conditions under which they are residually torsion–free nilpotent. We show that for every prime p, every geometric 3–manifold group is virtually residually p. We show that geometric 3–manifold groups are virtually residually torsion–free nilpotent precisely when they do not arise from Sol geometry.

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1. Introduction

1.1. **Statement of Results.** Let M be a prime, compact, orientable 3–manifold with $\chi(M) = 0$. The Geometrization Theorem (see [19], [20], [21], [25], [26]) says that there exists a finite collection of incompressible tori $\{T_i\} \subset M$ such that each component of

$$M \setminus \bigcup_i T_i$$

admits a geometric structure of finite volume. Precisely, let M_i be a component of

$$M \setminus \bigcup_i T_i.$$

Then there is a finite volume, complete Riemannian metric on the interior of M_i such that the universal cover $\widetilde{M_i}$ is isometric to exactly one of the following spaces, called **model geometries**:

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(1) S^3.
(2) $S^2 \times \mathbb{R}$.
(3) $\mathbb{R}^3$.
(4) Nil.
(5) Sol.
(6) $\mathbb{H}^2 \times \mathbb{R}$.
(7) $\widetilde{PSL_2}(\mathbb{R})$.
(8) $\mathbb{H}^3$.

Whenever the interior of M admits a complete, finite–volume metric for which $\widetilde{M}$ is isometric to one of these eight spaces, we will say that M is **geometric**, or that it **admits a geometric structure**. Thurston showed in [26] that given a 3–manifold which is known to be geometric, one need not produce any metrics to determine which one of the eight geometric structures M admits. It suffices to understand the structure of the fundamental group $\pi_1(M)$. The salient features of the fundamental group which determine the geometric structure in order are as follows. A geometric 3–manifold M admits the corresponding geometric structure above if and only if $\pi_1(M)$ is:

(1) Finite.
(2) Virtually cyclic but not finite.
(3) Virtually abelian but not virtually cyclic.
(4) Virtually nilpotent but not virtually abelian.
(5) Virtually solvable but not virtually nilpotent.
(6) Virtually split as a trivial central extension by $\mathbb{Z}$ and contains a nonabelian free group.
(7) Virtually a nonsplit central extension by $\mathbb{Z}$ and contains a nonabelian free group, but does not virtually split as a trivial central extension.
(8) Not virtually solvable and does not contain an infinite cyclic normal subgroup.

Here, a group G **virtually** has a property X if there is a finite index subgroup G' of G with property X.

In this article, we wish to explore some of the properties of the fundamental groups of geometric 3–manifolds. In particular, we will be interested in residual p–properties and residual torsion–free nilpotence. For a prime p, we say that a group G is **residually** p if every nontrivial element of G survives in a finite p–group quotient of G. We say that G is **residually torsion–free nilpotent** if every nontrivial element of G survives in a torsion–free nilpotent quotient of G. Throughout this paper, p will be used to denote a prime number.

The main result which we shall establish in this paper is the following:

Theorem 1.1. *Let M be a geometric 3–manifold. Then $\pi_1(M)$ is virtually residually p. Furthermore, $\pi_1(M)$ is virtually residually torsion–free nilpotent precisely when M does not admit Sol geometry.*

The proof of Theorem 1.1 we offer is not self–contained. It will rely on recent deep results in hyperbolic geometry, in particular [2]. Below, we will indicate which parts of Theorem 1.1 will have self–contained proofs given in this paper.

An important source of 3–manifolds comes from **fibered** 3–manifolds. These are 3–manifolds which can be described as surface bundles over the circle. Write S for an orientable surface of genus g and n punctures, and let $\Psi \in \mathrm{Homeo}^+(S)$ be an orientation–preserving homeomorphism. Assume without loss of generality that Ψ preserves at least one point of S, which we treat as a basepoint for $\pi_1(S)$. The **mapping torus** T_Ψ is obtained by taking $S \times [0,1]$ and identifying $(x,0)$ with $(\Psi(x),1)$. The 3–manifold T_Ψ fits into a fibration

$$S \to T_\Psi \to S^1.$$

The fundamental group of T_Ψ can be presented as a semidirect product of the form

$$\langle \pi_1(S), t \mid t^{-1}\pi_1(S)t = \Psi_*(\pi_1(S))\rangle.$$

Here, Ψ_* is the $\pi_1(S)$–automorphism induced by Ψ. It is a standard fact that the isomorphism type of $\pi_1(T_\Psi)$ and the homeomorphism type of T_Ψ depend only on the isotopy class of Ψ, and thus only on its image ψ in the mapping class group $\mathrm{Mod}(S)$. Thus, we may write T_ψ for the mapping torus of a mapping class ψ. We call ψ the **monodromy** of the fibration, S the **fiber**, and the generator t of $\pi_1(T_\psi)$ the **stable letter**.

We will establish Theorem 1.1 in several steps. We will first analyze the case of fibered 3–manifolds in detail and prove the following:

Theorem 1.2. *Let T_ψ be a fibered 3–manifold. Then $\pi_1(T_\psi)$ is virtually residually p.*

The case of Sol geometry, which is to say of hyperbolic torus bundles over the circle, requires special attention:

Theorem 1.3. *Let T_A be a hyperbolic torus bundle over the circle with monodromy $A \in SL_2(\mathbb{Z})$.*

1. *The group $\pi_1(T_A)$ is residually p if and only if p divides $\det(A - I)$.*
2. *The group $\pi_1(T_A)$ is not residually p for any p if and only if $\pi_1(T_A)$ is not residually nilpotent, if and only if A is conjugate over $\mathbb{Q}$ to*

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

We remark briefly the by **hyperbolic torus bundle**, we mean a torus bundle with hyperbolic monodromy, not that the corresponding bundle is a hyperbolic manifold.

Manifolds modeled on $\widetilde{PSL_2}(\mathbb{R})$ and $\mathbb{H}^2 \times \mathbb{R}$ geometry will be treated next. Recall that such manifolds are all finitely covered by circle bundles over orientable surfaces:

Theorem 1.4. *Let M be modeled on $\mathbb{H}^2 \times \mathbb{R}$ or on $\widetilde{PSL_2}(\mathbb{R})$ geometry. Then $\pi_1(M)$ virtually admits a faithful representation into $\pi_1(S) \times H$, where S is an orientable surface and where H is a torsion–free nilpotent group. In particular, $\pi_1(M)$ is virtually residually torsion–free nilpotent and virtually residually p for every prime.*

An straightforward consequence of Theorem 1.4 is the following fact, which was known (see [9]) but whose proof is, to the author's knowledge, new:

Corollary 1.5. *Let M be a manifold modeled on $\widetilde{PSL_2}(\mathbb{R})$ geometry. Then $\pi_1(M)$ is linear over $\mathbb{Z}$.*

Establishing residual torsion–free nilpotence for fibered 3–manifolds is generally harder than residual p–properties for any particular prime, since the former implies the latter. We can prove the following though:

Theorem 1.6. *Let T_ψ be a fibered 3–manifold with fiber S, and suppose that the action ψ_* of ψ on $H_1(S, \mathbb{Z})$ is unipotent. Then $\pi_1(T_\psi)$ is residually torsion–free nilpotent.*

Here, we say that ψ_* is **unipotent** if the minimal polynomial of ψ_* is $x - 1$. Finally, we will offer an approach to studying the residual properties of hyperbolic 3–manifold groups which essentially reduces the problem to the case of fibered 3–manifolds. The statement of the result for which we will give a self–contained proof is as follows:

Theorem 1.7. *Suppose that every virtually Haken hyperbolic 3–manifold is virtually fibered. Let M be a finite volume hyperbolic 3–manifold and let p be a prime. Then $\pi_1(M)$ is virtually residually p.*

1.2. **Notes and References.** Residual properties of 3–manifold groups have been independently studied by M. Aschenbrenner and S. Friedl in [3] and [4]. The recent resolution of the virtual Haken conjecture and the virtual fibering conjecture (see [1], [2], [15], [30], [31]) establishes residual torsion–free nilpotence of all finite volume hyperbolic 3–manifold groups. Aside from the establishment of the most general case of Theorem 1.1, our discussion will be independent of those results.

Residual freeness of geometric 3–manifold groups was studies by H. Wilton in [29]. Therein, he posed the question of whether geometric 3–manifold groups are residually torsion–free nilpotent.

It is a classical result of Mal'cev that finitely generated linear groups are virtually residually p at all but at most finitely many primes. Finite volume hyperbolic 3–manifold groups are linear by definition, and A. Lubotzky asked the author whether hyperbolic 3–manifold groups are virtually residually p for every prime. Theorem 1.7 is a partial answer to Lubotzky's question.

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3. Tools for analyzing nilpotent groups

In this section, we will present an account of the basic algebraic tools which we will be using. Let G be a group. We will write

$$G = \gamma_1(G) > \gamma_2(G) > \cdots$$

for the **lower central series** of G. For $i > 1$, we define

$$\gamma_{i+1}(G) = [G, \gamma_i(G)] = [\gamma_1(G), \gamma_i(G)].$$

Observe that a group is residually nilpotent if and only if

$$\bigcap_i \gamma_i(G) = \{1\}.$$

The following fact allows us to make algebraic the following idea: let X be a finite CW complex and let Y_1, Y_2 be two finite p–power, Galois (i.e. normal) covers of X. Then there exists a p–power refinement Z of Y_1 and Y_2, in the sense that Z is a Galois cover of X, Y_1 and Y_2, and Z has finite p–power degree over X.

Proposition 3.1. *Let G be a group, let p be a prime, and let $K_1, K_2 < G$ be finite index, normal subgroups of p–power index. Then the intersection $K_1 \cap K_2$ is normal and has p–power index in G.*

Proof. Since K_1 and K_2 normalize each other, we can form the subgroup $H = K_1K_2$ which sits between K_i and G for $i = 1, 2$. Notice that H is normal in G and has p–power index. The Second Isomorphism Theorem for groups asserts that

$$H/K_1 \cong K_2/(K_1 \cap K_2).$$

The left hand side is obviously a p–group, so that $K_1 \cap K_2$ has p–power index in K_2. It follows that $K_1 \cap K_2$ has p–power index in G. The normality of $K_1 \cap K_2$ in G is immediate. □

Observe that the assumption of normality in the previous proposition is essential. For example, take the standard copy of $A_4 < A_5$. This subgroup has index 5, as do all of its conjugates. Their intersection in A_5 itself is trivial since A_5 is simple. Therefore the index of the intersection is 60, which is not a power of 5.

In general, it is difficult to show that a group lacks a particular residual property. The following is useful for us in establishing certain such facts:

Proposition 3.2. *Let N be a nonabelian finitely generated nilpotent group.*

(1) *Suppose N has a cyclic abelianization. Then N is cyclic.*
(2) *Suppose N is torsion–free. Then $H_1(N, \mathbb{Q})$ has rank at least two.*
(3) *Suppose N is a finite p–group. Then $H_1(N, \mathbb{Z}/p\mathbb{Z})$ has rank at least two.*

Proof. For claim (1), let N_1 be the abelianization of N and let N_2 be the largest two–step nilpotent quotient of N (so that $N_1 \cong \gamma_1(N)/\gamma_2(N)$ and $N_2 = \gamma_1(N)/\gamma_3(N)$). Then N_2 fits into a central extension of the form

$$1 \to \gamma_2(N)/\gamma_3(N) \to N_2 \to N_1 \to 1.$$

The isomorphism type of this extension is classified by its Euler class

$$e \in H^2(N_1, \gamma_2(N)/\gamma_3(N)).$$

Since N_1 is cyclic, it admits a periodic resolution (see [6] for instance). In particular, the cohomology group

$$H^2(N_1, \gamma_2(N)/\gamma_3(N))$$

vanishes so that the extension is trivial and N_2 is abelian. Since N_1 is the abelianization of N, it follows that $N_2 \cong N_1$.

For the other two claims, we use the standard fact that for any finitely generated group the commutator bracket furnishes a surjective map

$$H_1(G, \mathbb{Z}) \otimes H_1(G, \mathbb{Z}) \to \gamma_2(G)/\gamma_3(G)$$

(see [17]). If the torsion–free part of the abelianization of N has rank one then the anti–symmetry of the commutator bracket implies that $\gamma_2(N)/\gamma_3(N)$ is finite. If N is torsion–free and nonabelian, this is impossible. Similarly, if the p–part of the abelianization of N is cyclic then the antisymmetry of the commutator bracket implies that the p–part of $\gamma_2(N)/\gamma_3(N)$ is trivial. If N is a nonabelian p–group, this is again impossible. □

Throughout this article, we will be repeatedly appealing to the fact that free groups and surface groups are residually p for every prime. These facts are well–known, and one of the first proofs for surfaces can be found in the paper [7] of G. Baumslag. We include another proof whose flavor is distinctly geometric.

Proposition 3.3. *Let G be a finitely generated free group or a surface group, and let p be a prime. Then G is residually p.*

Proof. First suppose that G is free. Identify G with the fundamental group of a finite wedge of circles X, which we endow with the graph metric. We let $X = X_0$ and we build a tower of finite covering spaces by taking X_{i+1} to be the cover of X_i corresponding to the natural surjective map

$$\pi_1(X_i) \to H_1(X_i, \mathbb{Z}/p\mathbb{Z}).$$

Note that $X_i \to X_0$ is a normal (in fact characteristic) covering space of p–power degree.

Let Γ be a finite graph. Let γ be a loop in Γ, which we view as a nontrivial path (without backtracking) which has the same initial and terminal vertex. We say that γ is a **simple loop** in Γ if, as an unbased loop, it visits each vertex at most once. We claim that any simple loop in Γ represents a primitive integral homology class. To see this, let e be an edge of γ. Since $\gamma \setminus e = T_0$ is a tree, we can extend T_0 to a maximal tree $T \subset \Gamma$. Collapsing T to a point, we see that the image of e in Γ/T is a free factor of the fundamental group of Γ. In particular, the homotopy class of γ is primitive over $\mathbb{Z}$. It follows that the modulo p homology class of γ is nontrivial for each prime.

Note also that in any finite graph with the graph metric, any minimal length nontrivial loop is always simple. It follows that any loop of length k in X_0 does not lift to X_k. In particular, the length of the shortest loop in X_k tends to infinity as k tends to infinity. In particular, $G = \pi_1(X)$ is residually p.

For closed surface groups there is an additional complication, which is that short loops may be homologically trivial. Let $G = \pi_1(S)$, and we will choose a discrete, cocompact embedding $G \to PSL_2(\mathbb{R})$ so that S is then a quotient of the hyperbolic plane $\mathbb{H}^2$. Each essential free homotopy class of curves in S is now represented by a hyperbolic geodesic. We again let $S = X_0$ and we construct the tower of p–power covering spaces $\{X_i\}$ as for graphs.

Note that with the hyperbolic metric, any shortest closed geodesic is still simple. If $\gamma \subset X_i$ is a shortest length closed geodesic, we record the homology class of γ. If it is nontrivial then γ is nonseparating and hence is nontrivial and primitive in $H_1(X_i, \mathbb{Z}/p\mathbb{Z})$. Otherwise, γ is separating. It is easy to check that each lift of γ to X_{i+1} is nonseparating. It follows that the shortest closed loop on X_i does not lift to a closed loop in X_{i+2}. The length spectrum of geodesics in X_0 is discrete, so that as i tends to infinity, the length of the shortest loop in X_0 which lifts to X_i also tends to infinity. □

Since we are interested in residually p properties of groups and in residual torsion–free nilpotence, it is useful to understand the relationship between these two properties. Let G be a finitely generated group, let $g \in G$, and let $\mathcal{P}$ be a set of primes. We say that $g \in G$ is **$\mathcal{P}$–good** if there is an $N = N(g)$ such that g survives in a p–group quotient of G of nilpotence degree no more than N for each $p \in \mathcal{P}$.

We will record the following proposition for its independent interest – it is not used in the sequel.

Proposition 3.4. *Let G be a finitely generated group. The following are equivalent:*

1. *G is residually torsion–free nilpotent.*
2. *There is an infinite set of primes $\mathcal{P}$ such that each nontrivial $g \in G$ is $\mathcal{P}$–good.*
3. *Each nontrivial $g \in G$ is $\mathcal{P}$–good with respect to the set of all primes.*

Proof. Suppose G is residually torsion–free nilpotent, and let $1 \neq g \in G$. Let N be a torsion–free nilpotent quotient of G where g survives. We may embed N as a group U of integral unipotent matrices in some finite dimensional general linear group, as is proved in [22], though this fact seems to be originally proved by P. Hall in [12]. Let p be any prime. Reducing the entries of U modulo p^n results in a finite p–group U_{p^n} whose nilpotence degree is no larger than that of N. Choosing n sufficiently large, we see that 1 implies 3. We have that 3 implies 2 trivially.

To see that 2 implies 1, let $1 \neq g \in G$. For each $p \in \mathcal{P}$, let P_g be a p–group in which g survives and which has nilpotence degree at most N, where N is the nilpotence degree guaranteed by $\mathcal{P}$–goodness. Write

$$X = \prod_{p \in \mathcal{P}} P_g,$$

and let H be the image of G in X. Observe that H is finitely generated and is a subgroup of X, so that the nilpotence degree of H is at most N. Furthermore, the image of g has infinite order in H, since g projects nontrivially to each of the factors X. Since H is finitely generated and nilpotent, the torsion elements of H form a normal subgroup of H (see [22]), so that there is a torsion–free quotient $\overline{H}$ of H in which g survives. In particular, G is residually torsion–free nilpotent. □

It is well–known (see [17] for instance) that free groups and surface groups are in fact residually torsion–free nilpotent. One can see this geometrically as in the proof of Proposition 3.3, using certain torsion–free homology covers instead modulo p homology covers, and analyzing injectivity radii. Some care must be taken when dealing with infinite covers and in ensuring that the resulting covers are nilpotent and not just solvable.

The final topic we will discuss in this section is that of automorphisms of p–groups. For a group G, write $\varphi(G)$ for the intersection of its maximal proper subgroups. We call $\varphi(G)$ the **Frattini subgroup** of G. By convention if G has no nontrivial proper maximal subgroups, we define $\varphi(G) = \{1\}$. The following result is standard and can be found in [16], for instance:

Theorem 3.5. *Let P be a finite p–group. Then $P/\varphi(P)$ is the largest elementary abelian quotient of P.*

Here, an **elementary abelian group** is one which is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$ for some prime p and some integer n. Thus, if P is a finite p–group then

$$P/\varphi(P) \cong H_1(P, \mathbb{Z}/p\mathbb{Z}).$$

The most important result we will discuss here is this:

Theorem 3.6. *Let P be a finite p–group and let $\Gamma < \operatorname{Aut}(P)$ be a subgroup which acts trivially on $P/\varphi(P)$. Then Γ is also a p–group.*

In the interest of space, we will not give a complete proof of Theorem 3.6 here. The reader may find a complete and detailed discussion in [10]. The reason we are interested in Theorem 3.6 is for the following easy consequence:

Corollary 3.7. *Let P be a p–group and let $\psi \in \operatorname{Aut}(P)$ induce a unipotent automorphism of $H_1(P, \mathbb{Z}/p\mathbb{Z})$. Then ψ has p–power order.*

As in the characteristic zero case, an automorphism of the vector space $H_1(P, \mathbb{Z}/p\mathbb{Z})$ is **unipotent** if its minimal polynomial is $x - 1$.

Proof of Corollary 3.7. Identify the automorphisms of $H_1(P, \mathbb{Z}/p\mathbb{Z})$ with $GL_n(\mathbb{F}_p)$ for an appropriate n. It is well–known that every unipotent subgroup of $GL_n(\mathbb{F}_p)$ is a p–group (see 0.8 of [10] for instance). Thus the automophism of $H_1(P, \mathbb{Z}/p\mathbb{Z})$ induced by ψ has p–power order. It follows that ψ has p–power order as an automorphism of P by Theorem 3.6. □

4. Fibered 3–manifolds

In this section, we will analyze the case of fibered 3–manifolds.

4.1. Fibered 3–manifold groups are virtually residually p.

The fact that fibered 3–manifold groups are virtually residually p for every prime is a consequence of a somewhat more general fact. Let G be any group and let $\Gamma < \operatorname{Aut}(G)$ be any subgroup. We can construct the semidirect product

$$1 \to G \to G_\Gamma \to \Gamma \to 1,$$

where the conjugation action of Γ on G is given by the action of $\operatorname{Aut}(G)$ on G.

Theorem 4.1. *Let G be a finitely generated and residually p, and let $\Gamma < \operatorname{Aut}(G)$ be a subgroup which acts unipotently on $H_1(G, \mathbb{Z}/p\mathbb{Z})$. Then G_Γ is also residually p.*

Proof. Suppose first that G is a finite p–group. By Corollary 3.7, every element of Γ has p–power order, so that Γ is also a p–group. It follows that G_Γ has p–power order.

For the general case, let P be a finite p–power quotient of G by a characteristic subgroup. Such quotients always exist by a fairly straightforward application of Proposition 3.1. We then obtain a map $\Gamma \to \operatorname{Aut}(P)$ whose image Π still acts unipotently on $H_1(P, \mathbb{Z}/p\mathbb{Z})$. Again, we have that P_Π is a p–group. By definition, G is exhausted by p–power index subgroups. It follows that G_Γ is residually p. □

The finite generation of G implies that $H_1(G, \mathbb{Z}/p\mathbb{Z})$ is a finite group. The following statement is an immediate consequence of Theorem 4.1.

Corollary 4.2. *Let G be a finitely generated and residually p, and let $\Gamma < \operatorname{Aut}(G)$ be any subgroup. Then G_Γ is virtually residually p.*

Theorem 1.2 follows immediately, replacing the general subgroup $\Gamma < \operatorname{Aut}(\pi_1(S))$ by the single automorphism of $\pi_1(S)$ given by a lift of the action of the mapping class ψ.

4.2. **Sol geometry.** Any compact 3–manifold which admits Sol geometry is finitely covered by a torus bundle over the circle with hyperbolic monodromy. Thus for virtually residually p considerations, it suffices to consider the case of a torus bundle with hyperbolic monodromy. Recall the statement of Theorem 1.3: let T_A be a hyperbolic torus bundle with monodromy $A \in SL_2(\mathbb{Z})$. Then $\pi_1(T_A)$ is residually p if and only if $p \mid \det(A - I)$, and $\pi_1(T_A)$ is not residually p for any prime if and only if $\pi_1(T_A)$ is not residually nilpotent, if and only if A is $\mathbb{Q}$–conjugate to

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Proof of Theorem 1.3. We first consider the case of the $\mathbb{Q}$–conjugacy class of

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

If Q is any invertible, rational 2×2 matrix, we have that

$$QAQ^{-1} - I = Q(A - I)Q^{-1}.$$

Observe that $A - I$ is an element of $GL_2(\mathbb{Z})$, so that the determinant of $QAQ^{-1} - I$ is a unit in $\mathbb{Z}$. Thus, if $B \in SL_2(\mathbb{Z})$ is in the rational conjugacy class of A then $B - I \in GL_2(\mathbb{Z})$. Let us now compute the lower central series of $G = \pi_1(T_B)$. We will use the presentation

$$G = \langle \mathbb{Z}^2, t \mid t^{-1}\mathbb{Z}^2 t = B(\mathbb{Z}^2)\rangle.$$

Observe that

$$[t, \mathbb{Z}^2] = (B - I)\mathbb{Z}^2 = \mathbb{Z}^2,$$

so that we have

$$G = \gamma_1(G) > \gamma_2(G) = \gamma_3(G) = \cdots = \mathbb{Z}^2.$$

In particular, the lower central series stabilizes at the second term. It follows that G is not residually nilpotent and is therefore not residually p for any prime.

Now let $A \in SL_2(\mathbb{Z})$ be a general hyperbolic element. One easily checks that A has two real eigenvalues $\lambda^{\pm 1}$ with $|\lambda| \neq 1$. It follows that $\det(A - I) \neq 0$. For our analysis, it is convenient for us to pass to

$$PSL_2(\mathbb{Z}) \cong SL_2(\mathbb{Z})/\{\pm I\}.$$

Write n for the trace of A, and $\overline{A}$ for the image of A in $PSL_2(\mathbb{Z})$. We can include $PSL_2(\mathbb{Z}) \subset PSL_2(\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$. From this perspective, we have that the $\mathbb{R}$–conjugacy class of $\overline{A}$ is determined by the absolute value of its trace $|n|$.

If $\overline{A}$ has trace $|n|$ then it is $\mathbb{R}$–conjugate to

$$X = \begin{pmatrix} |n| - 1 & 1 \\ |n| - 2 & 1 \end{pmatrix},$$

where this matrix is well–defined in $SL_2(\mathbb{Z})$ up to $-I$. So, there exists an element $Q \in PSL_2(\mathbb{R})$ such that $Q\overline{A} = XQ$ in $PSL_2(\mathbb{R})$. Finding entries for such a matrix Q is tantamount to solving a system of linear equations with integer entries. By Cramer's rule the solutions are rational, so we may assume Q has rational entries. The system of equations to be solved is underdetermined, so fix a particular element $Q \in PSL_2(\mathbb{Q})$ realizing the conjugacy. We will abuse notation and think of Q as an element of $SL_2(\mathbb{Q})$, as a choice of lift does not affect conjugation. Since X is well–defined in $SL_2(\mathbb{Z})$ up to $-I$, we have that A is conjugate to either

$$X = \begin{pmatrix} |n| - 1 & 1 \\ |n| - 2 & 1 \end{pmatrix}$$

or to

$$-X = \begin{pmatrix} -|n| + 1 & -1 \\ -|n| + 2 & -1 \end{pmatrix}$$

in $SL_2(\mathbb{Q})$. Now let $Y^{\pm} = \pm X - I$. We have that $A - I$ is conjugate to $Y^{\pm}$ in $SL_2(\mathbb{Q})$ by Q.

One verifies that

$$(Y^+)^2 = \begin{pmatrix} (|n|-2)^2 + |n| - 2 & |n| - 2 \\ (|n|-2)^2 & |n| - 2 \end{pmatrix}$$

and that

$$(Y^-)^2 = \begin{pmatrix} (|n|+2)(|n|-1) & (|n|+2)(|n|-2) \\ |n|+2 & |n|+2 \end{pmatrix}$$

In particular, each entry of $(Y^+)^k$ is divisible by $|n| - 2$ and each entry of $(Y^-)^k$ is divisible by $|n| + 2$ whenever $k \geq 2$. It follows that the matrices $Y^{\pm}$ are nilpotent modulo some prime when $|n| \neq 3$ (since we exclude the case $|n| \leq 2$ by hyperbolicity). Observe that only Y^+ can fail to be nilpotent modulo some prime, and from the discussion above we see that this can only happen if A is conjugate to

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

in $SL_2(\mathbb{Q})$.

Suppose $A - I$ is conjugate to Y^+. Notice that for any M, there is a $k = k(M)$ such that

$$(Y^+)^k = (|n| - 2)^M \cdot C,$$

where $C \in PSL_2(\mathbb{Z})$. If p is a prime dividing $|n| - 2$, we have that

$$(Y^+)^k \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \pmod{p^M}.$$

Since Q has fixed, rational entries, it follows that for some possibly larger $K \geq k$, we have that

$$(A - I)^K \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \pmod{p^M}.$$

The same argument applies when $A - I$ is conjugate to Y^-, replacing p by a prime dividing $|n| + 2$. It follows that A acts unipotently on $\mathbb{Z}^2$ modulo these primes and consequently that $\pi_1(T_A)$ is residually p by Theorem 4.1.

If q is a prime which does not divide $|n| - 2$, then the determinant of Y^+ is a unit modulo q. Similarly, if q does not divide $|n| + 2$ then the determinant of Y^- is a unit modulo q. It follows that when we abelianize $\pi_1(T_A)$, we obtain

$$H_1(T_A, \mathbb{Z}) \cong \mathbb{Z} \oplus F,$$

where q does not divide the order of F. In particular,

$$H_1(T_A, \mathbb{Z}/q\mathbb{Z}) \cong \mathbb{Z}/q\mathbb{Z}.$$

By Proposition 3.2, one sees that $\pi_1(T_A)$ cannot be residually q.

Finally, observe that if $A \in SL_2(\mathbb{Z})$ then $\det(A - I) = -\operatorname{tr}(A) + 2$. If $n = \operatorname{tr}(A)$ is positive, then $\det(A - I) = -(n - 2) = -(|n| - 2)$. If n is negative then $\det(A - I) = |n| + 2$. It follows that $\pi_1(T_A)$ is residually p exactly when p divides $\det(A - I)$. □

We remark that Aschenbrenner and Friedl obtained a characterization of residually p fundamental groups of Sol manifolds in [4]. One can check that their result is equivalent to ours.

4.3. **Unipotent homology actions and residual torsion–free nilpotence.** In this subsection, we would like to prove Theorem 1.6, namely that if $\psi \in \mathrm{Mod}(S)$ acts unipotently on $H_1(S,\mathbb{Z})$ then $\pi_1(T_\psi)$ is residually torsion–free nilpotent. Much like Theorem 1.2, Theorem 1.6 will follow from a more general fact:

Theorem 4.3. *Let G be a finitely generated, residually torsion–free nilpotent group and let $\Gamma < \mathrm{Aut}(G)$ be a subgroup acting unipotently on $H_1(G,\mathbb{Z})$. Then G_Γ is residually torsion–free nilpotent.*

Proof. Let $\{\gamma_i(G)\}$ denote the lower central series of G as usual. We will assume that the successive quotients $\gamma_i(G)/\gamma_{i+1}(G)$ are all torsion–free. In general this is not true, but one can define an appropriate "torsion–free lower central series" which is a filtration of G by characteristic subgroups that are commensurable with the lower central series in an appropriate sense, and for which successive quotients are all torsion–free. See [14] for more details.

It is a standard fact that the commutator furnishes a surjective map

$$H_1(G,\mathbb{Z}) \otimes \gamma_i(G)/\gamma_{i+1}(G) \to \gamma_{i+1}(G)/\gamma_{i+2}(G).$$

By induction, it follows that if Γ acts unipotently on $H_1(G,\mathbb{Z})$ then it acts unipotently on $\gamma_i(G)/\gamma_{i+1}(G)$ for all i.

Since $\gamma_i(G)$ is characteristic in G, we may consider the action of Γ on $G/\gamma_i(G)$, thus getting a map

$$\Gamma \to \Gamma_i < \mathrm{Aut}(G/\gamma_i(G)).$$

Let $N_i = (G/\gamma_i(G))_{\Gamma_i}$ be the associated semidirect product. Evidently, the kernels of the maps

$$G_\Gamma \to N_i$$

exhaust all of G_Γ. By the assumption that $\gamma_i(G)/\gamma_{i+1}(G)$ is torsion–free for each i, we have that each N_i is torsion–free. It suffices to show that N_i is nilpotent for each i.

First of all, Γ_2 is nilpotent since it is a unipotent subgroup of a finite dimensional general linear group. Observe that we obtain a conjugation action of N_i on $\gamma_i(G)/\gamma_{i+1}(G)$. This action is unipotent, since $G/\gamma_i(G)$ acts trivially on $\gamma_{i-1}(G)/\gamma_i(G)$ and Γ_i acts unipotently on $\gamma_{i-1}(G)/\gamma_i(G)$. It follows that there is a primitive fixed vector in $\gamma_{i-1}(G)/\gamma_i(G)$ under the action of N_i, say v. It follows that the center of N_i is nontrivial. We see that N_i still acts unipotently on

$$(\gamma_{i-1}(G)/\gamma_i(G))/\langle v\rangle,$$

so that $N_i/\langle v\rangle$ still has a nontrivial, infinite order center. By induction, it follows easily that N_i admits a filtration of the form

$$1 < Z_1 < Z_2 < \cdots Z_n = N_i,$$

where for each $j \leq n$ we have Z_j/Z_{j-1} is central in Z_n/Z_{j-1}. Thus, N_i is nilpotent. □

4.4. **Two examples.** One is naturally led to wonder whether there are examples of fibered 3–manifolds which are not virtually residually nilpotent. Hyperbolic torus bundles provide such examples, since the fundamental group of any such bundle is residually p for at most finitely many primes. Proposition 3.4 shows that residually torsion–free nilpotent groups are residually p at every prime. One can provide another proof:

Proposition 4.4. *The fundamental group of a hyperbolic torus bundle T_A is not virtually residually torsion–free nilpotent.*

Proof. One can easily show that if $T' \to T_A$ is a finite cover then $H_1(T',\mathbb{Z}) \cong \mathbb{Z} \oplus F$, where F is a finite abelian group. So, the abelianization of any torsion–free nilpotent quotient of $\pi_1(T')$ has rank at most one. By Proposition 3.2, we have that any torsion–free nilpotent quotient of $\pi_1(T')$ is cyclic. Thus, $\pi_1(T')$ is not residually torsion–free nilpotent. □

The following example is also worth noting:

Proposition 4.5. *Let K be any knot. Then $\pi_1(S^3 \setminus K)$ is not residually p for any prime. In particular, $\pi_1(S^3 \setminus K)$ is not residually torsion–free nilpotent.*

Proof. This also follows from Proposition 3.2, since $H_1(S^3 \setminus K, \mathbb{Z}) \cong \mathbb{Z}$. □

5. Hyperbolic manifolds and actions on trees

In this section, we will reduce our study of general finite volume hyperbolic 3–manifolds to that of fibered 3–manifolds. Let M be a finite volume hyperbolic 3–manifold and let Γ be its fundamental group. By a result due to Thurston, the representation

$$\Gamma \to PSL_2(\mathbb{C})$$

specifying the hyperbolic structure on M lifts to $SL_2(\mathbb{C})$ (cf. [8]). Let $\mathcal{R} = \mathcal{R}(\Gamma)$ denote the $SL_2(\mathbb{C})$ representation variety of Γ. It is a standard fact from algebraic geometry that $\mathcal{R}$ contains a point over $\overline{\mathbb{Q}}$ and in fact a faithful representation $\Gamma \to SL_2(\overline{\mathbb{Q}})$. Since Γ is finitely generated, there is a finite extension $K/\mathbb{Q}$ such that the image of Γ lands in $SL_2(K)$. We let $\mathcal{O}$ denote the ring of integers in K. Any entry of an element in the image of Γ can be written as a/b, where $a, b \in \mathcal{O}$ and $b \neq 0$ is either equal to 1 or is not a unit. Fixing a finite generating set for Γ, there are only finitely many denominators occurring in the image of the generating set in $SL_2(K)$. These denominators will be contained in finitely many maximal ideals in $\mathcal{O}$. Each maximal ideal of $\mathcal{O}$ lies over a unique nonzero prime ideal $p\mathbb{Z}$ of $\mathbb{Z}$.

For each prime ideal $P \subset \mathcal{O}$, we can complete $\mathcal{O}$ at P to get a DVR. We will denote this completion by $\widehat{\mathcal{O}}_P$. A nonzero prime ideal $p\mathbb{Z} \subset \mathbb{Z}$ is called **bad prime ideal** for the representation

$$\Gamma \to SL_2(K)$$

if there is no prime ideal $P \subset \mathcal{O}$ lying over $p\mathbb{Z}$ such that the image of Γ is conjugate into a subgroup of $SL_2(\widehat{\mathcal{O}}_P)$ upon completing. Write $\mathcal{B}$ for the set of primes $\{p \in \mathbb{Z}\}$ such that $p\mathbb{Z}$ is a bad prime ideal. We call $\mathcal{B}$ the set of **bad primes**.

The following observation of Mal'cev was pointed out to the author by the referee:

Lemma 5.1. *Let R be a Noetherian local ring and let $m \subset R$ be a maximal ideal such that R/m is finite of characteristic p. Then for all n, the group $GL_n(R)$ is virtually residually p.*

Proof. For each $i \geq 1$, write G_i for the kernel of the natural homomorphism

$$GL_n(R) \to GL_n(R/m^i).$$

The ring R/m^i is finite and has p–power order, so that G_i has finite index in G for each i. Furthermore,

$$\bigcap_i G_i = \{1\}$$

since

$$\bigcap_i m^i = \{0\}$$

by the Krull Intersection Theorem.

We claim that for $i \geq 1$, the quotients G_i/G_{i+1} are all p–groups. Write $A \in G_i$ as $I + B$, where B has all entries in m^i. Then $A^p - I = (I + B)^p - I$ is a sum of p matrices whose entries are all in m^{i+1}, by the Binomial Theorem. It follows that $A^p - I$ is a matrix whose entries are all in m^{i+1}, so that $A^p \in G_{i+1}$. □

We immediately obtain the following:

Lemma 5.2. *Suppose that Γ is the fundamental group of a finite volume hyperbolic 3–manifold, and that its set of bad primes $\mathcal{B}$ is empty. Then for each prime p, we have that Γ is virtually residually p.*

Now suppose that $\mathcal{B} \neq \emptyset$. Standard theory of group actions on trees implies that Γ acts on a tree with no global fixed point and without inversions. We recall the basics of this theory for the convenience of the reader, following [23]. Fix $p \in \mathcal{B}$ and a prime ideal $P \subset \mathcal{O}$ lying over $p\mathbb{Z}$, and let $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_P$. We let $\widehat{K}$ be the fraction field of $\widehat{\mathcal{O}}$. We have a canonical map $\mathcal{O} \to \widehat{\mathcal{O}}$ which is injective since

$$\bigcap_n P^n = 0,$$

since $\mathcal{O}$ is a Dedekind domain. Thus we obtain an injective map $K \to \widehat{K}$, and a faithful representation $\Gamma \to SL_2(\widehat{K})$ induced by the inclusion $SL_2(K) \to SL_2(\widehat{K})$.

By construction, the field $\widehat{K}$ comes equipped with a discrete valuation ν. Explicitly, it takes an equivalence class of fractions $\gamma = \alpha/\beta$, determines an i and j such that $\alpha \in P_i \setminus P^{i+1}$ and $\beta \in P^j \setminus P^{j+1}$ and sets $\nu(\gamma) = i - j$. The valuation ν thus defined is a discrete valuation, so that $\widehat{\mathcal{O}}$ is a DVR. We write $\widehat{P}$ for the unique maximal ideal in $\widehat{\mathcal{O}}$.

Let V be a two–dimensional vector space over $\widehat{K}$. Recall that at $\widehat{\mathcal{O}}$–lattice in V is a rank two $\widehat{\mathcal{O}}$–module which spans V as a $\widehat{K}$–vector space. Let L be a $\widehat{\mathcal{O}}$–lattice and L' a sublattice. Then L/L' is isomorphic to $\widehat{\mathcal{O}}/\widehat{P}^a \oplus \widehat{\mathcal{O}}/\widehat{P}^b$ for some nonnegative integers a and b. There is a natural action of $\widehat{K}$ on the set of $\widehat{\mathcal{O}}$–lattices in V, given by scalar multiplication.

Note that if L and L' are arbitrary lattices, we can find a $k \in \widehat{K}$ such that $kL' \subset L$. We will say that two lattices are equivalent if they are in the same $\widehat{K}$–multiplication orbit. There is a natural graph whose vertices are equivalence classes of lattices, and whose edges span pairs of equivalence classes for which there exist representatives satisfying $L/L' \cong \widehat{\mathcal{O}}/\widehat{P}$. It is shown in [23] that this graph is a tree, called the **lattice tree** of $\widehat{\mathcal{O}}$. We have that $SL_2(\widehat{K})$ acts on this tree via its action on $\widehat{K}^2$. The stabilizers of vertices are the conjugates of $SL_2(\widehat{\mathcal{O}})$ in $SL_2(\widehat{K})$.

The following is immediate:

Lemma 5.3. *Let Γ be as above and let $P \subset \mathcal{O}$ lie over a bad prime ideal $p\mathbb{Z}$. Then Γ acts on the lattice tree of $\widehat{O}$ without a global fixed point.*

The action in Proposition 5.3 gives rise to a nontrivial splitting of Γ. The final ingredient we need is the following, which is due to Epstein, Stallings and Waldhausen, and a proof can be found in [8]:

Lemma 5.4. *Let M be a compact, orientable 3–manifold. For any nontrivial splitting of $\pi_1(M)$ there exists a nonempty system S of incompressible, non–peripheral surfaces such that the image of the inclusion on fundamental groups is contained in an edge group. Furthermore, the image of the fundamental groups of the components of $M \setminus S$ are contained in a vertex group.*

Combining Lemmas 5.2, 5.3 and 5.4, we obtain the following:

Theorem 5.5. *Let $\Gamma = \pi_1(M)$ of a finite volume hyperbolic 3–manifold which is not Haken. Then for each prime p, the group Γ is virtually residually p.*

We obtain Theorem 1.7 by combining the previous result with Theorem 1.2. We briefly remark that a finitely generated linear group is virtually residually p for all but finitely many primes. One can fairly easily produce examples of finitely generated linear groups which are not virtually residually p for some particular prime, and in fact for any finite collection of primes (see [28]). As noted in the introduction, Lubotzky asked the author whether hyperbolic 3–manifold groups are virtually residually p for every prime, and his question was part of the motivation for this article. The recent resolution of the virtual Haken conjecture in [2] implies among other things that hyperbolic 3–manifold groups are linear over $\mathbb{Z}$ and are therefore virtually residually p for every prime.

6. Circle bundles over surfaces

In this section, we consider 3–manifolds which are finitely covered by circle bundles over orientable surfaces. Such a manifold M fits into a fibration sequence of the form

$$S^1 \to M \to S,$$

where S is an orientable surface. Passing to a finite cover of M if necessary, we have that $\pi_1(M)$ fits into a (not necessarily split) short exact sequence of the form

$$1 \to \mathbb{Z} \to \pi_1(M) \to \pi_1(S) \to 1,$$

where the left copy of $\mathbb{Z}$ is central in $\pi_1(M)$. As is standard from cohomology of groups (see [6], for instance), the isomorphism type of $\pi_1(M)$ is determined by the Euler class of the extension, which is an element $e \in H^2(S,\mathbb{Z})$. When the copy of $\mathbb{Z}$ is central in $\pi_1(M)$, we will say that the bundle has **trivial monodromy**.

Theorem 6.1. *Let M be a circle bundle over an orientable surface S with trivial monodromy. Then there is an injective homomorphism*

$$\pi_1(M) \to \pi_1(S) \times H,$$

where H is a finitely generated, torsion–free nilpotent group. In particular, any circle bundle over a surface has a virtually residually torsion–free nilpotent fundamental group.

Proof. Let $e \in H^2(S,\mathbb{Z})$ be the Euler class of the extension. If $e = 0$ then the extension is split, so that $\pi_1(M) \cong \pi_1(S) \times \mathbb{Z}$. Setting $H = \mathbb{Z}$, we obtain the desired result.

Now suppose that $e \neq 0$. Write p_1 for the map $\pi_1(M) \to \pi_1(S)$ induced by the fibration. We may assume that S is closed of genus g, for otherwise $H^2(S,\mathbb{Z}) = 0$. There is a natural map $\pi_1(S) \to \mathbb{Z}^{2g}$ given by the abelianization map. We obtain a map

$$H^2(\mathbb{Z}^{2g},\mathbb{Z}) \to H^2(S,\mathbb{Z})$$

induced by the abelianization. It is a standard fact that this map is surjective (see Griffiths and Harris' book [11], for instance). Thus we can choose a non–split central extension

$$1 \to \mathbb{Z} \to H \to \mathbb{Z}^{2g} \to 1$$

whose Euler class is in the preimage of e under the map

$$H^2(\mathbb{Z}^{2g},\mathbb{Z}) \to H^2(S,\mathbb{Z}).$$

It follows that H is a torsion–free nilpotent quotient of $\pi_1(M)$ and that the central copy of $\mathbb{Z} < \pi_1(M)$ maps injectively into H. Write p_2 for the surjection from $\pi_1(M)$ to H.

Thus, we see that each element of $\pi_1(M)$ can be written as $g = x \cdot z$, where x is a preimage of an element of $\pi_1(S)$ in $\pi_1(M)$ under p_1, and where z is an element of the central copy of $\mathbb{Z}$. Define a map

$$\iota : \pi_1(M) \to \pi_1(S) \times H$$

by $p_1 \times p_2$. If g maps to a nontrivial element of $\pi_1(S)$ under p_1 then $\iota(g) \neq 1$. If $1 \neq g$ maps to the identity in $\pi_1(S)$ then g is central in $\pi_1(M)$ and hence maps to a nontrivial element of H under p_2. It follows that ι is injective. □

Recall that Corollary 1.5 asserts that if M is modeled on $\widetilde{PSL_2(\mathbb{R})}$ geometry then $\pi_1(M)$ is linear over $\mathbb{Z}$. This is now more or less immediate by combining the proof of the previous result together with $\mathbb{Z}$–linearity of surface groups (see [18], for instance) and the fact that finite induction of $\mathbb{Z}$–linear representations gives $\mathbb{Z}$–linear representations.

7. Establishing Theorem 1.1

In this section, we will show how to deduce Theorem 1.1 from the other results proved in this paper combined with some other tools.

Proof of Theorem 1.1. The establishment of both residual properties for the first four geometries is trivial, since in those cases the fundamental groups are all either finite, virtually torsion–free abelian, or virtually torsion–free nilpotent.

Sol geometry: this is the content of Theorem 1.3 and the example immediately after its proof.

$\mathbb{H}^2 \times \mathbb{R}$ geometry and $\widetilde{PSL_2}(\mathbb{R})$ geometry: this is the content of Theorem 1.4.

$\mathbb{H}^3$ geometry: let M be a finite volume hyperbolic 3–manifold, and let p be a prime. The resolution of the virtual fibering conjecture shows that every virtually Haken hyperbolic 3–manifold is virtually fibered. By Theorem 1.7, we see that $\pi_1(M)$ is virtually residually p. Virtual residual torsion–free nilpotence of $\pi_1(M)$ follows from [2]. Indeed, the authors prove that $\pi_1(M)$ virtually embeds in a residually torsion–free nilpotent group (in particular a right-angled Artin group). □

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