RESIDUALLY FINITE RATIONALLY p GROUPS

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Abstract. In this article we develop the theory of residually finite rationally p (RFRp) groups, where p is a prime. We first prove a series of results about the structure of finitely generated RFRp groups (either for a single prime p, or for infinitely many primes), including torsion-freeness, a Tits alternative, and a restriction on the BNS invariant. Furthermore, we show that many groups which occur naturally in group theory, algebraic geometry, and in 3-manifold topology enjoy this residual property. We then prove a combination theorem for RFRp groups, which we use to study the boundary manifolds of algebraic curves $\mathbb{CP}^2$ and in $\mathbb{C}^2$. We show that boundary manifolds of a large class of curves in $\mathbb{C}^2$ (which includes all line arrangements) have RFRp fundamental groups, whereas boundary manifolds of curves in $\mathbb{CP}^2$ may fail to do so.

Contents

1. Introduction

In this paper, we develop a group theoretic property called *residually finite rationally p*. We study the class of finitely generated groups with this property for its

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own sake, and we study this property among several classes of groups which occur in algebraic geometry and in 3-manifold topology. Most notably, we show that this property is enjoyed by the boundary manifold of a curve arrangement with only type A singularities in $\mathbb{C}^2$, but that the analogous property for boundary manifolds of curve arrangements in $\mathbb{CP}^2$ does not necessarily hold.

1.1. **The class of RFRp groups.** Let p be a prime. A finitely generated group G is called *residually finite rationally* p of *RFRp* if there exists a sequence of nested, finite index subgroups $\{G_i\}_{i\geqslant 1}$ of G such that:

(1) $G = G_1$.
(2) The intersection of the groups $\{G_i\}$ is trivial.
(3) Each quotient G_i/G_{i+1} is an elementary abelian p-group.
(4) Every element $g \in G_i \setminus G_{i+1}$ represents a nonzero class in $H_1(G_i, \mathbb{Q})$.

Let us define the *RFRp topology* on G by choosing as a neighborhood basis for the identity the standard RFRp filtration of G (cf. Lemma 2.1). The group G is RFRp precisely when this topology is Hausdorff, or equivalently, the trivial group is a closed subgroup.

Many finitely generated groups which occur naturally in geometric group theory and in topology are RFRp. For instance, we have the following result, which is a combination of Propositions 4.1, 4.2, and 6.3.

Proposition 1.1. *The following groups are RFRp, for all primes p:*

(1) Finitely generated free groups.
(2) Closed, orientable surface groups.
(3) Right-angled Artin groups.

1.2. **Properties of RFRp groups.** The groups we study here enjoy many useful properties. We present some of these properties in the following theorem, which summarizes Proposition 3.1 and Theorems 3.3 and 3.4, as well as Corollary 5.3 and Theorems 5.6 and 5.8.

Theorem 1.2. *Let G be a finitely generated group which is RFRp for some prime p. Then:*

(1) G is residually finite. In particular, if G is finitely presented then G has a solvable word problem.
(2) G is torsion-free.
(3) G is residually torsion-free polycyclic.
(4) For each n, the maximal abelian subgroups of G of rank n are separable.

If, moreover, the group G is RFRp for infinitely many primes, finitely presented, and nonabelian, then:

(5) G is large, i.e., G virtually surjects to a nonabelian free group.

(6) The maximal k-step solvable quotients G/G^k are not finitely presented, for any $k \geqslant 2$.
(7) The derived subgroup G' is not finitely generated.
(8) The complement of the BNS invariant $\Sigma^1(G)$ is not empty.
(9) The group G is bi-orderable.

1.3. **A combination theorem.** Our main result about RFRp groups is a combination theorem which allows us to construct many new RFRp groups from old ones:

Theorem 1.3 (Theorem 6.1)**.** *Fix a prime p. Let $G = G_\Gamma$ be a finite graph of finitely generated groups with vertex groups $\{G_v\}_{v\in V(\Gamma)}$ and groups $\{G_e\}_{e\in E(\Gamma)}$ satisfying the following conditions:*

(1) For each $v \in V(\Gamma)$, the group G_v is RFRp.
(2) For each $v \in V(\Gamma)$, the RFRp topology on G induces the RFRp topology on G_v.
(3) For each $e \in E(\Gamma)$ and each $v \in e$, we have that the image of G_e in G_v given by the graph of groups structure of G is closed in the RFRp topology on G_v.

Then G is RFRp.

The reader is directed to Section 6 for the relevant technical definitions.

1.4. 3**-manifold topology.** The RFRp property also produces a new invariant of 3-manifold groups which is finer than previously studied residual properties enjoyed by 3-manifolds:

Theorem 1.4. *Let $G = \pi_1(M)$ be a geometric 3-manifold group, possibly with toroidal boundary. Then there is a finite index subgroup $G_0 < G$ which is RFRp for every prime p if and only if M admits one of the following geometries: $\{S^3, S^2 \times \mathbb{R}, \mathbb{R}^3, \mathbb{H}^2 \times \mathbb{R}, \mathbb{H}^3\}$. Otherwise, no finite index subgroup of G is RFRp for any prime.*

We remark that Theorem 1.4 relies on Agol's resolution of the virtual Haken conjecture [3] and on Dani Wise's work [43, 6, 44]. We further remark one subtlety concerning Theorem 1.4 to the reader. Namely, a circle bundle over a surface with nonempty boundary can admit a geometric structure modeled on both $\mathbb{H}^2 \times \mathbb{R}$ and on $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$ at the same time. Circle bundles over surfaces with nonempty boundary are always considered to be in the purview of Theorem 1.4.

Motivated by the topological study of plane algebraic curves (see Subsection 1.5 below) we isolate a class $\mathscr{X}$ of compact, 3-dimensional graph manifolds whose fundamental groups are RFRp. Namely, a graph manifold M lies in the class $\mathscr{X}$ if the following conditions are satisfied:

($\mathscr{X}_1$) The underlying graph Γ is finite, connected, and bipartite with colors $\mathscr{P}$ and $\mathscr{L}$, and each vertex in $\mathscr{P}$ has degree at least two.

($\mathscr{X}_2$) Each vertex manifold M_v is homeomorphic to a trivial circle bundle over an orientable surface with boundary.
($\mathscr{X}_3$) If M_v is colored by $\mathscr{L}$ then at least one boundary component of M_v is a boundary component of M, and the Euler number of M_v is zero.
($\mathscr{X}_4$) If M_v is colored by $\mathscr{P}$ then no boundary component of M_v is a boundary component of M, and the Euler number of M_v is nonzero;
($\mathscr{X}_5$) The gluing maps are given by flips.

We refer the reader to Section 7 for more details and precise definitions of all technical terms involved in this definition. Using Theorem 1.3, we prove the following result.

Theorem 1.5. *Let M be a compact graph manifold satisfying the above conditions. Then for each prime p, the group $\pi_1(M)$ is RFRp.*

In the above theorem, the assumption that the gluing maps of edge spaces be flips is not an assumption by itself. By a recent result of Doig and Horn in [15], any gluing map in a graph manifold can be made a flip map, at the expense of adding exceptional fibers to the vertex spaces. So, the combination of assumptions ($\mathscr{X}_2$) and ($\mathscr{X}_5$) in Theorem 1.5 do actually make for a nontrivial hypothesis. Graph manifolds satisfying just these two assumptions and some mild condition on the graph Γ were shown by Schroeder [38] to possess metrics of non-positive curvature.

Recently, many authors have studied graph manifolds which are *virtually special*, see for instance [36, 19, 29]. One of the algebraic consequences of a graph manifold being virtually special is that its fundamental group is virtually RFRp for each prime p, cf. [4, 5, 27]. It is important to note that, although the graph manifolds covered by Theorem 1.5 are generally virtually special, the conclusion of the theorem is not a virtual statement. In particular, the theorem does not follow formally from known results, since RFRp is a more refined property than virtual specialness. The reader is directed to Subsection 4.4 below.

Applying Agol's and Wise's results [1, 3, 43, 44], one obtains the following general fact quite easily:

Corollary 1.6. *Let M be a compact aspherical 3-manifold with $\chi(M) = 0$. Then there exists a finite cover $M' \to M$ such that for each prime p, the group $\pi_1(M')$ is RFRp.*

Note that Corollary 1.6 is a virtual statement and hence does not imply Theorem 1.5 formally. To see why Corollary 1.6 holds, it suffices to note that M as in the hypothesis of the corollary is proved by Agol to be virtually special, so that some finite index subgroup of $\pi_1(M)$ is RFRp for each prime by Proposition 1.1, part 3.

1.5. **Plane algebraic curves.** The naturality of the manifolds in the purview of Theorem 1.5 comes from the fact that they are an axiomatized version of boundary manifolds of curve arrangements in $\mathbb{C}^2$. More precisely, let $\mathscr{C}$ be a (reduced) algebraic curve in the complex affine plane. The *boundary manifold* of this curve, $M_{\mathscr{C}}$, is obtained by intersecting the boundary of a regular neighborhood of $\mathscr{C}$ with a 4-ball of sufficiently large radius, so that all singularities of $\mathscr{C}$ are contained in this ball. Clearly, $M_{\mathscr{C}}$ is a compact, connected, oriented 3-manifold. Moreover, if each irreducible component of the curve $\mathscr{C}$ is transverse to the line at infinity in $\mathbb{C}^2$, then the boundary components of $M_{\mathscr{C}}$ are tori.

In Theorem 8.6 we show that, except for a few easy-to-handle cases, all boundary manifolds arising in the above fashion belongs to the class $\mathscr{X}$ of graph-manifolds. As a consequence, we deduce from Theorem 1.5 the following result.

Theorem 1.7. *Let $\mathscr{C}$ be an algebraic curve in $\mathbb{C}^2$. Suppose each irreducible component of $\mathscr{C}$ is smooth and transverse to the line at infinity, and all singularities of $\mathscr{C}$ are of type* A. *Then $\pi_1(M_{\mathscr{C}})$ is RFRp, for all primes p.*

The following particular case is worth singling out.

Corollary 1.8. *If $\mathscr{A}$ is an arrangement of lines in $\mathbb{C}^2$, then the fundamental group of the boundary manifold of $\mathscr{A}$ is RFRp, for all primes p.*

We also show in Section 8 that Theorem 1.7 does not generalize to the compact case, namely, that the boundary manifold of an algebraic curve in $\mathbb{CP}^2$ (even one that satisfies the aforementioned conditions), does not always have an RFRp fundamental group.

The motivation behind studying the RFRp property for boundary manifolds of arrangements comes from the following fundamental unsolved problems:

Problem 1.9. *Let $G = \pi_1(\mathbb{C}^2 \backslash V(\mathscr{A}))$, where here $\mathscr{A}$ is a line arrangement and where $V(\mathscr{A})$ denotes the variety consisting of the union of the lines in $\mathscr{A}$.*

(1) Is G residually finite?
(2) Is G torsion-free?

The algebraic and topological structure of the complement $\mathbb{C}^2 \backslash V(\mathscr{A})$ is closely related to that of the boundary manifold $M_{\mathscr{A}}$ (see [22, 11], for instance), and Theorem 1.7 provides the first step in an approach to resolving Problem 1.9.

As for a self-contained approach to boundary manifolds of arrangements, this paper achieves (among many other things) two goals. The first is that it provides an axiomatic setup for studying boundary manifolds. The second is a subtlety of virtual versus non-virtual residual properties: the RFRp property of fundamental groups of boundary manifolds at every prime p holds on the nose, not just after passing to a finite index subgroup.

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2. Residually finite rationally p groups

In this section, we give a (very nearly) self-contained account of residually finite rationally p groups.

2.1. The RFRp filtration.

Let $G = G_1$ be a finitely generated group and let p be a prime. We say that G is *residually finite rationally p* or RFRp if there exists a sequence of subgroups $\{G_i\}_{i\geqslant 1}$ of G such that:

(1) For each i, the group G_{i+1} is a normal subgroup of G_i.
(2) We have
$$\bigcap_{i\geqslant 1} G_i = \{1\}.$$
(3) For each i, the group G_i/G_{i+1} is an elementary abelian p-group.
(4) For each i, we have that
$$\ker\{G_i \to H_1(G_i,\mathbb{Q})\} < G_{i+1}.$$

The reader may compare the RFRp condition with the RFRS condition developed by Agol in [1]. Agol requires each subgroup G_i to be normal in G and drops the requirement that G_i/G_{i+1} be a p-group.

For a general finitely generated, abelian group K, let $\operatorname{Tors}(K)$ denote the torsion subgroup of K, and let
$$\operatorname{TFr}(K) = K/\operatorname{Tors}(K). \tag{1}$$
be the maximal torsion-free quotient of K.

Lemma 2.1. *Let G be RFRp as above with a sequence $\{G_i\}$ of subgroups witnessing the statement that G is RFRp. Then there exists a sequence of subgroups $\{K_i\}$ of G which witness the fact that G is RFRp, and such that K_i is normal in G_1 for each i.*

Proof. We set $K_1 = G$, and we define
$$K_{i+1} = \ker\big\{K_i \to \operatorname{TFr} H_1(K_i,\mathbb{Z}) \to (\operatorname{TFr} H_1(K_i,\mathbb{Z}))\otimes \mathbb{Z}/p\mathbb{Z}\big\}. \tag{2}$$

By construction, each subgroup K_i is characteristic in G, thereby verifying condition (1). It is also clear that the sequence $\{K_i\}_{i\geqslant 1}$ satisfies conditions (3) and (4). To see that condition (2) holds, note that K_i/K_{i+1} is the largest elementary abelian quotient of K_i satisfying (4). It follows immediately that $K_i \leqslant G_i$, so that
$$\bigcap_i K_i \subset \bigcap_i G_i = \{1\}, \tag{3}$$
whence the conclusion. ∎

It follows from Lemma 2.1 that the RFRp condition is strictly stronger than Agol's RFRS condition. The reader may note that in [1], Agol shows that right-angled Artin groups are RFRp for $p = 2$, which we will show in Theorem 4.3 implies that all subgroups of right-angled Artin groups are RFR2.

The nested sequence of subgroups $\{K_i\}$ furnished by Lemma 2.1 will be called the *standard RFRp sequence* or the *standard RFRp filtration*. Passing between groups and spaces, if X is a connected CW-complex with $\pi_1(X) = G$, and if $\{X_i\}$ is a tower of covers such that $\pi_1(X_i) = K_i$, we call $\{X_i\}$ the *standard RFRp tower* of X. We will often call the quotients G/K_i the *RFRp quotients of* G, which is not to be confused with those quotients of G which are RFRp.

2.2. **The RFRp topology.** Let p be a fixed prime. If G is a finitely generated group, we take the natural definition for the *RFRp topology* on G. A neighborhood basis for the identity is given by the standard RFRp filtration of G, and a basis for the topology in general is given by the cosets of these subgroups. The group G is RFRp if and only if this topology is Hausdorff.

Let $H < G$ be a subgroup, let $\{G_i\}$ be the standard RFRp filtration on G, and let $\phi_i \colon G \to G/G_i$ be the canonical projection. Note that H is *closed* in the RFRp topology if and only if for each $g \in G \backslash H$, there is an i such that $\phi_i(g) \notin \phi_i(H)$.

If G is a finitely generated group and $H < G$ is a finitely generated subgroup, then the *RFRp topology on H induced by G by restriction* is the topology on H whose neighborhood basis for the identity is given by the subgroups $\{G_i \cap H\}_{i \geqslant 1}$, where $\{G_i\}_{i \geqslant 1}$ is the standard RFRp filtration of G. If $\{H_j\}_{j \geqslant 1}$ is the standard RFRp filtration on H, we say that G induces the RFRp topology on H if for each j there exists an i such that $H_j > G_i \cap H$.

Let p be a fixed prime, let G be a finitely generated group, and let $\{G_i\}$ be the standard RFRp filtration on G. We denote the *RFRp radical* of G by

$$\mathrm{rad}_p(G) = \bigcap_i G_i. \tag{4}$$

We have that G is RFRp if and only if $\mathrm{rad}_p(G)$ is trivial. Notice that if $i \leqslant j$ then $\mathrm{rad}_p(G_i) = \mathrm{rad}_p(G_j)$, by the definition of the standard RFRp filtration on G. The following fact is relatively straightforward, but is nevertheless useful:

Proposition 2.2. *Let G be a finitely generated group, and let $\phi \colon G \to H$ be a surjective homomorphism, where H is RFRp. If $g \notin \ker \phi$, then $g \notin \mathrm{rad}_p(G)$.*

Proof. Write $\{G_i\}$ denote the standard RFRp filtration on G. Let $1 \neq h = \phi(g)$, and let $\{H_i\}$ be the standard RFRp filtration of H. Then $h \in H_i \backslash H_{i+1}$ for some i. Pulling back the subgroups $\{H_i\}$ to a collection of subgroups $\{K_i\}$ of G, we have that $g \in K_i \backslash K_{i+1}$. Moreover, we have that K_i/K_{i+1} is an elementary abelian p-group, and the quotient map $K_i \to K_i/K_{i+1}$ factors through the torsion-free abelianization

$K_i \to \mathrm{TFr}\, H_1(K_i, \mathbb{Z})$. It follows that for each i, there exists a j such that $G_j < K_i$, by the same argument as in Lemma 2.1. It follows that

$$\mathrm{rad}_p(G) < \bigcap_i K_i, \tag{5}$$

so that $g \notin \mathrm{rad}_p(G)$. □

The following fact will be useful in the sequel:

Corollary 2.3. *Let G be a finitely generated group, and let $r\colon G \to H$ be a retraction to a subgroup $H < G$. Then the RFRp topology on G induces the RFRp topology on H.*

Proof. Let $\{G_i\}_{i\geqslant 1}$ be the RFRp filtration on G and let $\{H_i\}_{i\geqslant 1}$ be the RFRp filtration on H. Note that $H_1 = H \cap G_1$ by definition. Assume that $H_i = G_i \cap H$ for some i. The retraction r maps G_i onto H_i, since the inclusion map of H_i into G_i is a right inverse to the identity map on H_i.

Notice that H_i maps onto H_i/H_{i+1}, and that this group is a quotient of G_i which must factor through $\mathrm{TFr}(H_1(G_i, \mathbb{Z})) \otimes \mathbb{Z}/p\mathbb{Z}$. Hence, $H_{i+1} > H \cap G_{i+1}$. Thus the topology on H induced by the filtration $\{G_i\}_{i\geqslant 1}$ is the RFRp topology on H. □

Corollary 2.4. *Let G be a finitely generated RFRp group, and let $\phi\colon G \to H$ be a retraction. Then H is closed in the RFRp topology on G.*

Proof. The proof is identical to the proof of Lemma 3.9 in [23]. We recall a proof for the convenience of the reader. Let $\{G_i\}_{i\geqslant 1}$ be the standard RFRp filtration of G, let $N = \ker \phi$, and let $N_i = G_i \cap N$. Note that for each i, the subgroup $N_i < N$ has finite index. Observe that every element of G can be written uniquely as a product $n \cdot h$, where $n \in N$ and $h \in H$. It follows that the intersection of the subgroups $\{N_i H\}_{i\geqslant 1}$ is exactly H, so that H is closed in the RFRp topology on G. □

2.3. **Relationship to nilpotent groups.** We note the following fairly straightforward fact. The reader may wish to compare the proof of Proposition 2.5 below with [26, Lemma 8.3].

Proposition 2.5. *Let N be a non-abelian nilpotent group. Then N is not RFRp for any prime p.*

Proof. First, if N has torsion then N is not RFRp for any prime, as we will see below in Proposition 3.1. So, we may assume that N is torsion-free.

Let $\{\gamma_i(N)\}_{i\geqslant 1}$ denote the lower central series of N, so that $\gamma_1(N) = N$ and

$$\gamma_{i+1}(N) = [N, \gamma_i(N)]. \tag{6}$$

By assumption, this series terminates. Also let $\{N_j\}_{j\geqslant 1}$ be a sequence of subgroups of N which witnesses the claim that N is RFRp. Then, in fact, $\{N_j\}_{j\geqslant 1}$ is the standard RFRp filtration for N.

By induction on the length of the lower central series and on i, it is straightforward to verify that if $g \in \gamma_2(N)\backslash\gamma_3(N)$, then the image of g is either trivial or torsion in N_j^{ab}, so that $g \in N_{j+1}$ for each j. This last claim follows from the fact that for each j, some nonzero power of g is a product of commutators in the torsion-free group N_j. This contradicts the assumption that $\bigcap_j N_j = \{1\}$. □

Furthermore, whether or not a particular group enjoys the RFRp property depends on the prime p:

Proposition 2.6. *For each prime p, there exists a finitely presented group G_p which is RFRp, but G_p is not RFRq for any prime $q \neq p$.*

Proof. Fix a basis $\{v_1, \ldots, v_p\}$ for $\mathbb{Z}^p$. Let

$$\mathbb{Z}/p\mathbb{Z} \to \mathrm{GL}_p(\mathbb{Z}) = \mathrm{Aut}(\mathbb{Z}^p) \tag{7}$$

be the regular representation of $\mathbb{Z}/p\mathbb{Z}$ which permutes the coordinates of $\mathbb{Z}^p$. We consider the $\mathbb{Q}$-irreducible representation $V \cong \mathbb{Z}^{p-1} \otimes \mathbb{Q}$ of $\mathbb{Z}/p\mathbb{Z}$ given by vectors whose coordinates add up to zero, and let $A \subset V$ be the integral points. Furthermore, we let G_p be the semidirect product

$$1 \longrightarrow A \longrightarrow G_p \longrightarrow \mathbb{Z} \longrightarrow 1\ , \tag{8}$$

where the $\mathbb{Z}$-action on A is via the canonical projection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$.

Observe that $b_1(G_p) = 1$. Observe furthermore that the kernel of the map $G_p \to \mathbb{Z}/p\mathbb{Z}$ given by reducing the first homology of G_p modulo p is isomorphic to $\mathbb{Z}^p$. It is clear then that G_p is RFRp.

Now let $q \neq p$ be another prime. Let $G_p \twoheadrightarrow \mathbb{Z}/q\mathbb{Z}$ be the map given by reducing the homology of G_p modulo q, and let K_q be the kernel. Since q and p are relatively prime, and since V is an irreducible $\mathbb{Q}$-representation of $\mathbb{Z}/p\mathbb{Z}$, we have that $K_q \cong G_p$. It follows that G_p is not RFRq, since A is contained in any sequence of subgroups witnessing the claim that G_p is RFRq. □

Observe that in Proposition 2.6, the group G_p is virtually abelian. To obtain a non-virtually abelian example, observe that for each n, the natural homomorphism $\mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$ is surjective. Thus, one can mimic the construction of Proposition 2.6 in the free group case, obtaining a semidirect product

$$1 \longrightarrow F_{p-1} \longrightarrow H_p \longrightarrow \mathbb{Z} \longrightarrow 1 \tag{9}$$

of $\mathbb{Z}$ with a free group of rank $(p-1)$ which is RFRp for exactly one prime. It is easy to check that H_p is virtually a direct product, and that neither G_p nor H_p is residually torsion-free nilpotent.

3. Properties of RFRp groups

In this section we discuss further properties enjoyed by residually finite rationally p groups.

3.1. Torsion-free quotients of RFRp groups.

We start with some immediate consequences of the definition.

Proposition 3.1. *Let G be a finitely generated group which is RFRp. Then:*

(1) G is residually p. In particular, G is residually finite and residually nilpotent.
(2) G is torsion-free.
(3) If in addition G is finitely presented, then G has a solvable word problem.

Proof. Item (1) is straightforward from the definition. Item (2) follows from the fact that if $1 \neq g \in G_i \backslash G_{i+1}$, then g represents a torsion-free class in $H_1(G_i, \mathbb{Z})$, and therefore has infinite order in G. Item (3) is a completely standard result about finitely presented, residually finite groups (see for instance [31]). ◻

If a finitely generated group G is RFRp for every prime p, we have that G is residually p for every prime and torsion-free. Recall that a group G is *residually torsion-free nilpotent* if each non-identity element $g \in G$ survives in a torsion-free nilpotent quotient of G. Note that a group which is residually torsion-free nilpotent is torsion-free and residually p for every prime p.

Residual torsion-free nilpotence of a finitely generated group G is a rather strong property which has many useful consequences. For instance, if G is residually torsion-free nilpotent then G is bi-orderable and $\mathbb{Z}[G]$ is an integral domain (see [13, 12]).

Question 3.2. *Let G be a finitely generated group which is RFRp for every prime p. Is G residually torsion-free nilpotent?*

Recall that a group G is *polycyclic* if it admits a finite subnormal series with cyclic factors. Note that a finitely generated nilpotent group is polycyclic and that a polycyclic group is solvable, but that the reverse implications are generally false.

Theorem 3.3. *Let G be a finitely generated group which is RFRp for some prime p. Then G is residually torsion-free polycyclic. In particular, G is residually torsion-free solvable.*

Proof. Let $\{G_i\}$ be the standard RFRp sequence for G, with $G_1 = G$. We will define a sequence $\{K_i\}$ of subgroups of G such that $K_i \leqslant G_i$ for each i and such that G/K_i is a torsion-free polycyclic group for each i. Since $\bigcap_i G_i = \{1\}$, this will prove that G is residually torsion-free polycyclic.

Set $K_1 = G_1$ and set

$$K_2 = \ker\{G_1 \to \mathrm{TFr}\, H_1(G_1, \mathbb{Z})\} < G_2.$$

Since G is finitely generated, we have that G/K_2 is a finitely generated torsion-free abelian group and therefore torsion-free polycyclic. In general, we set

$$K_{i+1} = (\ker\{G_i \to \mathrm{TFr}\, H_1(G_i, \mathbb{Z})\}) \cap K_i < G_{i+1}.$$

By the Second Isomorphism Theorem for groups, we have that

$$K_i/K_{i+1} \cong \frac{K_i \cdot (\ker\{G_i \to \mathrm{TFr}\, H_1(G_i, \mathbb{Z})\})}{\ker\{G_i \to \mathrm{TFr}\, H_1(G_i, \mathbb{Z})\}}.$$

Since $K_i < G_i$, we have that K_i/K_{i+1} is a subgroup of the finitely generated abelian group $\mathrm{TFr}\, H_1(G_i, \mathbb{Z})$ and is therefore a finitely generated, torsion-free abelian group. By construction, K_i is normal in G for each i, so that by induction on i, we have that G/K_i is torsion-free polycyclic for each i. □

We note the residual torsion-free solvability of RFRp groups because of apparent connections to BNS invariants [18].

3.2. **Separability of maximal abelian subgroups.** In the proof of Theorem 6.1, we will require the separability of certain subgroups of RFRp groups. Let G be a group. We will say that a subgroup $H < G$ is *separable* if for every $g \in G \backslash H$, there is a finite quotient $\phi \colon G \to Q$ such that $\phi(g) \notin \phi(H)$. A subgroup H is *RFRp-separable* in G if we can assume that $Q = G/G_i$ for some term G_i in the standard RFRp filtration of G. In other words, a subgroup $H < G$ is RFRp-separable in G if and only if H is closed in the RFRp topology on G.

The following result about RFRp groups mirrors a result of E. Hamilton about hyperbolic 3-manifold groups (see [20]):

Theorem 3.4. *Let G be a finitely generated RFRp group and let $K < G$ be a finitely generated abelian subgroup which is maximal among abelian groups with the same rank as K. Then K is RFRp-separable in G.*

The maximality assumption on K simply means that if K is properly contained in an abelian subgroup $H < G$ then $\operatorname{rk} K < \operatorname{rk} H$. The necessity of this assumption results from the following example: suppose K and H are both torsion-free abelian groups of rank n, and that p does not divide $[H : K]$. Then K is not separable in the RFRp topology on H, even though the RFRp topology on K agrees with the RFRp topology induced from H.

Proof of Theorem 3.4. As usual, write $\{G_i\}_{i \geqslant 1}$ for the standard RFRp filtration of G, and write $\phi_i \colon G \to G/G_{i+1}$ for the canonical projection. Let $g \in G \backslash K$. Since K is maximal with respect to abelian subgroups of G of the same rank as K, the group

$H := \langle g, K\rangle$ is not isomorphic to K. We write n for the rank of K, so that either H is abelian of rank $n+1$, or g does not centralize K.

Notice that if g is not in the centralizer of K then there is some $k \in K$ such that $[g,k] \neq 1$. Then for some i, we have $[g,k] \in G_i \backslash G_{i+1}$. In particular, $\phi_i([g,k]) \neq 1$ in G/G_i, so that $\phi_i(g) \notin \phi_i(K)$. Thus, if g does not centralize K then we can separate g from K in the RFRp topology.

Thus, we may assume that g centralizes K, so that $H \cong \mathbb{Z}^{n+1}$. It suffices to find an i such that $\phi_i(H)$ is an abelian p-group of rank exactly $n+1$. This way, since $\phi_i(K)$ will be a p-group of rank at most n, we will immediately obtain that $\phi_i(g) \notin \phi_i(K)$, thereby showing that g may be separated from K in the RFRp topology.

We proceed by induction on n. The base case of the induction is clear. If H is a cyclic group, then because G is RFRp and $g \in G\backslash\{1\}$, there is an i for which $\phi_i(H)$ will be a p-group of rank exactly one.

For the inductive step, we may assume that there is an i such that $\phi_i(K)$ is a p-group of rank exactly n. Picking a basis $\{k_1, \dots, k_n, g\}$ for H, we may choose an index $j \geqslant i$ such that no basis element for H lies in in G_{j+1}. The basis elements themselves may not lie in G_j, but by replacing the chosen basis elements by positive powers if necessary, we may assume they do. We then consider the image $\bar{H}$ of H inside of $\operatorname{TFr}(G_j^{\mathrm{ab}})$.

Observe that since $j \geqslant i$, we have that $\operatorname{rk} \bar{H}$ is at least n, by the assumptions on i. Note furthermore that if $\operatorname{rk} \bar{H} = n+1$, then the image of $\bar{H}$ in

$$\operatorname{TFr}(G_j^{\mathrm{ab}}) \otimes \mathbb{Z}/p^m\mathbb{Z} \tag{10}$$

will have rank $n+1$ for some sufficiently large m. By the definition of the RFRp filtration on G, we have that the image of $\phi_k(H)$ in G/G_{k+1} will contain an abelian p-group of rank exactly $n+1$ for some $k \geqslant j$, and will thus itself have rank exactly $n+1$.

Thus, we may assume that $\operatorname{rk} \bar{H} = n$, so that there is an element $d \in H$ which is nontrivial in H but which is trivial in $\bar{H}$. We choose an index s such that $d \notin G_{s+1}$ and again replace the basis elements of H and d by suitable powers so that they lie in G_s. Then, (suitable powers of) the elements $k_1, \dots, k_n, h, d$ generate a subgroup of $\operatorname{TFr}(G_s^{\mathrm{ab}})$ of rank at least n, and d lies in the kernel of the natural map

$$\operatorname{TFr}(G_s^{\mathrm{ab}}) \longrightarrow \operatorname{TFr}(G_j^{\mathrm{ab}}) \tag{11}$$

induced by the inclusion $G_s \to G_j$. The image of H in $\operatorname{TFr}(G_j^{\mathrm{ab}})$ under this map has rank exactly n. Thus, the image of H in $\operatorname{TFr}(G_s^{\mathrm{ab}})$ must have rank exactly $n+1$.

Again, we see that the image of H in $\operatorname{TFr}(G_s^{\mathrm{ab}}) \otimes \mathbb{Z}/p^m\mathbb{Z}$ has rank $n+1$, for some sufficiently large m. In particular, $\phi_t(H)$ is an abelian p-group of rank $n+1$ for some $t \geqslant s$, and this completes the proof. $\square$

3.3. **Orderability.** In this subsection, we establish item (9) of Theorem 1.2. This follows from the following result of Rhemtulla [37]:

Theorem 3.5 ([37])**.** *Let G be a group which is residually locally residually p for infinitely many primes p. Then G is bi-orderable.*

The consequence of this result which is relevant for our discussion is the following:

Corollary 3.6. *Let G be a group which is RFRp for infinitely many primes p. Then G is bi-orderable.*

4. Classes of groups which are RFRp

We now populate the class of RFRp groups with several families of examples occurring in low-dimensional topology and geometric group theory.

4.1. **Free groups and surface groups.** We start by showing that finitely generated free groups are residually finite rationally p, based on an argument the first author gave in [25], which we will recall for the convenience of the reader.

Proposition 4.1. *Finitely generated free groups are RFRp, for all primes p.*

Proof. We realize a free group as the fundamental group of a wedge of circles X, which we think of as a graph equipped with the graph metric. In any simplicial graph Γ equipped with the graph metric, we have the following two observations: first, any shortest unbased (non-backtracking) loop γ in Γ is simple, i.e., γ has no self-intersections. Second, if γ is a simple, oriented loop, then the homology class $[\gamma] \in H_1(\Gamma, \mathbb{Z})$ is primitive. This can be seen by choosing any edge e of γ, extending $\gamma \backslash e$ to a maximal tree $T \subset \Gamma$, and considering the graph Γ/T.

Using these observations, we build a sequence of covers of X by setting $X_1 = X$ and letting X_{i+1} be the finite cover of X_i induced by the quotient $H_1(X_i, \mathbb{Z}/p\mathbb{Z})$ of $\pi_1(X_i)$. We see that the shortest unbased loop in X_i does not lift to X_{i+1}, so that by induction, the shortest loop in X_i has length at least i in the graph metric. □

More generally, we will show in Proposition 6.3 that right-angled Artin groups are residually finite rationally p, for all primes p.

Proposition 4.2. *Fundamental groups of closed, orientable surfaces are RFRp, for all primes p.*

Proof. The argument is nearly identical to that for free groups. For genus one, the claim is straightforward, so we assume the genus of the base surface to be at least two. We choose a hyperbolic metric on a surface $X = X_1$ and on all of its covers (by pullback). Again, any shortest geodesic on a hyperbolic surface is simple. If γ is a simple, oriented, closed geodesic on a hyperbolic surface, γ represents a primitive homology class if and only if it is non-separating. If $\gamma \subset X$ is a separating simple closed geodesic and p is any prime, then γ lifts to the universal modulo p homology cover $X_p \to X$, and any lift of γ is non-separating on X_p (though the union of all lifts of γ is separating).

We now build the tower of covers $\{X_i\}$ of $X = X_1$ in the same manner as in the case of free groups. The hyperbolic length spectrum of geodesics on X is a discrete subset of $\mathbb{R}$, since a hyperbolic metric is induced by a discrete representation of $\pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$. Thus, we again see that for any closed geodesic $\gamma \subset X$, we have that γ does not lift to X_i for $i \gg 1$. ◻

4.2. **Operations on RFR*p* groups.** Next, we show that the class of RFRp groups is closed under certain natural operations. A nearly verbatim statement for RFRS groups was established by Agol in [1], albeit our proof for part (3) is somewhat different.

Theorem 4.3. *Fix a prime p. The class of RFRp groups is closed under the following operations:*

(1) Passing to finitely generated subgroups.
(2) Taking finite direct products.
(3) Taking finite free products.

We remark that an arbitrary subgroup of an RFRp group will be RFRp in an appropriate sense; only the finite generation may be lost.

Proof of Theorem 4.3. We prove the items in order. Let G be a group which is RFRp, as witnessed by a sequence of nested subgroups $\{G_i\}$, and let $H < G$ be an arbitrary subgroup. We set $H_i = H \cap G_i$. Evidently, we have

$$\bigcap_i H_i = \{1\}.$$

Furthermore, H_i/H_{i+1} is the image of H_i inside of G_i/G_{i+1} and is therefore an elementary abelian p-group. Moreover, if $h \in H_i \backslash H_{i+1}$, then $h \in G_i \backslash G_{i+1}$ and therefore has infinite order in $H_1(G_i, \mathbb{Z})$. It follows that h must also have infinite order in $H_1(H_i, \mathbb{Z})$. Thus, the sequence of subgroups $\{H_i\}$ witnesses the claim that H is RFRp. Thus, the class of RFRp groups is closed under taking subgroups.

If G and H are groups, we have

$$H_1(G \times H, \mathbb{Z}) \cong H_1(G, \mathbb{Z}) \times H_1(H, \mathbb{Z}).$$

If G and H are RFRp with nested sequences of subgroups $\{G_i\}$ and $\{H_i\}$, we set $K = G \times H$ and $K_i = G_i \times H_i$ for each i. We have that

$$K_i/K_{i+1} \cong G_i/G_{i+1} \times H_i/H_{i+1},$$

as follows from an easy computation, so that K_i/K_{i+1} is an elementary abelian p-group. Furthermore, K_i/K_{i+1} is a quotient of TFr $H_1(G_i \times H_i, \mathbb{Z})$, and

$$\bigcap_i K_i = \bigcap_i G_i \times H_i = \{1\},$$

so that $\{K_i\}$ witnesses the fact that K is RFRp. By an easy induction, this shows that the class of RFRp groups is closed under taking finite direct products.

To prove that the class the class of RFRp groups is closed under taking finite free products, we will note that this claim is a special case of Theorem 6.1. It will not be circular to postpone the proof until then (see Corollary 6.2). $\square$

4.3. **Circle bundles over surfaces.** The following fact shows that there is a sharp dichotomy between groups which are RFRp and groups which are not RFRp in the class of cyclic central extensions of surface groups:

Theorem 4.4. *Let S be an aspherical, compact, orientable surface and let*

$$F \longrightarrow E \longrightarrow S$$

be a fiber bundle with fiber $F = S^1$, such that the total space E is orientable. Write $e \in H^2(S, \mathbb{Z})$ for the Euler class of the bundle.

(1) If $e = 0$, then $\pi_1(E)$ is RFRp for every prime p.

(2) If $e \neq 0$ then $\pi_1(E)$ is not RFRp for any prime p.

Proof. If $e = 0$ then $\pi_1(E) \cong \mathbb{Z} \times \pi_1(S)$. Thus, $\pi_1(E)$ is RFRp for every prime p by combining Proposition 1.1 and Theorem 4.3.

If $e \neq 0$ then we follow the argument given in [26], which we reproduce here for the reader's convenience. We have a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(E) \longrightarrow \pi_1(S) \longrightarrow 1\ , \tag{12}$$

where the leftmost copy of $\mathbb{Z}$ is central and is generated by an element t. We claim that for any prime p, we have $\mathrm{rad}_p(\pi_1(E)) = \langle t \rangle$.

First, if $g \in \pi_1(E)$ then we may write $g = h \cdot t^k$, where $h \in \pi_1(S)$. Since $\pi_1(S)$ is RFRp for each prime p, we have that if $h \neq 1$ then $h \notin \mathrm{rad}_p(\pi_1(E))$, by Proposition 2.2. Thus, we have an inclusion $\langle t \rangle \supset \mathrm{rad}_p(\pi_1(E))$, which holds for every prime p.

Conversely, write $G = \pi_1(E)$ and $\{G_i\}_{i \geqslant 1}$ for the standard RFRp filtration of G. The fact that $e \neq 0$ means that a nonzero power of t is a product of commutators in G (see [10]). In particular, we have that t maps to a torsion element of $H_1(G, \mathbb{Z})$, so that $t \in G_2$. By induction, we may suppose that $t \in G_i$ for some $i > 1$. Since $G_i < G_1 = G$ has finite index, we have that G_i again decomposes as a nonsplit central extension of a surface group, by a standard cohomology of groups argument using the fact that $H^2(\pi_1(S), \mathbb{Z})$ is torsion-free. In particular, t maps to a torsion element of $H_1(G_i, \mathbb{Z})$, so that $t \in G_{i+1}$. Hence $t \in \mathrm{rad}_p(\pi_1(E))$, establishing the reverse inclusion. $\square$

4.4. **Complements.** In this short subsection, we gather some classes of groups which fail to be RFRp, be it for some prime, for infinitely many primes, or for all primes. Many of the details have been discussed above (see Subsections 2.1 and 2.3, as well as Theorem 4.4), so we can safely omit most of the proofs.

Proposition 4.5. *The following classes of finitely generated groups fail to be RFRp for any prime p:*

(1) Groups with torsion.
(2) Nonabelian nilpotent groups.
(3) Central extensions which are not virtually split.
(4) Nonabelian groups G with $b_1(G) < 2$.

Proof. Only the last item has not been formally established above. If G is nonabelian and RFRp then G surjects onto a nonabelian p-group, whose abelianization cannot be cyclic by elementary group cohomology considerations. Thus, $b_1(G) \geqslant 2$. $\square$

All the statements in the following proposition have been (or will be) established (see Subsection 3.3 above and Subsection 5.3 below).

Proposition 4.6. *The following classes of finitely generated groups fail to be RFRp for infinitely many primes:*

(1) Groups which are not bi-orderable.
(2) Groups which are not large.

Since bi-orderable groups are somewhat rare in nature, the first item of Proposition 4.6 really does suggest that RFRp is a fine property for groups to enjoy. For instance, many virtually special groups arising in 3-manifold topology fail to be bi-orderable. Thus, whereas a graph manifold may be virtually special, knowing its fundamental group is RFRp for all primes is a significantly different bit of data, which gives further strength to the results above, such as Theorem 1.5.

5. Alexander varieties, BNS invariant, and largeness

We start this section by reviewing some background on the homology jump loci and the Alexander varieties of spaces and groups, following [21, 33, 40].

5.1. **Jump loci and Alexander invariants.** Let G be a finitely-generated group, and let X be a connected CW-complex with finite 1-skeleton such that $\pi_1(X) \cong G$. The *characteristic varieties* $V_i(X)$ are the jumping loci for the (degree 1) cohomology groups of X with coefficients in rank 1 local systems. We write $\widehat{G}$ for the group of complex characters of G. Let $\chi\colon G \to \mathbb{C}^*$ be a character of G and let $H^1(X, \mathbb{C}_\chi)$ be the twisted cohomology module corresponding to χ. For each $i \geqslant 0$, put

$$V_i(X) = \{\chi \in \widehat{G} \mid \dim H^1(X, \mathbb{C}_\chi) \geqslant i\}. \tag{13}$$

It is readily seen that each of these sets is a Zariski closed subset of the character group; moreover, $V_i(X) \supseteq V_{i+1}(X)$ for all i, and $V_i(X) = \emptyset$ for $i \gg 0$. Clearly, each of these sets depends only on the fundamental group of X, so we may define $V_i(G) := V_i(X)$. If $\phi\colon G_1 \to G_2$ is a surjective homomorphism, we obtain an injective morphism $\hat{\phi}\colon \widehat{G}_2 \to \widehat{G}_1$ by precomposition. It is readily verified that the map $\hat{\phi}$ takes $V_i(G_2)$ to $V_i(G_1)$.

By definition, the trivial representation $\widehat{1} \in \widehat{G}$ belongs to $V_i(G)$ if an only if $b_1(G) \geqslant i$. Away from $\widehat{1}$, the sets $V_i(G)$ coincide with the *Alexander varieties* of G.

To define these varieties, first consider the derived series of G, defined inductively by setting $G' = [G,G]$, $G^2 = G'' = [G',G']$, and $G^k = [G^{k-1}, G^{k-1}]$ for $k \geqslant 3$. The quotient G/G^k is the *universal k-step solvable quotient* of G.

Next, let $B(G) = G'/G''$ be the *Alexander invariant* of G, viewed as a module over $\mathbb{Z}[G/G']$ via the conjugation action of G/G' on G'/G''. Note that $\mathbb{C}[G/G']$ is the coordinate ring of the character group $\widehat{G}$. We then let the i-th Alexander variety of G be the support locus of the i-th exterior power of the complexified Alexander invariant, that is,

$$W_i(G) = V\Big(\operatorname{ann}\Big(\bigwedge^i B(G) \otimes \mathbb{C}\Big)\Big). \tag{14}$$

As shown in [21], the following equality holds, for each $i \geqslant 1$:

$$V_i(G)\setminus\widehat{1} = W_i(G)\setminus\widehat{1}. \tag{15}$$

This description makes it apparent that the characteristic varieties $V_i(G)$ only depend on G/G'', the maximal metabelian quotient of G. More precisely, we have the following lemma.

Lemma 5.1. *For any finitely generated group G, the projection map $\pi\colon G \to G/G''$ induces an isomorphism $\hat{\pi}\colon \widehat{G/G''} \to \widehat{G}$ which restricts to isomorphisms $V_i(G/G'') \to V_i(G)$ for all $i \geqslant 1$.*

Proof. Clearly, the map π induces an isomorphism on abelianizations, and thus an isomorphism between the respective character group.

Now note that $(G/G'')' = G'/G''$ and $(G/G'')''$ is trivial; thus, the map π also induces an isomorphism $B(G) \to B(G/G'')$. Applying formulas (14) and (15) proves the remaining claim. ∎

5.2. **Non-finitely presented metabelian quotients.** Once again, let X be a connected CW-complex with finite 1-skeleton, and set $\pi_1(X) = G$. If A is a finite abelian group and $\phi\colon G \to A$ is a surjective homomorphism, we obtain a finite cover $X_A \to X$ induced by ϕ. The complex Betti number $b_1(X_A)$ is related to $b_1(X)$

and the varieties $V_i(X)$ by the following well-known formula:

$$b_1(X_A) = b_1(X) + \sum_{i=1}^{k} \left| \widehat{\phi}(\widehat{A} \setminus \widehat{1}) \cap V_i(X) \right|, \tag{16}$$

where here $V_i(X) = \varnothing$ for $i > k$.

The following result relates torsion points on the Alexander variety to largeness for finitely presented groups:

Theorem 5.2 (See [28])**.** *Let G be a finitely presented group. The group G is large if and only if there exists a finite index subgroup $H < G$ such that $V_1(H)$ has infinitely many torsion points.*

The finite presentation assumption in the above theorem is essential. For instance, let F_n be a free group of rank $n \geqslant 2$. It is readily verified that $V_1(F_n) = (\mathbb{C}^*)^n$. Thus, by Lemma 5.1, we also have that $V_1(F_n/F_n'') = (\mathbb{C}^*)^n$. In particular, the variety $V_1(F_n/F_n'')$ has infinitely many torsion points, though the group F_n/F_n'' is solvable, and thus not large.

As an application, we obtain the following corollary, which recovers and generalizes the main result of Baumslag and Strebel [7].

Corollary 5.3. *Let G be a finitely generated group which is nonabelian and RFRp for infinitely many primes p. Then the universal metabelian quotient G/G'' is not finitely presented. In particular, G' is not finitely generated.*

Proof. Observe that since G is RFRp for infinitely many primes and not abelian, we have that $V_1(G)$ contains infinitely many torsion points. By Lemma 5.1, we have that $V_1(G/G'')$ also contains infinitely many torsion points.

Now suppose G/G'' is finitely presented. Then Theorem 5.2 implies that G/G'' is large. However, G/G'' is solvable, and this is a contradiction. □

In fact, the same proof works for all universal solvable quotients:

Corollary 5.4. *Let G be a finitely generated group which is nonabelian and RFRp for infinitely many primes p. Then the universal k-step solvable quotient G/G^k is not finitely presented, for any $k \geqslant 2$.*

5.3. **A Tits Alternative for RFRp groups.** We now connect the RFRp property of a group G to the aforementioned arithmetic property of $V_1(G)$.

Lemma 5.5. *Let G be a non-abelian, finitely generated group which is RFRp for infinitely many primes. Then $V_1(G)$ contains infinitely many torsion points.*

Proof. Suppose G is RFRp for infinitely many primes p. For each prime p, we write

$$K_{p,n+1} = \ker\{G \to (\operatorname{TFr} H_1(G,\mathbb{Z})) \otimes \mathbb{Z}/p^n\mathbb{Z}\}. \tag{17}$$

We claim that if G is nonabelian and RFRp, then $b_1(K_{p,n}) > b_1(G)$ for $n \gg 0$. Indeed, otherwise we can construct a sequence of subgroups $\{G_i\}_{i\geqslant 0}$ which witness the fact that G is RFRp, so that $G_1 = G$ and $G_2 = K_{p,2}$. Since $b_1(K_{p,n}) = b_1(G)$ for all n, an easy induction shows that $G_n = K_{p,n}$ for all n. In particular, $\bigcap_n K_{p,n} = \{1\}$, which implies G is abelian, since $G' < K_{p,n}$ for all n. This is a contradiction.

Thus, if G satisfies out hypothesis, we have that $V_1(G)$ contains at least one p-torsion point for infinitely many values of p, by (16). Since for primes $p \neq p'$, the p-torsion and p'-torsion points on $V_1(G)$ are disjoint, we have that $V_1(G)$ contains infinitely many torsion points. ◻

Agol asked the first author [2] whether a group which is RFRS and not virtually abelian is *large*, i.e., virtually surjects to a nonabelian free group. We give the following affirmative partial answer to Agol's question, which contrasts sharply with the example described in Proposition 2.6:

Theorem 5.6. *Let G be a finitely presented group which is RFRp for infinitely many primes. Then either:*

(1) G is abelian.
(2) G is large.

Proof. Follows at once from Theorem 5.2 and Lemma 5.5. ◻

5.4. **Σ-invariants.** We now relate the RFRp property to the Bieri–Neumann–Strebel invariant of [8]. Once again, let G be a finitely generated group. Without loss of generality, we may assume that G is generated by a finite, symmetric set Ω. We write $\mathrm{Cayley}(G,\Omega)$ for the Cayley graph of G with respect to Ω, and we write $S(G)$ for the unit sphere in the first real cohomology group of G:

$$S(G) = (H^1(G,\mathbb{R})\setminus\{0\})/\{\chi \sim \lambda\cdot\chi,\ \lambda \in \mathbb{R}_{>0}\}. \tag{18}$$

For $\chi \in S(G)$, we write $\mathrm{Cayley}_\chi(G,\Omega)$ for the subgraph consisting of vertices $g \in G$ such that $\chi(g) \geqslant 0$. A fundamental fact about this graph is that its connectivity is independent of the generating set Ω, so we may suppress Ω in our notation.

We write

$$\Sigma^1(G) = \{\chi \in S(G) \mid \mathrm{Cayley}_\chi(G) \text{ is connected}\}, \tag{19}$$

and $E^1(G)$ for the complement of $\Sigma^1(G)$. If N is a normal subgroup of G, we write $S(G,N)$ for the real characters in $S(G)$ which vanish on N. The following result is fundamental in BNS theory:

Theorem 5.7 ([8])**.** *Let G be a finitely generated group, and let G/N be an infinite abelian quotient. The group N is finitely generated if and only if $S(G,N) \subset \Sigma^1(G)$. In particular, G' is finitely generated if and only if $E^1(G) = \varnothing$.*

By analogy to a result of Beauville on the structure of Kähler groups, we have the following result:

Theorem 5.8. *Let G be a finitely generated group which is RFRp for infinitely many primes p. If $E^1(G) = \varnothing$ (or, if $E^1(G/G^k) = \varnothing$, for some $k \geqslant 2$), then G is abelian.*

Proof. This follows immediately from Theorem 5.6 and Corollary 5.3. □

6. A combination theorem for RFRp groups

In this section, we wish to give suitable hypotheses on vertex spaces, edge spaces, and gluing maps in a graph of spaces which guarantee that the resulting space has an RFRp fundamental group. The hypotheses in Theorem 6.1 may be difficult to verify in general, though we will show that within a certain natural class of graphs of spaces, the hypotheses are satisfied.

6.1. **Graphs of spaces.** Let Γ be a finite graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma) \subset V(\Gamma) \times V(\Gamma)$. To each vertex $v \in V(\Gamma)$, we associate a connected, finite CW-complex X_v. Let $e = \{s,t\} \in E(\Gamma)$ be an edge. To each such edge e we associate a connected, finite CW-complex $X_e \times [0,1]$, together with maps of CW complexes $\phi_{e,s}\colon X_e \times \{0\} \to X_s$ and $\phi_{e,t}\colon X_e \times \{1\} \to X_t$.

We build the *graph of spaces* X_Γ by identifying $X_e \times \{0\}$ and $X_e \times \{1\}$ with their images under $\phi_{e,s}$ and $\phi_{e,t}$, respectively, for each edge of Γ. Replacing the discussion of CW-complexes with groups, we obtain a *graph of groups*. Note that in the most general definition of a graph of spaces, we do not assume that Γ is a simplicial graph, nor that the maps $\{\phi_{e,v}\}_{e\in E(\Gamma),v\in V(\Gamma)}$ induce injective maps on fundamental groups.

If $Y \to X = X_\Gamma$ is a finite covering space, we will implicitly pull back the graph of spaces structure on X to Y. In particular, the vertex spaces of Y are the components of the preimages of the vertex spaces of X, and the edge spaces of Y are the components of the preimages of the edge spaces of X.

Observe that a graph of spaces $X = X_\Gamma$ is equipped with a natural *collapsing map* $\kappa\colon X \to \Gamma$, which collapses each vertex space X_v to a point and each thickened edge space $X_e \times [0,1]$ to the interval $[0,1]$. We choose an arbitrary splitting $\iota\colon \Gamma \to X$. For each vertex space X_v, we choose a basepoint $p_v \in \iota(\Gamma) \cap X_v$, which we identify with a basepoint for $\pi_1(X_v)$. For each free homotopy class of loops $\gamma \subset X$, we put γ into *standard form*. That is to say, γ is allowed to trace out any based homotopy class of loops in X_v, based at p_v, and is allowed to travel between two adjacent vertex spaces along $\iota(\Gamma)$ only. It is clear that any homotopy class of loops in X can be put into standard form.

Note that if $Y \to X$ is is a finite covering space, then the space Y admits a natural collapsing map $\kappa_Y\colon Y \to \Gamma_Y$, and the graph Γ_Y admits a natural map to $\Gamma = \Gamma_X$.

These four maps form a natural commutative square, though it is important to note that $\Gamma_Y \to \Gamma$ is generally not a covering map.

If $\gamma \subset X$ is a homotopy class of loops, we define the *combinatorial complexity* of γ to be the number of times γ travels between two adjacent vertex spaces, minimized over all representatives of γ which are in standard form. In other words, we count the number of times that γ traverses an edge space of X. We write $C(\gamma)$ for the combinatorial complexity of γ.

If $\gamma \subset X$ is a homotopy class of loops, we define the *backtracking number* of γ to be the number of times which γ enters a vertex space X_v through an edge space X_e, and then exits X_v through the same edge space X_e, summed over all vertices and edge spaces. In other words, the backtracking number of γ is the total number of times the combinatorial loop $\kappa(\gamma)$ backtracks inside Γ. If a vertex v contributes to the backtracking number of γ, we will say that the loop γ backtracks at the vertex space X_v. We write $B_X(\gamma)$ for the backtracking number of γ in X.

Note that if $Y \to X$ is a covering space to which γ lifts, then γ and any of its lifts γ' have backtracking numbers $B_X(\gamma)$ and $B_Y(\gamma')$. It is straightforward to see that $B_X(\gamma) \geqslant B_Y(\gamma')$. We may thus say that the backtracking number is non-increasing along covers. By convention, the backtracking number of a loop can only be positive if the combinatorial complexity of the loop is positive, so a loop of combinatorial complexity zero will have backtracking number zero.

6.2. **A combination theorem for RFRp groups.** We now establish the main result of this section:

Theorem 6.1. *Fix a prime p. Let $X = X_\Gamma$ be a finite graph of connected, finite CW-complexes with vertex spaces $\{X_v\}_{v\in V(\Gamma)}$ and edge spaces $\{X_e\}_{e\in E(\Gamma)}$ satisfying the following conditions:*

(1) For each $v \in V(\Gamma)$, the group $\pi_1(X_v)$ is RFRp.
(2) For each $v \in V(\Gamma)$, the RFRp topology on $\pi_1(X)$ induces the RFRp topology on $\pi_1(X_v)$ by restriction.
(3) For each $e \in E(\Gamma)$ and each $v \in e$, we have that the image

$$\phi_{e,v}(\pi_1(X_e)) < \pi_1(X_v)$$

is closed in the RFRp topology on $\pi_1(X_v)$.

Then $\pi_1(X)$ is RFRp.

The reader may compare the hypotheses of Theorem 6.1 to the notion of $\mathfrak{P}$-*efficiency* (see [5] and [42]).

Observe that we do not assume that the gluing maps of the edge spaces to the vertex spaces induce injections on the level of fundamental groups, which could at least in principle have disastrous algebraic consequences. However, the assumption that each vertex space has an RFRp fundamental group and that the RFRp topology on $\pi_1(X)$ induces the RFRp topology on the fundamental group of each vertex space implies that the inclusion $X_v \to X$ induces an injection on the level of fundamental groups.

The condition that the image of the edge space fundamental group is closed in the RFRp topology on the fundamental group of the vertex space may seem difficult to verify, though we will show in the sequel that under natural hypotheses, this condition is automatically satisfied.

Before proving the result, we note that we use very few specifics about the RFRp topology. Indeed, the proof we give below could be suitably adapted to prove a combination theorem for the following classes of groups:

- Residually finite groups;
- Residually solvable groups;
- Residually nilpotent groups;
- Residually p groups.

Proof of Theorem 6.1. We will prove the theorem by induction on the combinatorial complexity and the backtracking number, $(C(\gamma), B(\gamma))$, ordered lexicographically. We will denote the standard RFRp tower of X by $\{X_i\}_{i\geqslant 1}$, and we will show that for each nontrivial homotopy class of closed loops $\gamma \subset X$, there is some i such that γ does not lift to X_i.

For the base case, we suppose that γ has combinatorial complexity zero, so that γ remains inside of a single vertex space X_v for its entire itinerary. Viewing γ as a based homotopy class of loops, we identify γ with an element of $G_v = \pi_1(X_v)$. We write $\{G_{v,i}\}_{i\geqslant 1}$ for the standard RFRp filtration of G_v, so that $\gamma \in G_{v,i} \backslash G_{v,i+1}$ for some i. Similarly, we will write $G = \pi_1(X)$ and $\{G_i\}_{i\geqslant 1}$ for the standard RFRp filtration on G.

By assumption we have that the RFRp topology on X induces the RFRp topology on X_v. Thus for each i, there is a j such that $G_j \cap G_v < G_{v,i}$. Since $\gamma \in G_{v,i} \backslash G_{v,i+1}$ for some i, we have that $\gamma_i \in G_j \backslash G_{j+1}$ for some j, which establishes the base case of the induction.

We now assume that the combinatorial complexity of γ is $n > 0$. We may suppose for a contradiction that γ lifts to a loop $\gamma_i \subset X_i$ for each i. Writing $\kappa_i \colon X_i \to \Gamma_i$ for the collapsing map induced by pulling back the graph manifold structure of X to X_i, we may assume that $\kappa_i(\gamma_i)$ is nullhomotopic in Γ_i for each i and each lift γ_i of γ to X_i. Indeed, otherwise observe that $\pi_1(X_i)$ surjects to $\pi_1(\Gamma_i)$ via κ_i, the latter of which is an RFRp group. By Proposition 2.2, we have that any element of $\pi_1(X_i)$ which is not in $\ker \kappa_i$ does not lie in $\operatorname{rad}_p(\pi_1(X_i)) = \operatorname{rad}_p(\pi_1(X))$.

For a lift $\gamma_i \subset X_i$ of γ, we will write $B_i(\gamma_i)$ for its backtracking number. Note that because the cover $X_i \to X$ is regular, this number is independent of the choice of lift. Furthermore, the combinatorial complexity $C(\gamma_i)$ is constant under passing to covers, so we will just denote it by n. Observe that we have the *a priori* estimate

$B_i(\gamma_i) \leqslant n/2$ for all i. Furthermore, if γ lifts to each cover X_i of X, then we must have $B_i(\gamma_i) \geqslant 1$ for all i. This is simply because a nullhomotopic loop in a graph must backtrack at least once. Thus, we have that the pair $(C(\gamma_i), B_i(\gamma_i))$ is bounded below by $(n, 1)$ for all i.

We claim that if γ lifts to each X_i, then for any loop γ_i with associated data

$$(C(\gamma_i), B(\gamma_i)) = (n, B(\gamma_i)), \tag{20}$$

we can either decrease $B(\gamma_i)$ by one or we can decrease $C(\gamma_i)$ by two, after passing to a sufficiently high index i. This will prove the result by completing the induction.

For each cover X_i in the standard RFRp tower of X, we will fix a lift of γ, say γ_i. We choose these lifts at the beginning so that if $k > i$, then the cover $X_k \to X_i$ restricts to a map $\gamma_k \to \gamma_i$. Since each lift γ_i has combinatorial length exactly n, we write

$$\{\gamma_i^1, \ldots, \gamma_i^n\} \tag{21}$$

for the segments which are the intersections of γ_i with the vertex spaces of X_i, i.e.,

$$\gamma_i^j = \gamma_i \cap X_i^j \subset X_i. \tag{22}$$

We will label these segments (and *ipso facto* the corresponding vertex spaces) coherently, so that for $k > i$, the segment γ_i^j is covered by the segment γ_k^j. For each i, j, we have that γ_i^j and γ_i^{j+1} lie in different vertex spaces of X_i, by definition.

However, since $\kappa_i(\gamma_i)$ is nullhomotopic in Γ_i, we have that for some j, the segments γ_i^{j-1} and γ_i^{j+1} lie in the same vertex space

$$X_i^{j-1} = X_i^{j+1} \subset X_i, \tag{23}$$

and that the segments γ_i^{j-1} and γ_i^{j+1} meet the segment $\gamma_i^j \subset X_i^j$ in the same edge space Y_i. In other words, γ_i backtracks at X_i^j.

We claim that if the loop γ_i backtracks at the vertex space X_i^j for all i, then either we may deform the segment γ_i^j into X_i^{j-1} for $i \gg 1$, or we may decrease $B_i(\gamma_i)$ by at least one, for $i \gg 1$. Note that in the first case, it follows then that we have decreased the combinatorial complexity of γ by at least two (after passing to a sufficiently high index i), so that this will complete the induction.

For each i, let us again write Y_i for the edge space of X_i between X_i^{j-1} and X_i^j. Fixing a basepoint y_i in each Y_i, we may deform γ_i^j to be a closed loop in X_i^j which is based at y_i, because γ_i enters and exits X_i^j through the same edge space Y_i. We will denote this based loop inside of X_i^j by β_i. Of course, β_i is well-defined only up to an element of $\pi_1(Y_i, y_i)$.

Observe that by the minimality of the combinatorial complexity of γ, we may assume that it is not possible to deform β_i into Y_i for any i, because then γ_i could be pushed to avoid X_i^j entirely, decreasing the combinatorial complexity by two and

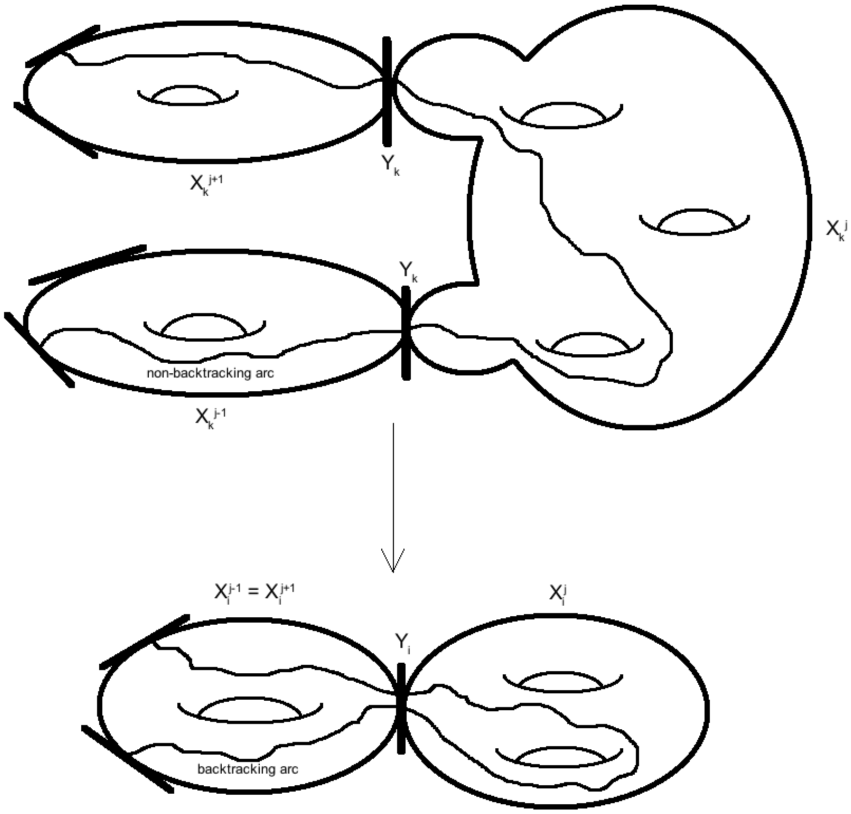

FIGURE 1. Lifting a backtracking arc to a non-backtracking arc

completing the induction. Fixing i, we claim that for $k \gg i$, each component β_k of the preimage of the loop β_i in the vertex space $X_k^j \subset X_k$ will be an arc traversing two distinct edge spaces of X_k. In particular, the loop γ_k no longer backtracks at the vertex space X_k^j (see Figure 1).

To see this last claim (and thus establish the theorem), we need to find a $k \gg i$ so that the deck transformation of the cover $X_k^j \to X_i^j$ corresponding to the image of the homotopy class of the loop β_i does not lie in the image of the subgroup $\phi_*(\pi_1(Y_i))$, where here we abuse notation and write ϕ for the gluing map which attaches Y_i to X_i^j.

By assumption, the subgroup $\phi_*(\pi_1(Y_i))$ is closed in the RFRp topology on $\pi_1(X_i^j)$, and the RFRp topology on $\pi_1(X_i^j)$ coincides with the restriction of the RFRp topology on $\pi_1(X_i)$. Writing $G_{i,k}$ for the deck group of $X_k^j \to X_i^j$, the statement that the subgroup $\phi_*(\pi_1(Y_i))$ is closed in the RFRp topology on $\pi_1(X_i^j)$ is exactly the statement that for every element

$$g \in \pi_1(X_i^j) \backslash \phi_*(\pi_1(Y_i)), \tag{24}$$

there is a k such that the image of g in $G_{i,k}$ does not lie in the image of $\phi_*(\pi_1(Y_i))$. Thus, for $k \gg i$, we have that the image of the homotopy class of β_i in $G_{i,k}$ does not lie in the image of $\phi_*(\pi_1(Y_i))$.

It follows that if β_k is a component of the preimage of β_i in such a cover $X_k^j \to X_i^j$, then the endpoints of β_k cannot both lie in a single component of the preimage of Y_i. This establishes the claim and proves the theorem. ◻

6.3. **Applications of the combination theorem.** We now give the promised missing part of Theorem 4.3:

Corollary 6.2. *The class of RFRp groups is closed under taking finite free products.*

Proof. By induction, it suffices to prove the corollary for two RFRp groups. Let $G = \pi_1(X)$ and $H = \pi_1(Y)$ be two such groups, where X and Y are connected, finite, based CW-complexes. The one-point union $Z = X \vee Y$ is homotopic to a CW-complex which has the structure of a graph of spaces, where X and Y are the vertex spaces and where the edge space is a point.

By assumption, G and H are RFRp groups; hence, the trivial group is closed in the RFRp topology on both G and H. It therefore suffices to prove that the RFRp topology on $G * H$ restricts to the RFRp topology on G and on H.

This last claim is straightforward, though. Setting $\{X_i\}_{i\geqslant 1}$, $\{Y_i\}_{i\geqslant 1}$, and $\{Z_i\}_{i\geqslant 1}$ to be the standard RFRp towers for X, Y, and Z respectively, we have that for each i, the space Z_i is homotopy equivalent to a wedge of circles, glued along one point to a finite collection of CW-complexes, each of which is homotopy equivalent to either X_i or Y_i. The integral first homology of Z_i is just the direct sum of the integral first homologies of these spaces. It follows easily then that the RFRp topology on $G * H$ restricts to the RFRp topology on both G and H.

The corollary now follows by Theorem 6.1. ◻

It is also possible to use Theorem 6.1 to prove that right-angled Artin groups enjoy the RFRp property, which is part (3) of Proposition 1.1 from the introduction.

Proposition 6.3. *Right-angled Artin groups are RFRp, for all primes p.*

Proof. First note that we may as well consider the case where the defining graph Γ is connected, since, as we just showed, the class of RFRp groups is closed under finite free products. The essential point is that for each $v \in V(\Gamma)$ one has a graph of groups decomposition

$$A(\Gamma) \cong A(\Gamma_v) *_{A(\mathrm{Lk}(v))} A(\mathrm{St}(v)), \tag{25}$$

where Γ_v is the subgraph of Γ spanned by $V(\Gamma)\backslash\{v\}$, and where $\mathrm{St}(v)$ and $\mathrm{Lk}(v)$ are the star and link of v, respectively.

The RFRp topology on $A(\Gamma)$ induces the RFRp topology on both vertex groups by induction on $|V(\Gamma)|$. We have that Γ_v is a proper subgraph of Γ and $A(\Gamma)$ retracts to $A(\Gamma_v)$, so we can apply Corollary 2.3. The right-angled Artin group $A(\mathrm{St}(v))$ is the direct product of $\mathbb{Z}$ with $A(\mathrm{Lk}(v))$, both of which are retracts of $A(\Gamma)$, so that Corollary 2.3 applies again.

Similarly by induction on $|V(\Gamma)|$, both vertex groups are RFRp. The final verification needed before applying Theorem 6.1 is to show that the edge group $A(\mathrm{Lk}(v))$ is closed in the RFRp topology on each vertex space. Since $A(\mathrm{Lk}(v))$ is a retract of $A(\Gamma)$, we apply Corollary 2.4 to confirm that fact.

The result now follows from Theorem 6.1. □

Remark 6.4. An alternative argument can be given, based on Lemma 3.9 from [23].

The proof of Proposition 6.3 has the following immediate corollary.

Corollary 6.5. *Let $\Lambda < \Gamma$ be a (full) subgraph and let p be a prime. Then $A(\Lambda) < A(\Gamma)$ is closed in the RFRp topology.*

7. 3-manifold groups and the RFRp property

7.1. Fundamental groups of 3-manifolds.

A group G is called a 3-*manifold group* if it can be realized as the fundamental group of a compact, connected, orientable 3-manifold M with $\chi(M) = 0$. In this section, we study 3-manifold groups and whether or not they are RFRp, for both geometric manifolds and non-geometric manifolds. In the first case, we can exactly characterize which geometric 3-manifolds groups are virtually RFRp, and in the second case we can give some hypotheses which guarantee that a non-geometric 3-manifold group is RFRp.

The reader will note that the hypotheses we place on the non-geometric 3-manifold groups are modeled on boundary manifolds of curve arrangements in $\mathbb{C}^2$, and indeed in this section we will prove that such a boundary manifold has an RFRp fundamental group.

We will restrict our discussion to *prime* 3-manifolds, namely ones which cannot be decomposed as nontrivial connected sums. Note that on the level of fundamental groups, connected sum corresponds to free product, and the free product of two finitely generated groups will be RFRp if and only if both free factors are RFRp (cf. Theorem 4.3 and Corollary 6.2).

7.2. Geometric 3-manifolds.

Recall that a 3-manifold M is *geometric* if it admits a finite volume complete metric modeled on one of the eight Thurston geometries,

$$\{S^3, S^2\times\mathbb{R}, \mathbb{R}^3, \mathrm{Nil}, \mathrm{Sol}, \mathbb{H}^2\times\mathbb{R}, \widetilde{\mathrm{PSL}_2(\mathbb{R})}, \mathbb{H}^3\},$$

see [34, 35, 39, 41]. Perelman's Geometrization Theorem says every prime 3-manifold can be cut up along a canonical collection of incompressible tori into finitely many pieces, every one of which is geometric. It is well-known (see [41]) that if a manifold is geometric, then its geometry can be read off from the structure of its fundamental group, and conversely the geometry of a 3-manifold determines the structure of its fundamental group. We are therefore prepared to give a proof of Theorem 1.4 as claimed in the introduction.

Recall that Theorem 1.4 asserts that certain geometric 3-manifold groups G are virtually RFRp, but not necessarily RFRp. This is an important distinction. For one, if we allow for orbifolds and orbifold fundamental groups, then G could potentially have torsion and therefore not be RFRp for any prime p. More essentially, there are geometric 3-manifold groups which fail to be RFRp for any prime, but which become RFRp for every prime after passing to a finite index subgroup. We illustrate this assertion with a class of examples.

Example 7.1. Let G be the fundamental group of a hyperbolic knot complement. Then G falls under the purview of Theorem 1.4, so that there is a finite index subgroup $K < G$ such that K is RFRp for every prime p. On the other hand, it is an easy exercise to check that for each prime p we have that $\mathrm{rad}_p(G) = [G, G] \neq \{1\}$, since a nonabelian p–group must have noncyclic abelianization.

Proof of Theorem 1.4. Let $G = \pi_1(M)$ be a geometric 3-manifold group. We begin with the geometries $\{S^3, S^2 \times \mathbb{R}, \mathbb{R}^3\}$. In the case of S^3, we have that G is finite and so there is nothing to show. In the other two cases, G either contains $\mathbb{Z}$ or $\mathbb{Z}^3$ with finite index, in which case it is clear that G is RFRp for each prime.

If M is modeled on $\mathbb{H}^2 \times \mathbb{R}$, then a finite index subgroup of G is isomorphic to $\pi_1(S) \times \mathbb{Z}$, where S is an orientable surface. Combining Proposition 1.1 and Theorem 4.3, we have that G is virtually RFRp for every prime.

If M is modeled on $\mathbb{H}^3$, then Agol's resolution of the Virtual Haken Conjecture [3] shows that a finite index subgroup of G lies as a finitely generated subgroup of a right-angled Artin group and is therefore RFRp for every prime, by Theorem 4.3.

If M is modeled in Nil geometry, then every finite index subgroup of G is nonabelian and nilpotent, and hence not RFRp for any prime p by Proposition 2.5.

If M is modeled on the Sol geometry, then G has a finite index subgroup H which is a semidirect product of $\mathbb{Z}^2$ with $\mathbb{Z}$, where the conjugation action of $\mathbb{Z}^2$ is by a hyperbolic matrix. Any finite index subgroup K of H has rank one abelianization, so that $\mathrm{rad}_p(K) \neq \{1\}$ for any p.

Finally, if M is modeled on $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$, then a finite index subgroup of G is a nonsplit central extension of $\pi_1(S)$ by $\mathbb{Z}$, where S is a closed, orientable surface of genus at least two. By Theorem 4.4, we have that G is not virtually RFRp for any prime p. This completes the proof. □

7.3. **Graph manifolds.** We wish to develop criteria which allow one to verify the hypotheses of Theorem 6.1, and thus prove that certain non-geometric 3-manifold groups are RFRp, and deduce Theorem 1.7. For our purposes, a prime 3-manifold

M is an *graph manifold* if it is a graph of spaces X satisfying the following conditions:

(1) Each vertex space X_v is a Seifert-fibered manifold, with $\deg(v)$ being at most the number of components of ∂X_v.
(2) Each edge space X_e is a torus.
(3) The gluing maps which assemble X are given by matching the two boundary components of $X_e \times [0, 1]$ via an orientation-preserving homeomorphism to a component of ∂X_v and ∂X_w respectively, where $e = \{v, w\}$.

For a general graph manifold M as we have defined it here, it may not be the case that $\pi_1(M)$ is RFRp, even if each of the vertex manifolds have RFRp fundamental groups. We illustrate this phenomenon with a family of examples.

Example 7.2. Let M_1 and M_2 be torus knot complements in S^3, which are well-known to be Seifert-fibered. We will write K_1 and K_2 for the respective fundamental groups. The cusps of M_1 and M_2 give rise to copies of $\mathbb{Z}^2$ inside of K_1 and K_2 respectively, and on the level of homology, the maps $\phi_i\colon \mathbb{Z}^2 \to H_1(M_i, \mathbb{Z})$ induced by inclusion have rank one. So, inside of K_i, we will decompose $\mathbb{Z}^2$ as a direct sum of two cyclic groups $A_i \oplus B_i$, where $B_i = \ker \phi_i$.

Let us glue now M_1 to M_2 along the cusps to get a new graph manifold M, in such a way that A_1 is identified with B_2 and A_2 is identified with B_1. The resulting 3-manifold has trivial first homology, and so does not have an RFRp fundamental group. In terms of Theorem 6.1, we see that the RFRp topology on $\pi_1(M)$ does not induce the RFRp topology on either K_1 or K_2.

7.4. **The vertex manifolds.** We now proceed with the construction of the graph manifolds comprising the class $\mathscr{X}$ described in the introduction. We go over each of the five axioms, and introduce some further notation and terminology along the way.

($\mathscr{X}_1$) Let Γ be a finite, connected, bipartite, simplicial graph such that each vertex has degree at least two. We color the vertices two colors, and we denote the resulting equivalence classes by $\mathscr{L}$ and $\mathscr{P}$. If $v = L \in \mathscr{L}$, we write $\mathscr{P}_L \subset \mathscr{P}$ for the set of vertices which are adjacent to L, and similarly if $v = P \in \mathscr{P}$, we write $\mathscr{L}_P \subset \mathscr{L}$ for the set of vertices which are adjacent to P.

($\mathscr{X}_2$) We build a graph manifold $X = X_\Gamma$ whose underlying graph is Γ as follows. For each vertex $v \in V(\Gamma)$, the vertex manifold X_v is homeomorphic to $S^1 \times S_v$, where S^1 is the circle, and where

$$S_v = S \setminus \bigcup_{i=1}^{m(v)} D_i^2, \tag{26}$$

where S is a closed, orientable surface, and where $\{D^2\}_{i=1}^{m(v)}$ denotes a disjoint union of $m(v)$ open disks. Thus, $\pi_1(X_v) \cong \mathbb{Z} \times F_{k(v)}$, where $F_{k(v)}$ denotes the free group of rank $k(v)$, a number which depends on $m(v)$ and on the genus of S. As part of axioms ($\mathscr{X}_3$) and ($\mathscr{X}_4$), will make the following assumptions on the graph Γ:

($\mathscr{X}_3'$) If $v = L \in \mathscr{L}$, then $m(v) \geqslant \deg v + 1$.

($\mathscr{X}_4'$) If $v = P \in \mathscr{P}$, then $m(v) = \deg v$.

In the first case, the boundary components of S_v will be denoted by

$$\{C_{L,P}\}_{P\in\mathscr{P}_L} \cup \{C_L^1, C_L^2, \dots, C_L^{r(L)}\}, \tag{27}$$

where $C_{L,P}$ corresponds to the edge $\{L, P\}$, and where $r(L) = m(L) - \deg(L)$. In the second case, the boundary components of S_v will be denoted by

$$\{C_{P,L}\}_{L\in\mathscr{L}_P}, \tag{28}$$

where $C_{P,L}$ corresponds to the edge $\{P, L\}$. The homology class of $C_{L,P}$ and $C_{P,L}$ in $H_1(S_v, \mathbb{Z})$ will be written $b_{L,P}$ and $b_{P,L}$ respectively, and the homology classes of $\{C_L^i\}$ will be written $\{b_L^i\}$.

7.5. **Euler numbers.** Next, we explain the role played by the Euler numbers in axioms ($\mathscr{X}_3$) and ($\mathscr{X}_4$). We start with a simple, motivating example.

Example 7.3. Suppose X is the exterior of an two-component Hopf link in S^3. Then X fibers overs the circle, with fiber an annulus, and with monodromy a Dehn twist around the core of the annulus. Alternatively, the Hopf fibration $S^3 \to S^2$ restricts to a fibration of X over $S^1 \times I$. Since the Hopf fibration has Euler class one, the Euler number of the Seifert manifold X is also one.

It follows that X is homeomorphic to a circle bundle over the annulus, but there is no trivialization preserving the annulus. We have that the group $H_1(X, \mathbb{Z})$ is generated by the homology class t of the fiber, together with the two boundary homology classes b_1, b_2 of the annulus, with the relation $t = b_1 + b_2$ corresponding to an Euler number of one. The homology classes b_1 and b_2 of the boundary components of the annulus are homologous in the annulus itself, but not in X.

We now return to an arbitrary graph manifold $X \in \mathscr{X}$. By assumption, each vertex manifold X_v is a Seifert manifold, since it is a circle bundle over an orientable surface with boundary with trivial monodromy, and thus homeomorphic to a product bundle. Yet, as we saw in the above example, the trivialization may not preserve the underlying surface, and the discussion of Euler numbers below reflects this fact.

We will write t_v for the homology generator of the S^1 factor of X_v, and B_V for the total homology span of the boundary components of $\{S_v\}_{v\in V(\Gamma)}$ inside of X_v. We then have that $H_1(X_v, \mathbb{Z})$ is a free abelian group, and the homology classes in $\langle t_v, B_v\rangle$ satisfy a single linear relation determined by the *Euler number* of the corresponding Seifert manifold. By definition, we take this Euler number, $e(v)$, to be the coefficient of t_v in this linear relation. In the terminology of Luecke and Wu [30], the integer $e(v)$ is the relative Euler number of X_v with respect to the chosen framing of ∂X_v, to wit, the curves in ∂X_v corresponding to the curves in ∂S_v specified in (27) and (28) under the homeomorphism $X_v \cong S^1 \times S_v$. In turn, this number coincides with the (orbifold) Euler number of the (closed) Seifert manifold obtained by filling in the boundary tori of X_v with solid tori, while matching the framing of ∂X_v with the meridians of these solid tori.

As the second part of axioms ($\mathscr{X}_3$) and ($\mathscr{X}_4$), we will make the following assumptions on the integers $e(v)$.

($\mathscr{X}_3''$) If $v = L$, we will assume that $e(v) = 0$, so that the relation reads

$$\sum_{P \in \mathscr{P}_L} b_{L,P} + \sum_{j=1}^{r(L)} b_L^j = 0. \tag{29}$$

($\mathscr{X}_4''$) If $v = P$, we will assume that $e(v) \neq 0$, so that the relation reads

$$\sum_{L \in \mathscr{L}_P} b_{P,L} = k_P \cdot t_P, \tag{30}$$

where $k_P = e(P)$ is a nonzero integer.

7.6. **Gluing maps.** Finally, we need to define the gluing maps which allow us to assemble our class $\mathscr{X}$ of graph manifolds. If $e = \{L, P\}$ forms an edge in Γ, axiom ($\mathscr{X}_5$) requires that we glue X_L to X_P via a *flip map*. That is to say, we choose homeomorphisms $\psi_e \colon S^1 \to C_{P,L}$ and $\bar{\psi}_e \colon C_{L,P} \to S^1$, and we glue X_L to X_P along $X_e \cong S^1 \times S^1$ via the homeomorphism

$$\psi_e \times \bar{\psi}_e \colon S^1 \times C_{L,P} \to C_{P,L} \times S^1. \tag{31}$$

Put another way, the chosen homeomorphisms of the vertex manifolds X_v with $S^1 \times S_v$ determine meridian-longitude pairs on each torus $X_{v,e}$, for each edge e incident to v. Given an edge $e = \{v, w\}$, we glue X_v to X_w by identifying the tori $X_{v,e}$ and $X_{w,e}$ via a diffeomorphism represented in the aforementioned basis by the matrix $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. For more information on this procedure, we refer to [15].

In this way, we have assembled a compact, connected, orientable graph manifold $X = X_\Gamma$. This completes the description of our class $\mathscr{X}$ of graph manifolds.

8. The boundary manifold of a plane algebraic curve

Let $\mathscr{X}$ be the class of all graph manifolds obtained by the procedure detailed in the previous section. Our next objective is to prove Theorem 1.5 from the introduction, which states that the fundamental groups of manifolds in this class enjoy the RFRp property, for all primes p. Before proceeding with the proof, we motivate our result by showing that certain 3-manifolds occurring in the topological study of complex plane algebraic curves belong to the class $\mathscr{X}$.

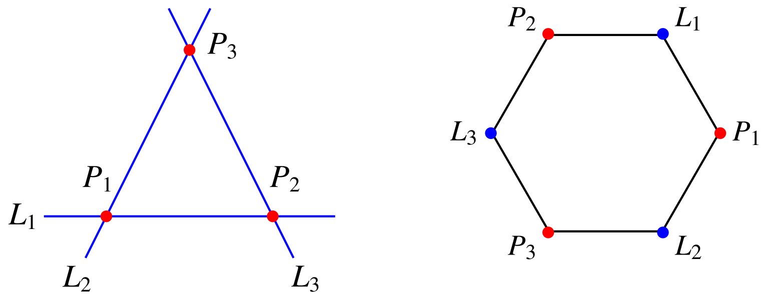

FIGURE 2. A generic arrangement of 3 lines and its graph

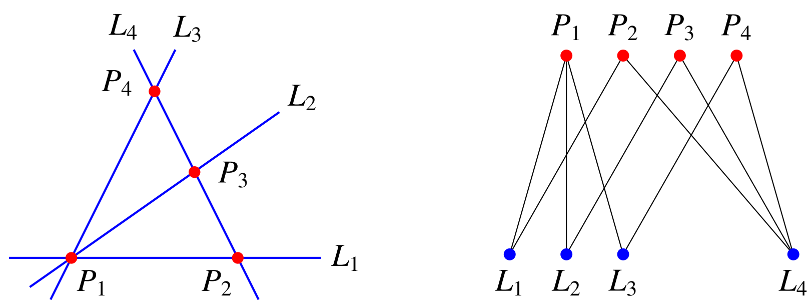

FIGURE 3. A near-pencil of lines and its associated graph

8.1. **Projective algebraic curves.** We refer the reader to [9] for background on the material in this subsection. Let $\mathscr{C}$ be an algebraic curve in the complex projective plane $\mathbb{CP}^2$, that is, the zero-locus of a homogeneous polynomial $f \in \mathbb{C}[x, y, z]$. Without essential loss of generality, we may assume $\mathscr{C}$ is reduced, i.e., f has no repeated factors. By definition, the degree of $\mathscr{C}$ is the degree of its defining polynomial f (which is uniquely defined, up to constants).

Let T be a regular neighborhood of $\mathscr{C}$, and let $M_{\mathscr{C}} = \partial T$ be its boundary. Then $M_{\mathscr{C}}$ is a closed, orientable 3-manifold, called the *boundary manifold* of the curve $\mathscr{C}$. As shown by Durfee in [16], the homeomorphism type of $M_{\mathscr{C}}$ is independent of the choices made in constructing the regular neighborhood T, and depends only on $\mathscr{C}$.

We will mainly be interested in the case when each irreducible component C is smooth, and all the singularities of $\mathscr{C}$ are simple, that is, any two distinct components intersect transversely. Here are a couple of well-known examples.

Example 8.1. Suppose $\mathscr{C}$ has a single irreducible component C, which we assume to be smooth. Then C is homeomorphic to an orientable surface Σ_g of genus $g = \binom{d-1}{2}$, where d is the degree of C. Moreover, by Bézout's theorem, $C \cdot C = d^2$. Thus, $M_{\mathscr{C}}$ is a circle bundle over Σ_g with Euler number $e = d^2$.

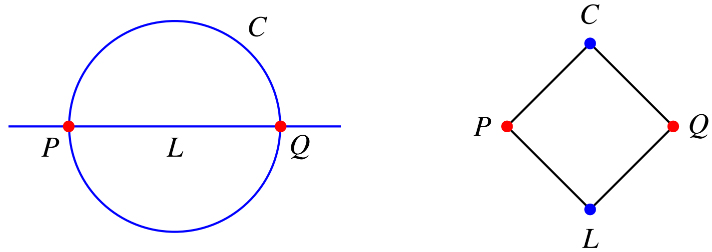

FIGURE 4. A conic-line arrangement and its intersection graph

Example 8.2. Suppose $\mathscr{C}$ is a pencil of n lines in $\mathbb{CP}^2$, defined by the polynomial $f = z_1^n - z_2^n$. Then $M_{\mathscr{C}} = \sharp^{n-1} S^1 \times S^2$; in particular, if $n = 1$ (the case $d = 1$ in the previous example), then $M_{\mathscr{C}} = S^3$.

Example 8.3. Suppose $\mathscr{C}$ is a near-pencil of n lines in $\mathbb{CP}^2$, defined by the polynomial $f = z_1(z_2^{n-1} - z_3^{n-1})$, then $M_{\mathscr{C}} = S^1 \times \Sigma_{n-2}$. The case $n = 3$ (for which $M_{\mathscr{C}}$ is the 3-torus) is depicted in Figure 2, while the case $n = 4$ is depicted in Figure 3.

Note that in the first example the group $\pi_1(M_{\mathscr{C}})$ is not RFRp, for any prime p, provided $d \geqslant 2$ (cf. Theorem 4.4), while in the second and third examples $\pi_1(M_{\mathscr{C}})$ is RFRp for all primes p.

It turns out that the boundary manifold of a plane algebraic curve is a graph manifold. This structure can be described in terms of Neumann's plumbing calculus [32]. We refer to [14, 17] for a detailed exposition of the subject, and to [24, 22, 11] for a more specific description in the case of line arrangements. Let us briefly review this construction, in the special context we consider here.

Given an algebraic curve $\mathscr{C} \subset \mathbb{CP}^2$, let $\mathscr{L}$ be the set of irreducible components, and let $\mathscr{P}$ be the set of multiple points, i.e., the set of points $P \in \mathbb{CP}^2$ where at least two distinct curves from $\mathscr{L}$ intersect. The underlying graph Γ is the incidence graph of the arrangement of irreducible curves comprising $\mathscr{L}$. The graph has vertex set $\mathscr{L} \cup \mathscr{P}$, and has an edge joining C to P precisely when C contains P (see Figures 2, 3, and 4). The case when the curve $\mathscr{C}$ is irreducible (and smooth) was treated in Example 8.1. So let us assume that $|\mathscr{L}| \geqslant 2$; in particular, $|\mathscr{P}| \geqslant 1$, and the graph Γ is bipartite. For each point $P \in \mathscr{P}$, the vertex manifold M_P is the exterior of a Hopf link on as many components as the multiplicity of P. Likewise, for each curve $C \in \mathscr{L}$, the vertex manifold M_C is a circle bundle whose base is C with a number of open 2-disks, one for each multiple point lying on C. Finally, the vertex manifolds are glued my means of flip maps along boundary tori, as specified by the plumbing graph Γ, to produce the boundary manifold $M_{\mathscr{C}} = M_{\Gamma}$.

Example 8.4. Suppose $\mathscr{C} = C \cup L$ consists of a smooth conic and a transverse line, as in Figure 4. The graph Γ is a square, and all vertex manifolds are thickened tori $S^1 \times S^1 \times I$. Following the algorithm from [32, Theorem 5.1], as sketched in [14, Figure 4.2], we see that the boundary manifold $M_{\mathscr{C}}$ is the mapping torus of a Dehn twist, or, alternatively, an S^1-bundle over $S^1 \times S^1$ with Euler number 1. Either description shows that $M_{\mathscr{C}}$ is the Heisenberg nilmanifold. In view of Proposition 2.5 (or, alternatively, Theorem 4.4), we conclude that the group $\pi_1(M_{\mathscr{C}})$ is not RFRp, for any prime p.

8.2. **Affine algebraic curves.** Similar considerations apply to affine, plane algebraic curves. More precisely, let $\mathscr{C}$ be a (reduced) algebraic curve in the affine plane $\mathbb{C}^2$, that is, the zero-locus of a polynomial $f \in \mathbb{C}[x, y]$ with no repeated factors. As before, we will only consider the case when each irreducible component of $\mathscr{C}$ is smooth, and all the singularities of $\mathscr{C}$ are of type A, that is, their germs are isomorphic to a pencil of lines. Furthermore, we shall assume that each irreducible component of $\mathscr{C}$ is transverse to the line at infinity.

Let ∂T be the boundary of a regular neighborhood of $\mathscr{C}$. We define the *boundary manifold* of the curve to be the intersection

$$M_{\mathscr{C}} := \partial T \cap B^4, \tag{32}$$

where B^4 is a ball of sufficiently large radius, so that all singularities of $\mathscr{C}$ are contained in this ball. Clearly, $M_{\mathscr{C}}$ is a smooth, connected, orientable 3-manifold, with boundary components tori $S^1 \times S^1$ in one-to-one correspondence with the irreducible components of $\mathscr{C}$.

Example 8.5. Suppose $\mathscr{C}$ has a single (smooth) irreducible component of degree d. Let $\overline{M}$ be the S^1-bundle with Euler number d^2 over the Riemann surface of genus $\binom{d-1}{2}$ from Example 8.1. The boundary manifold $M_{\mathscr{C}}$, then, is obtained by removing open tubular neighborhoods of d fibers of this bundle. Consequently, $\pi_1(M_{\mathscr{C}})$ is isomorphic to $\mathbb{Z} \times F_{(d-1)^2}$, and thus is RFR$p$, for all primes p.

We previously defined a class $\mathscr{X}$ of compact graph manifolds M for which the underlying graph Γ is connected, bipartite, and each vertex in one of the parts has degree at least two, such that each vertex manifold is a trivial circle bundle over an orientable surface with boundary obeying some technical conditions on the framings, and such that all the gluing maps are given by flips. The main result of this section shows that all boundary manifolds arising from this construction belong to this class.

Theorem 8.6. *Let $\mathscr{C}$ be a plane algebraic curve such that*

(1) Each irreducible component of $\mathscr{C}$ is smooth and transverse to the line at infinity.
(2) Each singular point of $\mathscr{C}$ is a type A *singularity.*

Then the boundary manifold $M_{\mathscr{C}}$ lies in $\mathscr{X}$.

Proof. We start by describing the underlying graph Γ of the graph-manifold $M_\Gamma = M_{\mathscr{C}}$. Let $\mathscr{L}$ be the set of irreducible components of $\mathscr{C}$, and let $\mathscr{P}$ be the set of multiple points of $\mathscr{C}$, i.e., the set of points P in $\mathbb{C}^2$ where at least two distinct curves from $\mathscr{L}$ meet. By our assumption on the singularities of $\mathscr{C}$, if L_1 and L_2 are two distinct components of $\mathscr{C}$ meeting at a point $P \in \mathscr{P}$, then L_1 and L_2 intersect transversely at P.

The graph Γ is the incidence graph of the resulting point–line configuration. This graph has vertex set $V(\Gamma) = \mathscr{L} \cup \mathscr{P}$ and edge set

$$E(\Gamma) = \{(L, P) \in \mathscr{L} \times \mathscr{P} \mid P \in L\}. \tag{33}$$

(See Figures 2, 3, and 4 for some illustrations.) Note that the link of a vertex P is $\mathscr{L}_P = \{L \in \mathscr{L} \mid P \in L\}$, whereas the link of a vertex L is $\mathscr{P}_L = \{P \in \mathscr{P} \mid P \in L\}$. In view of our assumptions, we have that $|\mathscr{L}_P| \geqslant 2$ and $|\mathscr{P}_L| \geqslant 1$, for all P and L. It follows that Γ is a connected, bipartite graph, and each vertex $P \in \mathscr{P}$ has degree at least two. Thus, axiom $(\mathscr{X}_1)$ is satisfied by the graph Γ.

By assumption, each component $L \in \mathscr{C}$ is a smooth, irreducible curve, transverse to the line at infinity. Let d be the degree of L. Then L can be viewed as an (orientable) Riemann surface of genus $g = \binom{d-1}{2}$, with d punctures corresponding to the points where L meets the line at infinity. By construction, the vertex manifold M_L is the boundary of a tubular neighborhood of L inside $\mathbb{C}^2$, intersected with a ball centered at 0 and containing all the points in $\mathscr{P}$. As the normal bundle of L is trivial, the vertex manifold M_L is homeomorphic to the product of S^1 with a copy of L from which several open disks (one for each point $P \in \mathscr{P}_L$) have been removed. It follows that $e(M_L) = 0$, and so axioms $(\mathscr{X}_2)$ and $(\mathscr{X}_3)$ hold for the vertex manifold M_L.

Likewise, each intersection point $P \in \mathscr{P}$ is a singularity of type A, and so its singularity link is the Hopf link on $|\mathscr{L}_P|$ components. Consequently, the vertex manifold M_P is the exterior of this link, and thus homeomorphic to $S^1 \times S_P$, where S_P is a sphere with a number of disks removed (one for each $L \in \mathscr{L}_P$). The idea outlined in Example 7.3 shows that the Euler number $e(M_P)$ is equal to one. Thus, axioms $(\mathscr{X}_2)$ and $(\mathscr{X}_4)$ hold for the vertex manifold M_P.

Finally, as shown in the aforementioned references, the vertex manifolds are glued along tori via flip maps, in a manner specified by the edges of the plumbing graph Γ. Thus, axiom $(\mathscr{X}_5)$ is verified, and this completes the proof. ◻

We single out an immediate corollary, for future use.

Corollary 8.7. *Let $\mathscr{A}$ be an arrangement of lines in $\mathbb{C}^2$. Then the boundary manifold $M_{\mathscr{A}}$ lies in $\mathscr{X}$.*

9. Applying the combination theorem to boundary manifolds

This section is devoted to proving Theorem 1.5 from the introduction, using Theorem 6.1 as the main tool.

9.1. **A Mayer–Vietoris sequence.** In an arbitrary graph manifold X, even within the class $\mathscr{X}$ we have defined previously, it is generally still not true that the inclusion $X_v \to X$ of a vertex manifold induces an injection $H_1(X_v, \mathbb{Z}) \to H_1(X, \mathbb{Z})$. We will now give conditions on the graph Γ which will guarantee that the inclusion of a vertex manifold induces an injection on first homology, for graph manifolds within the class $\mathscr{X}$.

First observe that the Mayer–Vietoris sequence for $X = X_\Gamma$ implies that

$$H_1(X, \mathbb{Z}) \cong H_V \oplus \mathbb{Z}^{b_1(\Gamma)}, \tag{34}$$

where H_V is the image of the map induced on homology by the inclusion

$$\coprod_{v \in V(\Gamma)} X_v \to X. \tag{35}$$

Recall we are assuming that each vertex manifold X_v is homeomorphic to $S^1 \times S_v$, where S_v is obtained by deleting a number of disjoint, open disks from a closed, orientable surface S.

For each vertex v, we will write $B_v \subset H_1(X_v, \mathbb{Z})$ for the subgroup generated by the homology classes of the boundary components of S_v. Specifically, we consider the inclusion $\bigcup_{i=1}^{m(v)} S^1_i \to S_v \subset X_v$ given by sending $S^1_i \to \partial D^2_i$, and we set B_v to be the image of the induced map on first homology. We then have a direct sum decomposition

$$H_1(S_v, \mathbb{Z}) \cong B_v \oplus W_v, \tag{36}$$

where roughly W_v is "generated by the genus" of S_v. Note that when we assemble X, all the gluing maps are performed along boundary components of X_v. The Mayer–Vietoris sequence implies then that

$$H_V \cong \langle B_V, t_v \mid v \in V(\Gamma) \rangle \oplus \Big(\bigoplus_{v \in V(\Gamma)} W_v \Big), \tag{37}$$

where B_V is the total homology span of the boundary components of $\{S_v\}_{v \in V(\Gamma)}$ inside of X.

Recall from §7.4 that for each vertex $L \in \mathscr{L}$, the surface S_L has boundary curves $C_{L,P}$ indexed by $P \in \mathscr{P}_L$, and some extra boundary curves $C^1_L, \ldots, C^{r(L)}_L$. Recall also that we denote the corresponding homology classes in $H_1(S_L, \mathbb{Z})$ by $b_{L,P}$ and b^i_L, respectively. Likewise, for each vertex $P \in \mathscr{P}$, the surface S_P has boundary curves

$C_{P,L}$ indexed by $L \in \mathscr{L}_P$, and the homology classes of these curves are denoted by $b_{P,L}$. With this notation, we have that

$$B_L = \langle \{b_{L,P}\}_{P\in\mathscr{P}_L} \cup \{b_L^1, b_L^2, \ldots, b_L^{r(L)}\}\rangle \text{ and } B_P = \langle \{b_{P,L}\}_{L\in\mathscr{L}_P}\rangle. \tag{38}$$

Recall from §7.5 that, for each vertex v, we denote by t_v the homology generator of the S^1 factor of X_v. For $v = L$, we will write Ξ_L for the subgroup of B_L generated by the classes $b_L^2, \ldots, b_L^{r(L)}$, and we will set

$$\Xi = \bigoplus_{L\in\mathscr{L}} \Xi_L. \tag{39}$$

Observe that if $\deg L = 1$ then $\Xi_L = 0$.

With this notation, the group B_V is the span of the images of $\{b_{L,P}\}$, $\{b_{P,L}\}$, and Ξ, where L and P range over $\mathscr{L}$ and $\mathscr{P}$ respectively. We note that it is immediate from the Mayer–Vietoris sequence that the subgroup Ξ breaks off as a direct summand of B_V.

Lemma 9.1. *Let $X \in \mathscr{X}$. Then there is a finite index subgroup of the abelian group*

$$\langle B_V, t_v \mid_{v\in V(\Gamma)}\rangle/\Xi$$

which is freely generated by the homology classes $\{t_L\}_{L\in\mathscr{L}}$.

Proof. The Mayer–Vietoris sequence for X says that the image of $\langle B_V, t_v \mid_{v\in V(\Gamma)}\rangle$ is a quotient of

$$\bigoplus_{v\in V(\Gamma)} \langle B_v, t_v\rangle, \tag{40}$$

with relations given by the gluing maps $\psi_e \times \bar{\psi}_e$. Let $L, K \in \mathscr{L}_P$. The gluing maps impose the relations

$$b_{L,P} = t_P = b_{K,P}. \tag{41}$$

Similarly, let $P, Q \in \mathscr{P}_L$. The gluing maps impose the relations

$$b_{P,L} = t_L = b_{Q,L}. \tag{42}$$

It follows that $\langle B_V, t_v \mid_{v\in V(\Gamma)}\rangle$ is in fact generated by $\{t_v\}_{v\in V(\Gamma)}$. The only remaining relations come from the nonzero Euler numbers $e(P) = k_P$ of the vertex spaces $\{X_P\}_{P\in\mathscr{P}}$. Recall from (30) that these relations say that

$$k_P \cdot t_P = \sum_{L\in\mathscr{L}_P} t_L. \tag{43}$$

It follows immediately that $\langle B_V, t_v \mid_{v\in V(\Gamma)}\rangle/\Xi$ has a finite index subgroup which is generated by $\{t_L\}_{L\in\mathscr{L}}$, and that no further relations among these generators hold. $\square$

9.2. **Girth and homological injectivity.** Recall that the *girth* of a graph Γ is the length of the shortest non-backtracking loop in Γ.

Theorem 9.2. *Let $X \in \mathscr{X}$, and suppose that the girth of the defining graph Γ is at least six. Then the inclusion $X_v \to X$ induces an injection $H_1(X_v,\mathbb{Z}) \to H_1(X,\mathbb{Z})$.*

Proof. First, suppose that $v = P$. We have that

$$H_1(X_P,\mathbb{Z}) = \Big\langle t_P, \{b_{P,L}\}_{L\in\mathscr{L}_P} \mid k_P \cdot t_P = \sum_{L\in\mathscr{L}_P} b_{P,L} \Big\rangle. \tag{44}$$

In $H_1(X,\mathbb{Z})$, we have that the image of the finite index subgroup of $H_1(X_P,\mathbb{Z})$ generated by $k_P \cdot t_P$ and by $\{b_{P,L}\}_{L\in\mathscr{L}_P}$ is in fact generated by $\{t_L\}_{L\in\mathscr{L}_P}$. These latter elements generate a free abelian group of the same rank as $H_1(X_P,\mathbb{Z})$, by Lemma 9.1. It follows that the inclusion $X_P \to X$ induces an injective map $H_1(X_P,\mathbb{Z}) \to H_1(X,\mathbb{Z})$.

Now, suppose that $v = L$. Consider the finite index subgroup of $H_1(X_L,\mathbb{Z})/\Xi_L$ generated by (the images of) t_L and by $\{k_P \cdot b_{L,P}\}_{P\in\mathscr{P}_L}$. Since $b_{L,P}$ is identified with t_P, the image of $k_P \cdot b_{L,P}$ in $H_1(X,\mathbb{Z})$ is given by

$$\beta_{L,P} := \sum_{K\in\mathscr{L}_P} t_K, \tag{45}$$

as follows from the gluing relations in X.

It suffices to show that the classes t_L and $\{\beta_{L,P}\}_{P\in\mathscr{P}_L}$ are linearly independent in $H_1(X,\mathbb{Z})$. This will establish the theorem, because the inclusion $X_L \to X$ then induces a map $H_1(X_L,\mathbb{Z}) \to H_1(X,\mathbb{Z})$, whose image contains a free abelian subgroup whose rank is the same as that of $H_1(X_L,\mathbb{Z})$, so that this map must be injective.

For each class $\beta_{L,P}$, choose a vertex $K \in \mathscr{L}_P \backslash \{L\}$. Such a vertex exists, because we assumed that the degree of P is at least two. Now let $Q \in \mathscr{P}_L \backslash \{P\}$. Notice that $\mathrm{Lk}(P) \cap \mathrm{Lk}(Q) = \{L\}$, since the girth of Γ is at least six. Therefore, for each $P, Q \in \mathrm{Lk}(L)$, we can find a vertex $K_P \in \mathscr{L}_P \backslash \{L\}$ and $K_Q \in \mathscr{L}_Q \backslash \{L\}$ such that $K_P \neq K_Q$ for $P \neq Q$. Furthermore, in the expressions

$$\beta_{L,P} = \sum_{K\in\mathscr{L}_P} t_K \quad \text{and} \quad \beta_{L,Q} = \sum_{H\in\mathscr{L}_Q} t_H, \tag{46}$$

we have that

$$\{t_K\}_{K\in\mathscr{L}_P} \cap \{t_H\}_{H\in\mathscr{L}_Q} = \{t_L\}, \tag{47}$$

since $\mathscr{L}_P \cap \mathscr{L}_Q = \{L\}$. Thus, the generator t_{K_P} occurs in the expression of $\beta_{L,P}$ and no other class $\beta_{L,Q}$ for $P \neq Q$. Since

$$\langle B_V, t_v \mid_{v\in V(\Gamma)} \rangle / \Xi \tag{48}$$

is virtually freely generated by $\{t_L\}_{L\in\mathscr{L}}$, it follows immediately that the classes t_L and $\{\beta_{L,P}\}_{P\in\mathscr{P}_L}$ are linearly independent. $\square$

9.3. **Primitive lattices.** Before proceeding, we need to establish a couple of lemmas. Let A be a finitely generated abelian group, and let $B < A$ be a subgroup. We say that B is *primitive* if the inclusion $B \to A$ is a split injection.

Lemma 9.3. *Let A, B, C be finitely generated, torsion-free abelian groups, and let $i_A\colon C \to A$ and $i_B\colon C \to B$ be injective maps such that $i_A(C)$ and $i_B(C)$ are primitive. Then the natural copies of A and B in the pushout*

$$P = (A \oplus B)/(i_A(C) = i_B(C))$$

are primitive. Furthermore, P is torsion-free.

Proof. Since $i_B(C) < B$ is primitive, we have that $B \cong i_B(C) \oplus B'$ for some complement B'. Similarly, $A \cong i_A(C) \oplus A'$, so that $A' \cap B' = \{0\}$ in P, and so that $i_A(C)$ and $i_B(C)$ are identified in P via the inverses of i_A and i_B, respectively. We therefore have an isomorphism $P/B' \cong A$. Composing this isomorphism with the canonical projection $P \twoheadrightarrow P/B'$, we obtain an epimorphism $P \twoheadrightarrow A$ which splits the natural inclusion of A into P. Switching the roles of A and B, we have the conclusion of the lemma. It is clear that P is torsion-free. $\square$

Lemma 9.4. *Let A be a finitely generated abelian group, let $B, C < A$ be subgroups with an isomorphism $\phi\colon B \to C$, and let G_ϕ be the abelian HNN extension of A along ϕ, i.e.,*

$$G_\phi \cong \langle A, t \mid [t, A] = 1, C = \phi(B)\rangle.$$

Let $D < A$ be a primitive, torsion-free subgroup such that the inclusion $D \to A$ descends to an injection $D \to G_\phi$. Then $D < G_\phi$ is primitive.

Proof. The kernel of the canonical projection map $A \times \langle t\rangle \to G_\phi$ is the group

$$K = \langle\{\phi(b) - b \mid b \in B\}\rangle. \tag{49}$$

Since the inclusion $D \to A$ projects to an inclusion $D \to G_\phi$, we have that $K \cap D = \{0\}$. Thus, we have that D and K span a subgroup of A isomorphic to $D \oplus K$. We claim that K can be extended to a complement H for D in A, so that the inclusion of D into G_ϕ is split.

Let $T < A$ be the torsion subgroup. We have that D is still primitive in A/T. It is easy to check that the images of K and D in A/T still have trivial intersection. Indeed, suppose $x \in D$ and $y \in K$ differ by a torsion element t of order n, so that $x = y \cdot t$. Then since D is torsion-free, we have that

$$0 \neq x^n = y^n \in K \cap D,$$

a contradiction. Thus, we may find a map $A/T \to D$ which splits the inclusion of D into A, and for which K lies in the kernel. This map factors through G_ϕ, so that the inclusion $D \to G_\phi$ is primitive. $\square$

9.4. **Promoting injectivity to split-injectivity.** We now refine Theorem 9.2 slightly, which will allow us to prove that the class $\mathscr{X}$ has certain desirable closure properties with respect to taking finite covers. Namely, we will now show that under the hypothesis that the girth of the defining graph Γ is at least six, the inclusion $X_v \to X$ induces a split injection on the level of first integral homology.

We wish to show that under certain general conditions, the vertex groups $\{G_v\}$ and the edge groups $\{G_e\}$ in a graph of groups G_Γ split on the level of homology. That is to say, we will give some general conditions under which the inclusion $G_v \to G_\Gamma$ induces a split map $H_1(G_v, \mathbb{Z}) \to H_1(G_\Gamma, \mathbb{Z})$. We will generally assume that the groups $\{G_v\}$ and $\{G_e\}$ all include into G_Γ, and that all these groups are finitely generated.

Theorem 9.5. *Let G_Γ be a graph of groups with vertex groups $\{G_v\}$ and edge groups $\{G_e\}$. Suppose that:*

1. *For each v, the inclusion $G_v \to G_\Gamma$ induces an injection $H_1(G_v, \mathbb{Z}) \to H_1(G_\Gamma, \mathbb{Z})$.*
2. *The group $H_1(G_v, \mathbb{Z})$ is torsion-free for each $v \in V$.*
3. *The inclusion $G_e \to G_v$ induces a split injection $H_1(G_e, Z) \to H_1(G_v, Z)$.*

Then the inclusion $G_v \to G_\Gamma$ induces a split injection $H_1(G_v, \mathbb{Z}) \to H_1(G_\Gamma, \mathbb{Z})$.

Proof. It suffices to compute the abelianization of the group G_Γ, which is described as an iterated amalgamated product and HNN extension via the gluing data specified by the edge groups.

Let $T \subset \Gamma$ be a maximal tree, and let G_T be the associated graph of groups. On the level of homology, an easy induction using the Mayer–Vietoris sequence and Lemma 9.3 implies the conclusion for $H_1(G_v, \mathbb{Z}) < H_1(G_T, \mathbb{Z})$.

The conclusion for G_Γ now follows from Lemma 9.4 and an easy induction on $|E(\Gamma) \backslash E(T)|$. ◻

Corollary 9.6. *Let $X \in \mathscr{X}$ have defining graph Γ of girth at least six. Then for each $v \in V(\Gamma)$, the inclusion $X_v \to X$ induces a split injection $H_1(X_v, \mathbb{Z}) \to H_1(X, \mathbb{Z})$.*

9.5. **Propagating homological injectivity to finite covers.** Let Z have the homotopy type of a finite CW-complex, let p be a prime, and let $Z' \to Z$ be the finite cover classified by the natural map

$$\pi_1(Z) \longrightarrow H_1(Z, \mathbb{Z}/p\mathbb{Z}) \ . \tag{50}$$

We say this cover is the *p-congruence cover* of Z. The cover Z' is the *torsion-free p-congruence cover* of Z if instead we consider the natural map

$$\pi_1(Z) \longrightarrow (\mathrm{TFr}\, H_1(Z, \mathbb{Z})) \otimes \mathbb{Z}/p\mathbb{Z} \ . \tag{51}$$

Let X is a graph of spaces with underlying graph Γ, and let $\Gamma' \to \Gamma$ be a finite p-cover classified by a surjective homomorphism $\pi_1(\Gamma) \to G$, where G is a finite

p-group. We say that $X' \to X$ is a *girth-fixing p-cover* if it is classified by a composition of (surjective) homomorphisms

$$\pi_1(X) \twoheadrightarrow \pi_1(\Gamma) \twoheadrightarrow G\,, \tag{52}$$

where the first map is induced by the collapsing map κ, and where the corresponding cover Γ' of Γ has girth at least six.

Lemma 9.7. *Let $X \in \mathscr{X}$ with underlying graph of girth at least six, let $X' \to X$ be the torsion-free p-congruence cover, and let $X'' \to X'$ be a girth-fixing p-cover. Then $X', X'' \in \mathscr{X}$. Furthermore, the natural inclusion $X''_v \to X''$ of a vertex space induces a split injection on the level of first integral homology.*

Proof. For the first statement, we need only check that X' and X'' satisfy membership criteria for $\mathscr{X}$. Write Γ' and Γ'' for the respective underlying graphs of the natural pulled back graph of spaces structure. The coloring of the graph Γ pulls back to colorings of Γ' and Γ'', and the degrees of vertices of Γ' and Γ'' cannot decrease from those of Γ.

If X_v is a vertex space of X, then each component X'_v of the preimage of X_v in X' is simply the p-congruence cover of X_v, and similarly for X'', as follows from Corollary 9.6. Thus, for each $v \in V(\Gamma)$, the vertex space X_v will be a product of a circle with an orientable surface with boundary.

Let $v = L$. Then it is evident that the zero Euler number relation (29) for X_L pulls back to a zero Euler number relation for X'_L, and that X'_L will have boundary components which are boundary components of both X' and of X''.

Let $v = P$. Then since no boundary component of X_P is a boundary component of X, the same will be true of X'_P. Furthermore, the nonzero Euler number relation (30) simply replaces k_P by a nonzero integer multiple.

The fact that the gluing maps are flips in X immediately implies that the gluing maps are flips in X' and X'', since the covers of the vertex spaces preserve the circle and surface directions. Thus, X' and X'' lie in $\mathscr{X}$.

The second claim of the lemma follows from Corollary 9.6. $\square$

9.6. **Edge groups are closed in the RFRp topology.** Let $X = X_v$ be a trivial circle bundle over a compact, orientable surface with boundary, and let $X_e \subset X$ be a boundary component. We identify $\pi_1(X)$ with $\mathbb{Z} \times F$, where F is a finitely generated free group and where $\mathbb{Z}$ is generated by the circle direction. Furthermore, we identify $\pi_1(X_e)$ with a copy of $\mathbb{Z}^2$ inside of $\mathbb{Z} \times F$. We claim that this subgroup is always closed in the RFRp topology on $\mathbb{Z} \times F$. This fact follows easily from Theorem 3.4, but we give a direct argument which is more elementary:

Lemma 9.8. *Let $\mathbb{Z}^2 < \mathbb{Z} \times F$ be a maximal rank two abelian subgroup and let p be a prime. Then $\mathbb{Z}^2$ is closed in the RFRp topology on $\mathbb{Z} \times F$.*

Proof. We will write t for a generator of the central copy of $\mathbb{Z}$ in $\mathbb{Z} \times F$. It is easy to check that if $\mathbb{Z}^2 < \mathbb{Z} \times F$ is maximal then $t \in \mathbb{Z}^2$. Thus we may suppose that $\mathbb{Z}^2$ is generated by t and by $x \in F$, where $\langle x \rangle$ is a maximal cyclic subgroup of F.

Let $G_1 = \mathbb{Z} \times F$ and let $\{G_i\}_{i \geqslant 1}$ be the standard RFRp filtration on $\mathbb{Z} \times F$. It suffices to show that if $g \notin \mathbb{Z}^2$ then the image of g in G_1/G_i does not coincide with the image of $\mathbb{Z}^2$ in G_1/G_i, for $i \gg 1$.

We may suppose that $g = t^n y$, where $1 \neq y \in F$ is not contained in $\langle x \rangle$. Note that $[x, g] \neq 1$, so that there is some i such that the image of $[x, g]$ is nontrivial in G/G_i. But then the image of g in G/G_i does not commute with x and therefore cannot lie in the image of the abelian group $\mathbb{Z}^2$. ◻

9.7. **Graph manifolds of type $\mathscr{X}$ have the RFRp property.** We are now ready to complete the proof of Theorem 1.5 from the introduction, stating that the fundamental group of a graph manifold which belongs to the class $\mathscr{X}$ defined in §1.4 is RFRp, for all primes p.

Proof of Theorem 1.5. We only need to verify the hypotheses of Theorem 6.1. Let $X \in \mathscr{X}$, and let $\{X_i\}_{i \geqslant 1}$ be the standard RFRp tower of X, as usual. We refine the tower $\{X_i\}_{i \geqslant 1}$ slightly to a tower $\{Y_i\}_{i \geqslant 1}$ by taking an intermediate girth-fixing p-cover at each stage. Namely, we first let $Y_1 \to X_1$ be an arbitrary girth-fixing p-cover. In general, define $Z_{i+1} \to Y_i$ to be the usual torsion-free p-congruence cover of Y_i, and let $Y_{i+1} \to Z_{i+1}$ be an arbitrary girth-fixing p-cover. It is easy to check that for $j \gg i$, we have $\pi_1(X_j) < \pi_1(Y_i)$.

By Lemma 9.7, we have that $Y_i \in \mathscr{X}$ for each i, and for each vertex space $Y_{v,i}$ of Y_i, the inclusion map induces a split injection on the level of homology. But then it follows immediately that $\{Y_{v,i}\}_{i \geqslant 1}$ is equal to the usual RFRp tower for X_v, where X_v is a vertex space of X covered by each level in the tower $\{Y_{v,i}\}_{i \geqslant 1}$. We then have that the RFRp topology on $\pi_1(X)$ induces the RFRp topology on $\pi_1(X_v)$, for each vertex subspace $X_v \subset X$.

Thus, we need only see that the edge spaces are closed in the RFRp topology. This follows immediately from the topological description of the edge spaces and Lemma 9.8. ◻

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