Quotients of surface groups and homology of finite covers via quantum representations

THOMAS KOBERDA AND RAMANUJAN SANTHAROUBANE

Abstract. We prove that for each sufficiently complicated orientable surface S, there exists an infinite image linear representation ρ of $\pi_1(S)$ such that if $\gamma \in \pi_1(S)$ is freely homotopic to a simple closed curve on S, then $\rho(\gamma)$ has finite order. Furthermore, we prove that given a sufficiently complicated orientable surface S, there exists a regular finite cover $S' \to S$ such that $H_1(S', \mathbb{Z})$ is not generated by lifts of simple closed curves on S, and we give a lower bound estimate on the index of the subgroup generated by lifts of simple closed curves. We thus answer two questions posed by Looijenga, and independently by Kent, Kisin, Marché, and McMullen. The construction of these representations and covers relies on quantum SO(3) representations of mapping class groups.

1. Introduction

1.1. **Main results.** Let $S = S_{g,n}$ be an orientable surface of genus $g \geq 0$ and $n \geq 0$ boundary components, which we denote by $S_{g,n}$. A *simple closed curve* on S is an essential embedding of the circle S^1 into S. We will call an element $1 \neq g \in \pi_1(S)$ *simple* if there is a simple representative in its conjugacy class. Note that contrary to a common convention, we are declaring curves which are freely homotopic to boundary components to be simple.

Our main result is the following:

Theorem 1.1. *Let $S = S_{g,n}$ be a surface of genus g and with n boundary components, excluding the (g, n) pairs $\{(0, 0), (0, 1), (0, 2), (1, 0)\}$. There exists a linear representation*

$$\rho\colon \pi_1(S) \to \mathrm{GL}_d(\mathbb{C})$$

such that:

(1) *The image of ρ is infinite.*
(2) *If $g \in \pi_1(S)$ is simple then $\rho(g)$ has finite order.*

Thus, Theorem 1.1 holds for all surface groups except for those which are abelian, in which case the result obviously does not hold. In Subsection 4.4, we will give an estimate on the dimension of the representation ρ.

Date: January 20, 2016.

We will show that the representation ρ we produce in fact contains a nonabelian free group in its image (see Corollary 4.4, cf. [10]).

If $S' \to S$ is a finite covering space, we define the subspace

$$H_1^s(S', \mathbb{Z}) \subseteq H_1(S', \mathbb{Z})$$

to be *simple loop homology* of S'. Precisely, let $g \in \pi_1(S)$ be simple, and let $n(g)$ be the smallest positive integer such that $g^{n(g)}$ lifts to S'. Then

$$H_1^s(S', \mathbb{Z}) := \langle [g^{n(g)}] \mid g \in \pi_1(S) \text{ simple} \rangle \subseteq H_1(S', \mathbb{Z}),$$

where $[g^{n(g)}]$ denotes the homology class of $g^{n(g)}$ in S'. It is easy to check that $H_1^s(S, \mathbb{Z}) = H_1(S, \mathbb{Z})$.

Identifying $\pi_1(S')$ with a subgroup of $\pi_1(S)$, we write $\pi_1^s(S')$ for the *simple loop subgroup* of $\pi_1(S')$, which is generated by elements of the form $g^{n(g)}$. Here again, g ranges over simple elements of $\pi_1(S)$. Observe that $H_1^s(S', \mathbb{Z})$ is exactly the image of $\pi_1^s(S')$ inside of $H_1(S', \mathbb{Z})$.

We obtain the following result as a corollary to Theorem 1.1:

Theorem 1.2. *Let $S = S_{g,n}$ be a genus g surface with n boundary components, excluding the (g, n) pairs $\{(0, 0), (0, 1), (0, 2), (1, 0)\}$. Then there exists a finite cover $S' \to S$ such that*

$$H_1^s(S', \mathbb{Z}) \subsetneq H_1(S', \mathbb{Z}),$$

i.e. the simple loop homology of S' is properly contained in the full homology of S'.

Again, Theorem 1.2 holds for all nonabelian surface groups, and in the abelian case the result obviously cannot hold. We remark that I. Irmer has proposed a version of Theorem 1.2 in [15], and she proves that $H_1^s(S', \mathbb{Z}) = H_1(S', \mathbb{Z})$ whenever the deck group $S' \to S$ is abelian.

We obtain the following immediate corollary from Theorem 1.2:

Corollary 1.3. *Let $S = S_{g,n}$ be a genus g surface with n boundary components, excluding the (g, n) pairs $\{(0, 0), (0, 1), (0, 2), (1, 0)\}$. Then there exists a finite cover $S' \to S$ such that $\pi_1^s(S')$ is properly contained in $\pi_1(S')$.*

The representation ρ in Theorem 1.1 is produced using the quantum SO(3) representations of the mapping class group of S. We then use the Birman Exact Sequence to produce representations of the fundamental group of S. We use integral TQFT representations in order to approximate the representation ρ by finite image representations $\{\rho_k\}_{k \in \mathbb{N}}$ which converge to ρ in some suitable sense. Since the representations $\{\rho_k\}_{k \in \mathbb{N}}$ each have finite image, each such homomorphism classifies a finite cover $S_k \to S$. The cover $S' \to S$ furnished in Theorem 1.2 is any one of the covers $S_k \to S$, where $k \gg 0$.

Thus, the covers coming from integral TQFT representations will produce an infinite sequence of covers for which $H_1^s(S_k, \mathbb{Z})$ is a proper subgroup of $H_1(S_k, \mathbb{Z})$.

It is unclear whether $H_1^s(S_k, \mathbb{Z})$ has finite or infinite index inside of $H_1(S_k, \mathbb{Z})$, or equivalently if Theorem 1.2 holds when integral coefficients are replaced by rational coefficients. However we will show that for a fixed S, the index of $H_1^s(S_k, \mathbb{Z})$ in $H_1(S_k, \mathbb{Z})$ can be arbitrarily large. I. Agol has observed that Theorem 1.2 holds for the surface $S_{0,3}$ even when integral coefficients are replaced by rational coefficients (see [16]). We will show that if Theorem 1.2 holds over $\mathbb{Q}$, then it in fact implies Theorem 1.1: see Proposition 6.1.

1.2. **Notes and references.** The question of whether the homology of a regular finite cover $S' \to S$ is generated by pullbacks of simple closed curves on S appears to be well–known though not well–documented in literature (see for instance [16] and [17], and especially the footnote at the end of the latter). The problem itself is closely related to the congruence subgroup conjecture for mapping class groups ([16], [4], [5]), to the virtually positive first Betti number problem for mapping class groups (see [21], and also [7] for a free group–oriented discussion), and to the study of arithmetic quotients of mapping class groups (see [14]). It appears to have been resistant to various "classical" approaches up to now.

The problem of finding an infinite image linear representation of a surface group in which simple closed curves have finite order has applications to certain arithmetic problems, and the question was posed to the authors by M. Kisin and C. McMullen (see also Questions 5 and 6 of [17]). The existence of such a representation for the free group on two generators is an unpublished result of O. Gabber. Our work recovers Gabber's result, though our representation is somewhat different. In general, the locus of representations

$$X_s \subset \mathcal{R}(\pi_1(S), \mathrm{GL}_d(\mathbb{C}))$$

of the representation variety of $\pi_1(S)$ which have infinite image but under which every simple closed curve on $\pi_1(S)$ has finite image is invariant under the action of $\mathrm{Aut}(\pi_1(S))$. Theorem 1.1 shows that this locus is nonempty, and it may have interesting dynamical properties. For instance, Kisin has asked whether this X_s has any finite orbits under the action of $\mathrm{Aut}(\pi_1(S))$, up to conjugation by elements of $\pi_1(S)$.

At least on a superficial level, Theorem 1.1 is related to the generalized Burnside problem, i.e. whether or not there exist infinite, finitely generated, torsion groups. The classical Selberg's Lemma (see [22] for instance) implies that any finitely generated linear group has a finite index subgroup which is torsion–free, so that a finitely generated, torsion, linear group is finite. In the context of Theorem 1.1, we produce a finitely generated linear group which is not only generated by torsion elements (i.e. the images of finitely many simple closed curves), but the image of every element in the mapping class group orbit of these generators is torsion.

Finally, we note that G. Masbaum used explicit computations of TQFT representations to show that, under certain conditions, the image of quantum representations have an infinite order element (see [18]). This idea was generalized in [1] where Andersen, Masbaum, and Ueno conjectured that a mapping class with a pseudo-Anosov component will have infinite image under a sufficiently deep level of the TQFT representations. In [1], the authors prove their conjecture for a four–times punctured sphere (cf. Theorem 4.1). It turns out that their computation does not imply Theorem 4.1, for rather technical reasons.

Although the papers [18] and [1] do not imply our results, these computations of explicit mapping classes whose images under the TQFT representations are of infinite order is similar in spirit to our work in this paper.

The reader may also consult the work of L. Funar (see [9]) who proved independently (using methods different from those of Masbaum) that, under certain conditions, the images of quantum representations are infinite. Again, Funar's work does not imply our results.

2. Acknowledgements

The authors thank B. Farb, L. Funar, M. Kisin, V. Krushkal, E. Looijenga, J. Marché, G. Masbaum, and C. McMullen for helpful conversations. The authors thank the hospitality of the Matematisches Forschungsinstitut Oberwolfach during the workshop "New Perspectives on the Interplay between Discrete Groups in Low-Dimensional Topology and Arithmetic Lattices", where this research was initiated. The authors are grateful to an anonymous referee for a report which contained many helpful comments.

3. Background

In this section we give a very brief summary of facts we will require from the theory of TQFT representations of mapping class groups. We have included references for the reader to consult, but in the interest of brevity, we have kept the discussion here to a minimum.

3.1. **From representations of mapping class groups to representations of surface groups.** Let S be an oriented surface with or without boundary and let $x_0 \in S$ be a marked point, which we will assume lies in the interior of S. Recall that we can consider two mapping class groups, namely $\mathrm{Mod}(S)$ and $\mathrm{Mod}^1(S) = \mathrm{Mod}^1(S, x_0)$, the usual mapping class group of S and the mapping class group of S preserving the marked point x_0, respectively. By convention, we will require that mapping classes preserve ∂S pointwise.

When the Euler characteristic of S is strictly negative, these two mapping class groups are related by the Birman Exact Sequence (see [2] or [8], for instance):

$$1 \to \pi_1(S, x_0) \to \mathrm{Mod}^1(S) \to \mathrm{Mod}(S) \to 1.$$

Thus, from any representation of $\mathrm{Mod}^1(S)$, we obtain a representation of $\pi_1(S)$ by restriction. The subgroup $\pi_1(S) \cong \pi_1(S, x_0) < \mathrm{Mod}^1(S)$ is called the *point–pushing subgroup* of $\mathrm{Mod}^1(S)$.

When S has a boundary component B, one can consider the *boundary–pushing subgroup* of $\mathrm{Mod}(S)$. There is a natural operation on S which caps off the boundary component B and replaces it with a marked point b, resulting in a surface $\hat{S}$ with one fewer boundary components and one marked point. There is thus a natural map $\mathrm{Mod}(S) \to \mathrm{Mod}^1(\hat{S}, b)$, whose kernel is cyclic and generated by a Dehn twist about B. The boundary–pushing subgroup $\mathrm{BP}(S)$ of $\mathrm{Mod}(S)$ is defined to be the preimage of the point–pushing subgroup of $\mathrm{Mod}^1(\hat{S}, b)$. In general whenever $\hat{S}$ has negative Euler characteristic, we have an exact sequence

$$1 \to \mathbb{Z} \to \mathrm{BP}(S) \to \pi_1(\hat{S}, b) \to 1.$$

The left copy of $\mathbb{Z}$ is central, and this extension is never split if $\hat{S}$ is closed and has negative Euler characteristic. In fact, $\mathrm{BP}(S)$ is isomorphic to the fundamental group of the unit tangent bundle of $\hat{S}$. The reader is again referred to [2] or [8] for more detail.

Lemma 3.1. *Let $\rho\colon \mathrm{Mod}^1(S) \to Q$ be a quotient such that for each Dehn twist $T \in \mathrm{Mod}^1(S)$, we have $\rho(T)$ has finite order in Q. Then for every simple element g in the point–pushing subgroup of $\mathrm{Mod}^1(S)$, we have that $\rho(g)$ has finite order in Q.*

Proof. Let γ be an oriented simple loop in S based at x_0. Identifying γ with a simple or boundary parallel element g of the point–pushing subgroup of $\mathrm{Mod}^1(S)$, we can express g as a product of two Dehn twists thus: let $\gamma_1, \gamma_2 \subset S$ be parallel copies of the loop γ, separated by the marked point x_0 (i.e. one component of $S \setminus \{\gamma_1, \gamma_2\}$ is an annulus containing the marked point x_0). Then the point pushing map about γ is given, up to a sign, by $g = T_{\gamma_1} T_{\gamma_2}^{-1}$. Since γ_1 and γ_2 are disjoint, the corresponding Dehn twists commute with each other. Since $\rho(T_{\gamma_i})$ has finite order for each i, the element $\rho(g)$ has finite order as well. □

We will see in the sequel that if ρ is a TQFT representation of $\mathrm{Mod}^1(S)$, then by Theorem 3.2 below, Lemma 3.1 applies.

3.2. **SO(3)–TQFT representations.** Let $S_{g,n}$ be a genus g closed, oriented surface with n boundary components. The SO(3) topological quantum field theories (TQFTs) take as an input an odd integer $p \geq 3$ and a $2p^{th}$ primitive root of unity. As an output, they give a projective representation

$$\rho_p\colon \mathrm{Mod}^1(S_{g,n}) \to \mathrm{PGL}_d(\mathbb{C}),$$

which moreover depends on certain coloring data (which will be specified later on), and where here the dimension d depends the input data. The notion of a TQFT was introduced by Witten (see [25]). His ideas were based on a physical interpretation of the Jones polynomial involving the Feynman path integral, and the geometric quantization of the 3–dimensional Chern–Simons theory. The first rigorous construction of a TQFT was carried out by Reshetikhin and Turaev, using the category of semisimple representations of the universal enveloping algebra for the quantum Lie algebra $\mathrm{SL}(2)_q$ (see [23] and [24]). We will work in the TQFT constructed by Blanchet, Habegger, Masbaum, and Vogel in [3], wherein an explicit representation associated to a TQFT is constructed using skein theory. Perhaps the most important feature of these representations is the following well–known fact (see [3]):

Theorem 3.2. *Let $T \in \mathrm{Mod}^1(S_{g,n})$ be a Dehn twist about a simple closed curve. Then $\rho_p(T)$ is a finite order element of $\mathrm{PGL}_d(\mathbb{C})$.*

It is verifying that certain mapping classes have infinite order under TQFT representations which is often nontrivial and makes up most of the content of this paper. Here and in Subsection 3.3 we will survey some basic properties and computational methods for TQFT representations which we will require.

One can define a certain cobordism category C of closed surfaces with colored banded points, in which the cobordisms are decorated by uni-trivalent colored banded graphs. The details of this category are not essential to our discussion; for details we direct the reader to [3]. The SO(3)–TQFT is a functor Z_p from the category C to the category of finite dimensional vector spaces over $\mathbb{C}$.

A *banded point* (or an *ribbon point*) on a closed oriented surface is an oriented submanifold which is homeomorphic to the unit interval. If a surface has multiple banded points, we will assume that these intervals are disjoint. A banded point provides a good substitute for a boundary component within a closed surface, and a simple loop on S which encloses a single banded point can be thought of a boundary parallel loop. When one wants to study a surface with boundary from the point of view of TQFTs, one customarily attaches a disk to each boundary component and places a single banded point in the interior of each such disk. The banded points are moreover colored, which is to say equipped with an integer.

By capping off boundary components, we can start with a surface $S_{g,n}$ and produce a closed surface $\hat{S}_{g,n}$ equipped with n colored banded points. We will include a further colored banded point x in the interior of the surface with boundary $S_{g,n}$, which will play the role of a basepoint. We denote by $(\hat{S}_{g,n}, x)$ the closed surface with $n + 1$ colored banded points thus obtained.

Now for $p \geq 3$ odd, the SO(3)–TQFT defines a finite dimensional vector space

$$V_p(\hat{S}_{g,n}, x).$$

For the sake of computations, it is useful to write down an explicit basis for the space $V_p(\hat{S}_{g,n}, x)$.

Denote by y the set of $n+1$ colored banded points on $(\hat{S}_{g,n}, x)$, and by S_g the underlying closed surface without colored banded points. Let $\mathcal{H}$ be a handlebody such that $\partial\mathcal{H} = S_g$, and let G be a uni-trivalent banded graph such that $\mathcal{H}$ retracts to G. We suppose that G meets the boundary of $\mathcal{H}$ exactly at the banded points y and this intersection consists exactly of the degree one ends of G. A p–admissible coloring of G is a coloring, i.e. an assignment of an integer, to each edge of G such that at each degree three vertex v of G, the three (non–negative integer) colors $\{a, b, c\}$ coloring edges meeting at v satisfy the following conditions:

(1) $|a-c| \leq b \leq a+c$;
(2) $a+b+c \leq 2p-4$;
(3) each color lies between 0 and $p-2$;
(4) the color of an edge terminating at a banded point y_i must have the same color as y_i.

To any p-admissible coloring c of G, there is a canonical way to associate an element of the skein module

$$S_{A_p}(\mathcal{H}, (\hat{S}_{g,n}, x)),$$

where here the notation refers to the usual skein module with the indeterminate evaluated at A_p, which in turn is a $2p^{th}$ primitive root of unity. The skein module element is produced by cabling the edges of G by appropriate Jones-Wenzl idempotents (see [3, Section 4] for more detail). If moreover all the colors are required to be even, it turns out that the vectors associated to p–admissible colorings give a basis for $V_p(\hat{S}_{g,n}, x)$ (see [3, Theorem 4.14]).

We can now coarsely sketch the construction of TQFT representations. If $x_0 \in x$, we may contract x down to a point in order to obtain a surface with a marked point $(S_{g,n}, x_0)$. To construct this representation, one takes a mapping class $f \in \mathrm{Mod}^1(S_{g,n}, x_0)$ and one considers the mapping cylinder of f^{-1}. The TQFT functor gives as an output a linear automorphism $\rho_p(f)$ of $V_p(\hat{S}_{g,n}, x)$.

This procedure gives us a projective representation

$$\rho_p : \mathrm{Mod}^1(S_{g,n}, x_0) \to \mathrm{PAut}(V_p(S_{g,n}, x)),$$

since the composition law is well–defined only up to multiplication by a root of unity.

Our primary interest in the representation ρ_p lies in its restriction to the point pushing subgroup of $\mathrm{Mod}^1(S_{g,n}, x_0)$. More precisely, we need to compute explicitly the action of a given element of $\pi_1(S_{g,n}, x_0)$ on the basis of $V_p(\hat{S}_{g,n}, x)$ described above. We will explain how to do these calculations in Subsection 4.1, and some illustrative examples will be given in the proof of Theorem 4.1.

3.3. **Integral TQFT representations.** In this Subsection we follow some of the introductory material of [12] and of [13]. The TQFT representations of mapping class groups discussed in Subsection 3.2 are defined over $\mathbb{C}$ and may not have good integrality properties. In our proof of Theorem 1.2, we will require an "integral refinement" of the TQFT representations which was constructed by Gilmer [11] and Gilmer–Masbaum [12]. These integral TQFT representations have all the properties of general SO(3)–TQFTs which we will require, and in particular Theorem 1.1 holds for them.

To an oriented closed surface S equipped with finitely many colored banded points and an odd prime p (with $p \equiv 3 \pmod 4$) , one can associate a free, finitely generated module over the ring of integers $\mathcal{O}_p = \mathbb{Z}[\zeta_p]$, where ζ_p is a primitive p^{th} root of unity. We will denote this module by $\mathcal{S}_p(S)$. This module is stable by the action of the mapping class group, and moreover tensoring this action with $\mathbb{C}$ gives us the usual SO(3)–TQFT representation.

Most of the intricacies of the construction and the properties of the integral TQFT representations are irrelevant for our purposes, and we therefore direct the reader to the references mentioned above for more detail. We will briefly remark that the construction of the integral TQFT representations is performed using the skein module, so that the computations in integral TQFT are identical to those in the TQFTs described in Subsection 3.2. The feature of these representations which we will require is the following filtration by finite image representations.

Let $h = 1 - \zeta_p$, which is a prime in $\mathbb{Z}[\zeta_p]$. We consider the modules

$$\mathcal{S}_{p,k}(S) = \mathcal{S}_p(S)/h^{k+1}\mathcal{S}_p(S),$$

which are finite abelian groups for each $k \geq 0$. The representation ρ_p has a natural action on $\mathcal{S}_{p,k}(S)$, and we denote the corresponding representation of the mapping class group by $\rho_{p,k}$.

Observe (see [13]) that the natural map

$$\mathcal{O}_p \to \varprojlim \mathcal{O}_p/h^{k+1}\mathcal{O}_p$$

is injective, where here the right hand side is sometimes called the *h–adic completion* of $\mathcal{O}_p$. We immediately see that

$$\bigcap_k \ker \rho_{p,k} = \ker \rho_p.$$

4. Infinite image TQFT representations of surface groups

4.1. **Quantum representations of surface groups.** In this Subsection we describe the key idea of this paper, which is the use of the Birman Exact Sequence (see Subsection 3.1) together with TQFT representations in order to produce exotic representations of surface groups. Using the notation of Subsection 3.2, we have a projective representation

$$\rho_p : \mathrm{Mod}^1(S_{g,n}, x_0) \to \mathrm{PAut}(V_p(\hat{S}_{g,n}, x)).$$

Restriction to the point–pushing subgroup gives a projective representation of the fundamental group of $S_{g,n}$:

$$\rho_p : \pi_1(S_{g,n}, x_0) \to \mathrm{PAut}(V_p(\hat{S}_{g,n}, x)),$$

which is defined whenever the Euler characteristic of $S_{g,n}$ is strictly negative. In order to make the computations more tractable, let us describe this action more precisely. Let

$$\gamma : [0,1] \to S_{g,n}$$

be a loop based at x_0. By the Birman Exact Sequence, γ can be seen as a diffeomorphism f_γ of $(S_{g,n}, x_0)$. Pick a lift $\tilde{f}_\gamma$ of f_γ to the *ribbon mapping class group* of $(S_{g,n}, x)$, i.e. the mapping class group preserving the orientation on the banded point x, together with any coloring data that might be present. Note that any two lifts of f_γ differ by a twist about the banded point x, which is a central element of the ribbon mapping class group. The preimage of the point pushing subgroup inside of the ribbon mapping class group is easily seen to be isomorphic to the boundary pushing subgroup $\mathrm{BP}(S_{g,n+1})$, where the extra boundary component is the boundary of a small neighborhood of the banded point x.

By definition, $\tilde{f}_\gamma^{-1}$ is isotopic to the identity via an isotopy which is allowed to move x. Following the trajectory of the colored banded point x in this isotopy gives a colored banded tangle $\tilde{\gamma}$ inside $S_g \times [0,1]$. Observe that $\tilde{\gamma}$ is just a thickening of the tangle $t \in [0,1] \mapsto (\gamma(1-t), t) \in S_{g,n} \times [0,1]$.

We can naturally form the decorated cobordism $C_{\tilde{\gamma}}$ by considering $S_g \times [0,1]$ equipped with the colored banded tangle

$$(\tilde{\gamma}, a_1 \times [0,1], \ldots, a_n \times [0,1]),$$

where $(a_1, \ldots, a_n)$ are the n colored banded points on $\hat{S}_{g,n}$. By applying the TQFT functor, we obtain an automorphism

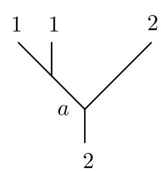
$$\rho_p(\gamma) := \rho_p(\tilde{f}_\gamma) = Z_p(C_{\tilde{\gamma}}).$$

We crucially note that this automorphism generally depends on the choice of the lift $\tilde{f}_\gamma$, but that changing the lift will only result in $\rho_p(\gamma)$ being multiplied by a root of unity, since the Dehn twist about x acts by multiplication by a root of unity. In particular, different choices of lift give rise to the same element $\mathrm{PAut}(V_p(S_{g,n}, x))$. Thus, we do indeed obtain a representation of $\pi_1(S_{g,n})$, and not of the boundary pushing subgroup.

We now briefly describe how to compute the action of the loop γ on the basis of TQFT described in Subsection 3.2. Let $\mathcal{H}$ be a handlebody with $\partial\mathcal{H} = S_g$ and let G be a uni-trivalent banded graph such that $\mathcal{H}$ retracts to G. Let G_c be a coloring of this graph as explained in Subsection 3.2. In the TQFT language, applying $Z_p(C_{\tilde{\gamma}})$ to G_c simply means that we glue the cobordism $C_{\tilde{\gamma}}$ to the handlebody $\mathcal{H}$ equipped with G_c. This gluing operation gives the same handlebody $\mathcal{H}$, but with a different colored banded graph inside. To express this new banded colored graph in terms of the basis mentioned above, we have to use a set of local relations on colored banded graphs, namely a colored version of the Kauffman bracket and the application of Jones–Wenzl idempotents. For the sake of brevity, we refer the reader to the formulas computed by Masbaum and Vogel (see [20]).

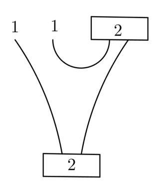
4.2. **The three-holed sphere.** The main technical case needed to establish Theorem 1.1 is the case of the surface $S_{0,3}$, a sphere with three boundary components. Let $\mathrm{Mod}^1(S_{0,3})$ be the mapping class group of $S_{0,3}$, preserving a fourth marked point in the interior of $S_{0,3}$.

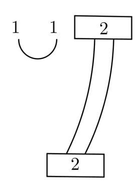
The correct setup for TQFT representations of $\mathrm{Mod}^1(S_{0,3})$ is that of spheres with banded points. Let $(S^2, 1, 1, 2, 2)$ be a sphere equipped with two banded points colored by 1 and two banded points colored by 2 (where one of these two serves as the base point). Let $p \geq 5$ and A_p be a $2p^{th}$ primitive root of unity. According to [3, Theorem 4.14] and the survey given in Subsection 3.2, the vector space $V_p(S^2, 1, 1, 2, 2)$ has a basis described by the following two colored banded graphs

(1) $G_a =$ 

with $a \in \{0, 2\}$,

where these graphs live in the 3–ball whose boundary is $(S^2, 1, 1, 2, 2)$. The computations can be done in this basis as described in Subsection 4.1. In order to simplify the computations, we will work in a different basis. We note that these colored banded graphs can be expanded to banded tangles in the 3-ball using a well–studied procedure (see [20] or [3, section 4]). It is easy to check that the following elements form a basis of $V_p(S^2, 1, 1, 2, 2)$:

$u_1 =$ 

$u_2 =$ 

More precisely,

$$u_1 = \frac{G_0}{A_p^2 + A_p^{-2}} + G_2 \quad \text{and} \quad u_2 = G_0.$$

Here again, the ambient 3–ball in which these tangles live is not drawn. The arcs drawn stand for banded arcs with the blackboard framing. The two end points of the arcs in the top left of the picture are attached to the points colored by 1 on the boundary of the 3–ball. The two rectangles labeled by 2 represent the second Jones–Wenzl idempotent, and these are attached to the two points colored by 2 on the boundary of the 3–ball. The construction and most properties of the Jones–Wenzl idempotents are irrelevant for our purposes, and the interested reader is directed to [3, Section 3].

The TQFT representation of $\mathrm{Mod}^1(S_{0,3})$ furnishes a homomorphism

$$\rho_p\colon\ \mathrm{Mod}^1(S_{0,3}) \to \mathrm{PAut}(V_p(S^2, 1, 1, 2, 2)) \cong \mathrm{PGL}_2(\mathbb{C})$$

in whose image all Dehn twists have finite order by Theorem 3.2, which then by restriction gives us a homomorphism

$$\rho_p\colon \pi_1(S_{0,3}) \to \mathrm{PGL}_2(\mathbb{C})$$

in whose image all simple loops have finite order, by Lemma 3.1.

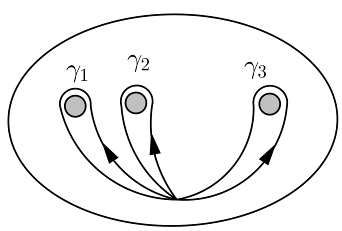
Theorem 4.1. *For all $p \gg 0$, the image of the representation*

$$\rho_p\colon \pi_1(S_{0,3}) \to \mathrm{PGL}_2(\mathbb{C})$$

contains an element of infinite order.

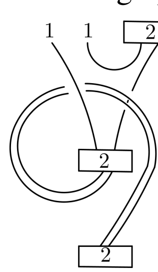
Proof. We can compute the action of $\pi_1(S_{0,3})$ on $V_p(S^2, 1, 1, 2, 2)$ explicitly, and thus find an element of $g \in \pi_1(S_{0,3})$ such that $\rho_p(g)$ has infinite order.

Let $\{\gamma_1, \gamma_2, \gamma_3\}$ be the usual generators of the fundamental group of the three–holed sphere:

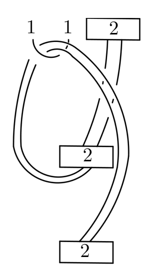


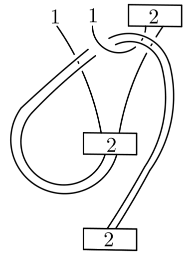
Following the discussion in Subsection 4.1, we can write down matrices for the action of $\rho(\gamma_i)$ for each i. A graphical representation is as follows:

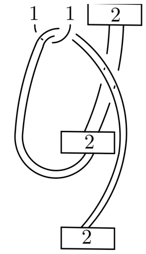
$\rho_p(\gamma_1)u_2 \quad =$



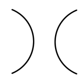
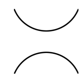
$\rho_p(\gamma_1)u_2 \quad =$

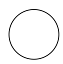


$\rho_p(\gamma_2)u_1 =$ 

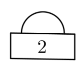
$\rho_p(\gamma_2)u_2 =$ 

We can then reduce these diagrams using the skein relations

 $= A_p$  $+ A_p^{-1}$ 

 $= -A_p^2 - A_p^{-2}$

in order to obtain diagrams without crossing and without trivial circles. We then use the Jones–Wenzl idempotents, and in particular the rule



$= 0,$

in order to simplify the diagrams further. We thus obtain for each diagram a linear combination of the tangles $\{u_1, u_2\}$. One easily checks that we have the following matrices:

$$\rho_p(\gamma_1) = \begin{pmatrix} 1 & A_p^{-10} - A_p^{-2} \\ 0 & A_p^{-12} \end{pmatrix} \quad \rho_p(\gamma_2) = \begin{pmatrix} A_p^{-8} & A_p^2 - A_p^{-6} \\ A_p^{-10} - A_p^{-14} & 1 - A_p^{-8} + A_p^{-12} \end{pmatrix}.$$

Similarly, we can compute

$$\rho_p(\gamma_3) = \begin{pmatrix} A_p^{-8} & 0 \\ -A_p^{-2} + A_p^{-6} & 1 \end{pmatrix}.$$

Here we recall that $p \geq 5$ is an odd integer and A_p is $2p^{th}$ primitive root of unity. Now, one checks that $\mathrm{tr}(\rho_p(\gamma_1)\rho_p(\gamma_2)^{-1}) = A_p^{12} - A_p^4 + 2 - A_p^{-4} + A_p^{-12}$. So we have that

$$|\mathrm{tr}(\rho_p(\gamma_1\gamma_2^{-1}))| \underset{A_p \to e^{\frac{i\pi}{6}}}{\longrightarrow} 5 > 2.$$

If we take a sequence of $2p^{th}$ primitive roots of unity $\{A_p\}$ such that $A_p \to e^{\frac{i\pi}{6}}$ as $p \to \infty$, we see that $|\mathrm{tr}(\rho_p(\gamma_1\gamma_2^{-1}))| > 2$ for $p \gg 0$. Whenever $|\mathrm{tr}(\rho_p(\gamma_1\gamma_2^{-1}))| > 2$, an elementary calculation shows that $\rho_p(\gamma_1\gamma_2^{-1})$ has an eigenvalue which lies off the unit circle. It follows that no power of $\rho_p(\gamma_1\gamma_2^{-1})$ is a scalar matrix, since the determinant of this matrix is itself a root of unity. Thus, $\rho_p(\gamma_1\gamma_2^{-1})$ has infinite order for $p \gg 0$. □

In the proof above, we note that the based loop $\gamma_1\gamma_2^{-1}$ is freely homotopic to a "figure eight" which encircles two punctures, and such a loop is not freely homotopic to a simple loop. In light of the discussion above, Theorem 4.1 could be viewed as giving another proof that $\gamma_1\gamma_2^{-1}$ is in fact not represented by a simple loop.

Corollary 4.2. *Let ρ_p be as above. For $p \gg 0$, the image of ρ_p contains a non-abelian free group.*

Proof. Let $g = \gamma_1\gamma_2^{-1}$ as in the proof of Theorem 4.1, and let $p \gg 0$ be chosen so that $\rho_p(g)$ has infinite order. We have shown that $\rho_p(g)$ in fact admits an eigenvalue which does not lie on the unit circle, so that $\lambda = \rho_p(g) \in \mathrm{PGL}_2(\mathbb{C})$ can be viewed as a loxodromic isometry of hyperbolic 3–space. By a standard Ping–Pong Lemma argument, it suffices to show that there exists a loxodromic isometry μ in the image of ρ_p whose fixed point set on the Riemann sphere $\hat{\mathbb{C}}$ is disjoint from that of λ. Indeed, then sufficiently high powers of λ and μ will generate a free subgroup of $\mathrm{PGL}_2(\mathbb{C})$, which will in fact be a classical Schottky subgroup of $\mathrm{PGL}_2(\mathbb{C})$. See [6], for instance.

To produce μ, one can just conjugate λ by an element in the image of ρ_p. It is easy to check that for $p \gg 0$, the element $\rho_p(\gamma_1) \in \mathrm{PGL}_2(\mathbb{C})$ has two fixed points, namely ∞ and the point

$$z_0 = \frac{A_p^{-2} - A_p^{-10}}{1 - A_p^{-12}},$$

which is distinct from infinity if A_p is not a twelfth root of unity. An easy computation using

$$\lambda = \begin{pmatrix} A_p^{24} - A_p^4 + 2 - A_p^{-4} - A_p^{-8} + A_p^{-12} & -A_p^{14} + A_p^6 - A_p^2 + A_p^{-6} \\ -A_p^{-10} + A_p^{-14} & A_p^{-8} \end{pmatrix}$$

shows that λ does not fix ∞. Similarly, a direct computation shows that λ does not fix z_0. If $p \gg 0$, the isometry $\rho_p(\gamma_1)$ will not preserve the fixed point set of λ setwise, so that we can then conjugate λ by a power of $\rho_p(\gamma_1)$ in order to get the desired μ. □

4.3. **General surfaces.** In this Subsection, we bootstrap Theorem 4.1 in order to establish the main case of Theorem 1.1.

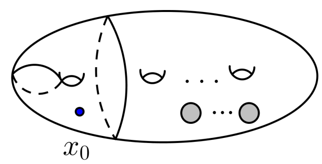
Proof of Theorem 1.1. Retaining the notation of Subsection 4.1, we start with the data of a surface $S_{g,n}$ which is equipped with a marked point x_0 in its interior. We then cap off the boundary components to get $(\hat{S}_{g,n}, x)$, which is a closed surface with $n + 1$ colored banded points. Recall that x is a thickening of the marked point x_0. Suppose furthermore that x is colored by 2.

Restricting the quantum representation of $\mathrm{Mod}^1(S_{g,n}, x_0)$ to the point pushing subgroup gives a representation

$$\rho_p : \pi_1(S_{g,n}, x_0) \to \mathrm{PAut}(V_p(\hat{S}_{g,n}, x))$$

Proving that ρ_p has infinite image can be done by combining Theorem 4.1 with a standard TQFT argument which has already appeared in [18]. For the sake of concreteness, we now give this argument in the case $g \geq 2$, and the other cases covered by Theorem 1.1 can be obtained in a similar fashion. We adopt the standing assumption that all banded points on $\hat{S}_{g,n}$ are colored by 2.

Consider the three-holed sphere inside $S_{g,n}$ whose boundary components are the two curves drawn in the following diagram:



Thus, we can map $\pi_1(S_{0,3})$ into $\pi_1(S_{g,n})$ and get an action of $\pi_1(S_{0,3})$ on $V_p(\hat{S}_{g,n}, x)$. From Theorem [3, Theorem 1.14], we see that this action of $\pi_1(S_{0,3})$ on $V_p(\hat{S}_{g,n}, x)$ contains as a direct summand a vector space $V \otimes W$, where $\pi_1(S_{0,3})$ acts on V as on $V_p(S^2, 1, 1, 2, 2)$ as discussed in Subsection 4.2 and where W is another representation. The conclusion of the theorem follows from observation that some element of $\pi_1(S_{0,3})$ acts with infinite order on

$$V = V_p(S^2, 1, 1, 2, 2)$$

for $p \gg 0$ by Theorem 4.1, and thus this element also acts with infinite order on $V \otimes W$. □

We briefly remark that, as mentioned in Subsection 3.3, Theorem 1.1 also holds for integral TQFT representations, with the same proof carrying over. Thus we have:

Corollary 4.3. *Let p be an odd prime, let ρ_p be the associated integral TQFT representation of $\mathrm{Mod}^1(S)$, and let $\pi_1(S) < \mathrm{Mod}^1(S)$ be the point–pushing subgroup. Then for all $p \gg 0$, we have that:*

1. *The representation ρ_p has infinite image.*
2. *If $g \in \pi_1(S)$ is simple or boundary parallel then $\rho_p(g)$ has finite order.*

A direct consequence of Corollary 4.2 and the splitting argument in TQFT used in the proof of Theorem 1.1 is the following fact:

Corollary 4.4. *The image of the representation*

$$\rho_p : \pi_1(S_{g,n}) \to PAut(V_p(\hat{S}_{g,n}, x))$$

contains a nonabelian free group for $p \gg 0$ as soon as $\pi_1(S_{g,n})$ is not abelian.

4.4. **Dimensions of the representations.** In this subsection, we give a quick and coarse estimate on the dimension of the representation ρ in Theorem 1.1. It suffices to estimate the dimension of the TQFT representation ρ_p of $\mathrm{Mod}^1(S)$ which restricts to an infinite image representation of $\pi_1(S)$, and then embed the corresponding projective general linear group in a general linear group.

The dimension of the space $V_p(S, x)$ is given by a *Verlinde formula*, which simply counts the number of even p-admissible colorings of trivalent ribbon graphs in a handlebody bounded by S, as sketched in Subsection 3.2. In principle, it is possible to compute the dimension of $V_p(S, x)$, though a closed formula is often quite complicated. See [3] and [12] for instance.

We note that, fixing p, the dimension of $V_p(S, x)$ grows exponentially in the genus of S, since in the proof of Theorem 1.1 we have that the color assigned to each boundary component of S is 2.

The argument for Theorem 1.1 furnishes one p which works for all genera, since the infinitude of the image of ρ_p is proven by considering the restriction of ρ_p to a certain three–holed sphere inside of S. It follows that the target dimension of

$$\rho_p \colon \pi_1(S) \to \mathrm{PGL}_d(\mathbb{C})$$

is $d \sim C^g$ for some constant $C > 1$. Since $\mathrm{PGL}_d(\mathbb{C})$ can be embedded in $\mathrm{GL}_{d^2}(\mathbb{C})$ using the adjoint action, we obtain the following consequence:

Corollary 4.5. *There is a constant $C > 1$ such that if $S = S_g$ is a closed surface of genus g, there is a representation ρ of $\pi_1(S)$ satisfying the conclusions of Theorem 1.1, of dimension bounded by C^g.*

5. Homology and finite covers

In this section we use integral TQFT representations to prove Theorem 1.2 and Corollary 1.3.

5.1. **From infinite image representations to finite covers.** Before using integral TQFT to establish Theorem 1.2 we show how from any projective representation $\rho : \pi_1(S) \to \mathrm{PGL}_d(\mathbb{C})$ satisfying the conclusions of Theorem 1.1 we can build a covering $S' \to S$ satisfying the conclusions of Corollary 1.3. Thus, any representation $\rho : \pi_1(S) \to \mathrm{PGL}_d(\mathbb{C})$ satisfying the conclusions of Theorem 1.1, even one not coming from TQFTs, already gives a somewhat counterintuitive result. More precisely:

Theorem 5.1. *Let $\rho : \pi_1(S) \to \mathrm{PGL}_d(\mathbb{C})$ be a projective representation such that*

(1) The image of ρ is infinite.
(2) If $g \in \pi_1(S)$ is simple then $\rho(g)$ has finite order.

Then there exists a finite cover $S' \to S$ such that $\pi_1^s(S')$ is an infinite index subgroup of $\pi_1(S')$.

Proof. Let $\rho : \pi_1(S) \to \mathrm{PGL}_d(\mathbb{C})$ as in the statement of the present theorem. The image of ρ is an infinite, finitely generated, linear group. Using Selberg's Lemma, we can find a finite index torsion–free subgroup $H \lhd \rho(\pi_1(S))$. So $H' = \rho^{-1}(H) \lhd \pi_1(S)$ is a finite index subgroup which classifies a finite cover $S' \to S$.

Now let $g \in \pi_1(S)$ be simple and let $n(g)$ be the smallest integer such that $g^{n(g)} \in H'$. We note that the element

$$\rho(g^{n(g)}) \in \rho(H') = H$$

is torsion since $\rho(g)$ has finite order, which forces $\rho(g^{n(g)}) = 1$, since H is torsion–free. On the one hand, we have that $n(g)$ is precisely the order of $\rho(g)$, and that $\pi_1^s(S') < \ker\rho$. On the other hand $H' = \rho^{-1}(H)$ so $\ker\rho < H'$. But ρ has infinite image, so that $\ker\rho$ is an infinite index subgroup of $\pi_1(S)$, from which we can conclude that $\pi_1^s(S')$ is an infinite index subgroup of $H' = \pi_1(S')$. □

5.2. **Homology of finite covers.** Let $p \geq 7$ be an odd integer and let $S = S_{g,n}$ be compact surface as in the statement of Theorem 1.2. Using the notation of Subsection 4.1 we consider the representation

$$\rho_p : \pi_1(S) \to \mathrm{PAut}(V_p(\hat{S}_{g,n}, x)).$$

which depends on a $2p^{th}$ primitive root of unity A_p. We suppose, as in the proof of Theorem 1.1, that all the banded points on $(\hat{S}_{g,n}, x)$ are colored by 2. For compactness of notation, the space $V_p(\hat{S}_{g,n}, x)$ will be denoted by $V_p(S)$. By Theorem 1.1, we may take p such that ρ_p has infinite image. In particular, $R := \ker\rho_p$ is an infinite index subgroup of $\pi_1(S)$.

Now we suppose that p is a prime number such that $p \equiv 3 \pmod 4$ and we define $\zeta_p = A_p^2$ which is a p^{th} root of unity. The integral TQFT as described in Subsection 3.3 defines a representation

$$\rho_p : \pi_1(S) \to \mathrm{PAut}(\mathcal{S}_p(S)),$$

where $\mathcal{S}_p(S)$ is a free $\mathbb{Z}[\zeta_p]$–module of finite dimension d_p. If $k \geq 0$ is an integer, we can consider the representations

$$\rho_{p,k} : \pi_1(S) \to \mathrm{PAut}(\mathcal{S}_p(S)/h^{k+1}\mathcal{S}_p(S)),$$

where here $h = 1 - \zeta_p$. Since the abelian groups $\mathcal{S}_p(S)/h^{k+1}\mathcal{S}_p(S)$ are finite, the groups $R_k := \ker\rho_{p,k}$ are finite index subgroups of $\pi_1(S)$.

Let $\mathcal{D}$ be the normal subgroup of $\pi_1(S)$ generated by

$$\{g^{n(g)} \mid g \in \pi_1(S) \text{ simple and } n(g) \text{ the order of } \rho_p(g)\}.$$

Note that since simple elements of $\pi_1(S)$ have finite order image under ρ_p, the value of $n(g)$ is always finite, so that the definition of $\mathcal{D}$ makes sense.

Similarly for $k \geq 0$ let $\mathcal{D}_k$ be the normal subgroup generated by

$$\{g^{n(g,k)} \mid g \in \pi_1(S) \text{ simple}, \; n(g,k) \text{ the order of } g \text{ in } \pi_1(S)/R_k\}.$$

Observe that if the subgroup $R_k < \pi_1(S)$ classifies a cover $S_k \to S$, then $\mathcal{D}_k$ is identified with the subgroup $\pi_1^s(S_k) < \pi_1(S)$.

We have the following filtration

$$\mathcal{D} \subset R \subset \cdots \subset R_{k+1} \subset R_k \subset \cdots \subset R_1 \subset R_0,$$

and we have that

$$\bigcap_k R_k = R$$

(see Subsection 3.3), so that $f \notin R$ if and only if $f \notin R_k$ for $k \gg 0$.

Lemma 5.2. *For all $k \gg 0$, we have that $\mathcal{D}_k = \mathcal{D}$.*

Proof. If $g \in \pi_1(S)$ is simple then $\rho_p(g)$ has finite order $n(g)$. Since

$$\bigcap_k R_k = R,$$

we have that the order of the image of g in $\pi_1(S)/R_k$ is exactly $n(g)$ for all $k \gg 0$. Finally, we have that ρ_p is the restriction of a representation of the whole mapping class group $\mathrm{Mod}^1(S)$, under whose action there are only finitely many orbits of simple closed curves. In particular, for all $k \gg 0$ and all simple $g \in \pi_1(S)$, the order of the image of g in $\pi_1(S)/R_k$ is exactly $n(g)$. □

With our choice of p fixed, let us write N_0 for the smallest k for which $\mathcal{D}_k = \mathcal{D}$, as in Lemma 5.2. Let $\phi \in \pi_1(S)$ such that $\rho_p(\phi)$ has infinite order. There exists m_0 such that $\phi^{m_0} \in R_{N_0}$, since R_{N_0} has finite index inside the group $\pi_1(S)$. For compactness of notation, we will write ψ for ϕ^{m_0}. Observe that $\psi \notin R$, since $\rho_p(\phi)$ has infinite order, and it follows that for $k \gg N_0$, we have that $\psi \notin R_k$. We set $N \geq N_0$ to be the integer such that $\psi \in R_N \setminus R_{N+1}$. Notice that R_N is a finite index subgroup of $\pi_1(S)$, whereas $\mathcal{D}_N = \mathcal{D}$ is an infinite index subgroup of R_N. In particular, R_N can be naturally identified with the fundamental group of a finite regular cover $S_N \to S$, and $\mathcal{D}$ can be naturally identified with a subgroup of $\pi_1(S_N)$, i.e. the subgroup $\pi_1^s(S_N)$.

For each N, we may write

$$q_N \colon R_N \to R_N/[R_N, R_N]$$

for the abelianization map.

Theorem 5.3. *There is a proper inclusion*

$$q_N(\mathcal{D}) \subsetneq R_N/[R_N, R_N].$$

In fact, for every $\delta \in [R_N, R_N]$, we have that $\delta \cdot \psi \notin \mathcal{D}$.

Theorem 5.3 implies Theorem 1.2 fairly quickly:

Proof of Theorem 1.2. Setting S' in the statement of Theorem 1.2 to be the cover $S_N \to S$ classified by the subgroup R_N, we have that $\mathcal{D}$ can be identified with $\pi_1^s(S')$. The image of $\mathcal{D}$ in $R_N/[R_N, R_N]$ under q_N is exactly $H_1^s(S', \mathbb{Z})$. The conclusion of Theorem 1.2 now follows from Theorem 5.3. □

Proof of Theorem 5.3. We will fix ψ and N as in the discussion before the theorem. First, a standard and straightforward computation shows that $[R_N, R_N] \subset R_{2N+1}$. Now, suppose there exists an element $\delta \in [R_N, R_N]$ such that $\delta \cdot \psi \in \mathcal{D} \subset R_{N+1}$. Then, we would obtain $\psi \in \delta^{-1} R_{N+1}$. Since $[R_N, R_N] \subset R_{2N+1}$ by the claim above, we have that $\delta \in R_{2N+1}$. But then we must have that $\psi \in R_{N+1}$ as well, which violates our choice of N, i.e. $\psi \in R_N \setminus R_{N+1}$. □

5.3. **Index estimates.** In this subsection, we estimate the index of $H_1^s(S_N, \mathbb{Z})$ inside of $H_1(S_N, \mathbb{Z})$ as a function of p and of N. In particular, we will show that the index can be made arbitrarily large by varying both p and N.

Proposition 5.4. *The index of $q_N(\mathcal{D})$ in $R_N/[R_N, R_N]$ is at least p^e, where here*

$$e = \left\lfloor \frac{N}{p-1} \right\rfloor + 1.$$

We need the following number–theoretic fact which can be found in [19, Lemma 3.1]:

Lemma 5.5. *There exists an invertible element $z \in \mathbb{Z}[\zeta_p]$ such that $p = z \cdot h^{p-1}$.*

The following is an easy number–theoretic fact, whose proof we include for the convenience of the reader:

Lemma 5.6. *If k and p are relatively prime, then k is invertible modulo h^n, for all $n \geq 1$.*

Proof. Since $p \equiv 0 \pmod{h}$, we have that $p^n \equiv 0 \pmod{h^n}$. Since k and p are relatively prime, we have that for each $n \geq 1$, there exist integers a and b such that

$$a \cdot k + b \cdot p^n = 1.$$

Thus, $a \cdot k \equiv 1 \pmod{h^n}$. □

Lemma 5.7. *Let $1 \leq k \leq p^e - 1$. With the notation of Theorem 5.3, we have that $\psi^k \notin R_{2N+1}$.*

Proof. We compute the "h–adic" expansion of $\rho_p(\psi)$ in a basis, as in the proof of Theorem 5.3. Up to an invertible element of $\mathbb{Z}[\zeta_p]$, we have

$$\rho_p(\psi) = I + h^{N+1}\Delta,$$

where here $\Delta \not\equiv 0 \pmod h$, since $\psi \in R_N \setminus R_{N+1}$. If $1 \le k \le p^e - 1$, we obtain the expansion

$$\rho_p(\psi)^k \equiv I + k \cdot h^{N+1}\Delta \pmod{h^{2N+2}}.$$

Since $1 \le k \le p^e - 1$, we have that k can be written $k = m \cdot p^l$, with $0 \le l < e$ and with m relatively prime to p.

Lemma 5.5 implies that $p = z \cdot h^{p-1}$, so that

$$k \cdot h^{N+1}\Delta = m \cdot z^l \cdot h^{l(p-1)+N+1}\Delta.$$

Now if

$$k \cdot h^{N+1}\Delta \equiv 0 \pmod{h^{2N+2}},$$

we see that

$$h^{l(p-1)+N+1}\Delta \equiv 0 \pmod{h^{2N+2}},$$

since m and z are invertible modulo h^{2N+2}. Note however that $l < e$, so that

$$l(p-1) < N + 1.$$

The expression

$$h^{l(p-1)+N+1}\Delta \equiv 0 \pmod{h^{2N+2}}$$

now implies that

$$\Delta \equiv 0 \pmod{h^{N+1-l(p-1)}},$$

which is impossible since $N + 1 - l(p-1) > 0$ and since $\Delta \not\equiv 0 \pmod h$.

It follows that

$$k \cdot h^{N+1}\Delta \not\equiv 0 \pmod{h^{2N+2}},$$

which in turn implies that $\psi^k \notin R_{2N+1}$. □

Proof of Proposition 5.4. In Theorem 5.3, we established that for each

$$\delta \in [R_N, R_N] \subset R_{2N+1},$$

we have that $\psi \cdot \delta \notin \mathcal{D} \subset R_{2N+1}$. From Lemma 5.7, it follows that powers of ψ represent at least p^e distinct cosets of R_{2N+1} in R_N, and hence of $[R_N, R_N]$ in R_N, whence the claim of the proposition. □

6. Representations of surface groups, revisited

In this final short section, we illustrate how a rational version of Theorem 1.2 implies Theorem 1.1, thus further underlining the interrelatedness of the two results. We will write S for a surface as above.

Proposition 6.1. *Let $S' \to S$ be a finite regular cover of S, and suppose that $rk(H_1^s(S',\mathbb{Q})) < rk(H_1(S',\mathbb{Q}))$. Then there exists a linear representation*

$$\rho\colon \pi_1(S) \to \mathrm{GL}_d(\mathbb{Z})$$

such that:

(1) The image of ρ is infinite.

(2) The image of every simple element of $\pi_1(S)$ has finite order.

In fact, the image of ρ can be virtually abelian (i.e. the image of ρ has a finite index subgroup which is abelian).

We remark again that our general version of Theorem 1.2 implies a proper inclusion between $H_1^s(S',\mathbb{Z})$ and $H_1(S',\mathbb{Z})$, which may no longer be proper when tensored with $\mathbb{Q}$. Therefore, we do not get that Theorem 1.1 and Theorem 1.2 are logically equivalent. Indeed, in order to deduce Theorem 1.2, we had to use specific properties of the SO(3)–TQFT representations.

Proof of Proposition 6.1. Let $S' \to S$ be a finite regular cover as furnished by the hypotheses of the proposition, and let G be the deck group of the cover. Write

$$H_1(S',\mathbb{Q}) \cong A \oplus B,$$

where $A \cong H_1^s(S',\mathbb{Q})$ and $B \neq 0$. Note that the natural action of G on $H_1(S',\mathbb{Q})$ respects the summand A, since being simple is a conjugacy invariant in $\pi_1(S)$. Write $A^{\mathbb{Z}}$ and $B^{\mathbb{Z}}$ for the intersections of these summands with $H_1(S',\mathbb{Z})$. Note that $H_1^s(S',\mathbb{Z}) \subset A^{\mathbb{Z}}$. Notice that for each integer $m \geq 1$, the subgroup $m \cdot A^{\mathbb{Z}}$ is characteristic in $A^{\mathbb{Z}}$, and is hence stable under the G–action on $H_1(S',\mathbb{Z})$. Let Γ be the group defined by the extension

$$1 \to H_1(S',\mathbb{Z}) \to \Gamma \to G \to 1,$$

which is precisely the group

$$\Gamma \cong \pi_1(S)/[\pi_1(S'),\pi_1(S')].$$

Write $\Gamma_m = \Gamma/(m \cdot A^{\mathbb{Z}})$. This group is naturally a quotient of $\pi_1(S)$.

Note that for all m, the group Γ_m is virtually a finitely generated abelian group, since it contains a quotient of $H_1(S',\mathbb{Z})$ with finite index. Note that every finitely generated abelian group is linear over $\mathbb{Z}$, as is easily checked. Moreover, if a group K contains a finite index subgroup $H < K$ which is linear over $\mathbb{Z}$, then K is also linear over $\mathbb{Z}$, as is seen by taking the induced representation. It follows that for all m, the group Γ_m linear over $\mathbb{Z}$. Furthermore, there is a natural injective map $B^{\mathbb{Z}} \to \Gamma_m$, so that Γ_m is infinite. Finally, if $g \in \pi_1(S)$ is simple, then for some $n = n(g) > 0$, we have that $[g^n] \in H_1^s(S', \mathbb{Z}) \subset A^{\mathbb{Z}}$. It follows that g has finite order in Γ_m. Thus, the group Γ_m has the properties claimed by the proposition. □

References

[1] J.E. Andersen, G. Masbaum, K. Ueno. Topological quantum field theory and the Nielsen-Thurston classification of M(0, 4). *Math. Proc. Cam. Phil. Soc.* 141(2006) 447–488.

[2] J. Birman. *Braids, links, and mapping class groups*. Annals of Mathematics Studies, No. 82. *Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo*, 1974.

[3] C. Blanchet, N. Habegger, G. Masbaum, P. Vogel. Topological quantum field theories derived from the Kauffman bracket. *Topology* 34(4) (1992), no. 4, 883–927.

[4] M. Boggi. The congruence subgroup property for the hyperelliptic modular group: the open surface case. *Hiroshima Mathematical Journal*, 39(3):351–362, 2009.

[5] M. Boggi. The congruence subgroup property for the hyperelliptic modular group: the closed surface case. Preprint, 2014.

[6] P. de la Harpe. *Topics in geometric group theory*. Chicago Lectures in Mathematics. *University of Chicago Press, Chicago, IL*, 2000.

[7] B. Farb and S. Hensel. Moving homology classes in finite covers of graphs. Preprint, 2015.

[8] B. Farb and D. Margalit. *A primer on mapping class groups*. Princeton Mathematical Series, 49. *Princeton University Press, Princeton, NJ*, 2012.

[9] L. Funar. On the TQFT representations of the mapping class groups. *Pacific J. Math.* 188(1999), 251–274.

[10] L. Funar and T. Kohno. Free subgroups within the images of quantum representations. *Forum Math.* 26 (2014), no. 2, 337–355.

[11] P. Gilmer. Integrality for TQFTs. *Duke Math. J.* 125 (2004), no. 2, 389–413.

[12] P. Gilmer and G. Masbaum. Integral lattices in TQFT. *Ann. Sci. École Norm. Sup. (4)* 40 (2007), no. 5, 815–844.

[13] P. Gilmer and G. Masbaum. Integral TQFT for the one–holed torus. *Pacific J. Math.* (252) no. 1 (2011), 93–112.

[14] F. Grunewald, M. Larsen, A. Lubotzky, and J. Malestein. Arithmetic quotients of the mapping class group. *Geom. Funct. Anal.*, to appear.

[15] I. Irmer. Lifts of simple curves in finite regular coverings of closed surfaces. Preprint, 2015.

[16] R. Kent and J. Marché, with comments by I. Agol. Math Overflow post, http://mathoverflow.net/questions/86894/homology-generated-by-lifts-of-simple-curves, 2012.

[17] E. Looijenga. Some algebraic geometry related to the mapping class group. *Oberwolfach Reports*, June 2015.

[18] G. Masbaum. An element of infinite order in TQFT-representations of mapping class groups. *Low-dimensional topology (Funchal, 1998), volume 233 of Contemp. Math.*, pages 137–139. Amer. Math. Soc., Providence, RI, 1999.

[19] G. Masbaum and J. D. Roberts. A simple proof of integrality of quantum invariants at prime roots of unity. *Math. Proc. Cambridge Philos. Soc.* 121 (1997) no. 3, 443-454

[20] G. Masbaum and P. Vogel. 3-valent graphs and the Kauffman bracket. *Pacific J. Math.* 162(2) : 361–381, 1994

[21] A. Putman and B. Wieland. Abelian quotients of subgroups of the mapping class group and higher Prym representations. *J. London Math. Soc.* (2) 88 (2013), no. 1, 79–96.
[22] M.S. Raghunathan. *Discrete subgroups of Lie groups*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68. *Springer-Verlag, New York–Heidelberg*, 1972.
[23] N. Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.,* 103(3):547–597, 1991.
[24] V. G. Turaev. Quantum invariants of knots and 3-manifolds. *de Gruyter Studies in Mathematics*, vol. 18. *Walter de Gruyter and Co., Berlin*, 2010.
[25] E. Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.,* 121:351–399, 1989.

Department of Mathematics, University of Virginia, Charlottesville, VA 22904-4137, USA
E-mail address: thomas.koberda@gmail.com

Institut de Mathématiques de Jussieu (UMR 7586 du CNRS), Equipe Topologie et Géométrie Algébriques, Case 247, 4 pl. Jussieu, 75252 Paris Cedex 5, France
E-mail address: ramanujan.santharoubane@imj-prg.fr