WHAT IS ... an acylindrical group action?

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The group $\mathbb{Z}$ acts on the real line $\mathbb{R}$ by translation. It is difficult to find a nontrivial group action which is easier to understand: the orbit of every point moves off to infinity at a steady and predictable rate, and the group action preserves the usual Euclidean metric on $\mathbb{R}$. Of course, this action is a covering space action, and the quotient space of the action is the circle, which is completely free of any topological pathologies.

Regular covering spaces in algebraic topology give rise to prototypically nice group actions. Among the most important features of a deck group action on a covering space is that it is free (i.e. no nontrivial element of the deck group has a fixed point) and properly discontinuous (i.e. for every compact subset K of the cover, there are at most finitely many deck group elements g such that $g \cdot K \cap K \neq \varnothing$, at least in the case where the base space is locally compact).

For certain purposes in topology and geometry, one can relax the freeness of an action without introducing insurmountable difficulties. Discrete groups of isometries of hyperbolic space, for example, oftentimes contain torsion elements such as rotations, and finite order isometries of Euclidean or hyperbolic spaces always have a fixed point. By considering quotients of Euclidean and hyperbolic spaces by discrete groups of isometries, one naturally obtains the class of Euclidean and hyperbolic orbifolds, thus enlarging the class of Euclidean and hyperbolic manifolds. Orbifolds enjoy many of the salient features of manifolds, so that a mild relaxation of freeness of group actions still allows for reasonable geometry to persist.

Relaxing proper discontinuity can lead to some pathological phenomena, for instance quotient maps whose quotient topologies fail to be Hausdorff or even fail to have any nontrivial open sets. Consider, for example, a group of rotations of the circle generated by an irrational multiple of π. Since the circle is compact, this action of $\mathbb{Z}$ cannot be properly discontinuous – indeed, every orbit is countably infinite and dense. Hence, the quotient is an uncountable space with no open sets except the empty set and the whole space. Nevertheless, group actions which are not properly discontinuous abound in mathematics and have led to the development of entire fields, such as noncommutative geometry in the sense of A. Connes. Group actions which are not properly discontinuous are also important and common in geometric group theory, with the following example being of central importance:

Let S be an orientable surface and let $\gamma \subset S$ be a *simple closed curve*, as illustrated in Figure 1 or in Figure 2. A curve is *essential* if it is not contractible to a point, and *nonperipheral* if it is not homotopic to a puncture or boundary component of S. The *curve graph* of S, denoted $\mathscr{C}(S)$, is the graph whose vertices are nontrivial homotopy classes of essential, nonperipheral, simple closed curves, and whose edge relation is given by disjoint realization. That is, γ_1 and γ_2 are adjacent if they admit representatives which are disjoint. Thus, the curve graph encodes the combinatorial topology of one–dimensional submanifolds of S. Note that for relatively simple surfaces, $\mathscr{C}(S)$ may be empty or may fail to have any edges as they are defined here. For sufficiently complicated surfaces however, $\mathscr{C}(S)$ has very intricate and interesting structure.

Whereas the curve graph as defined here is a manifestly combinatorial object, it is also a geometric object with the metric being given by the graph metric.

It is an interesting exercise for the reader to prove that if $\mathscr{C}(S)$ admits at least one edge, then $\mathscr{C}(S)$ is connected, is locally infinite, and has infinite diameter.

The *mapping class group* of S is the group of homotopy classes of orientation preserving homeomorphisms of S, and is written $\mathrm{Mod}(S)$. Mapping class groups are of central interest to geometric group theorists, as well as of significant interest to algebraic geometers, to topologists, and to homotopy theorists. From the point of view of geometric group theory, mapping class groups are studied via the geometric objects on which

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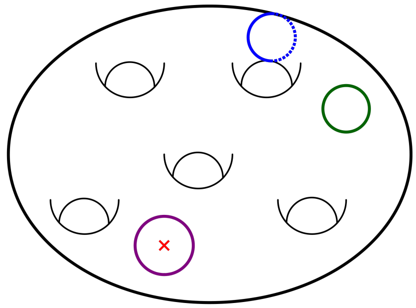

Figure 1. A surface of genus 5. The blue curve is essential and non–separating. The green curve is inessential. The purple curve bounds a puncture or boundary component, denoted by a red x, and is therefore peripheral.

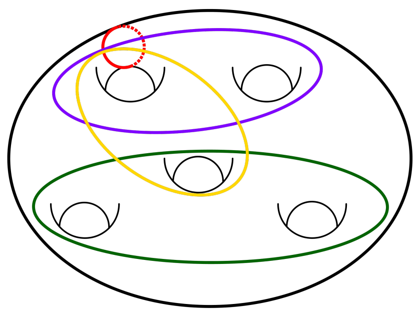

Figure 2. A surface of genus 5 with four essential curves drawn. The subgraph of $\mathscr{C}(S)$ spanned by them is given in Figure 3.

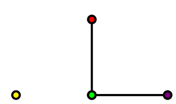

Figure 3. The subgraph of $\mathscr{C}(S)$ spanned by the curves in Figure 2. The colored curves are represented by vertices of the corresponding color. The graph metric distance from the purple curve to the red curve in $\mathscr{C}(S)$ is exactly two.

they act. Homeomorphisms of S act on the set of embedded loops on S, and similarly homotopy classes of homeomorphisms act on homotopy classes of embedded loops on S, and hence of simple closed curves. Since the adjacency relation in $\mathscr{C}(S)$ is a topological property, $\mathrm{Mod}(S)$ acts by graph automorphisms and hence by graph metric isometries on $\mathscr{C}(S)$.

As natural as the action of $\mathrm{Mod}(S)$ on $\mathscr{C}(S)$ is, its geometry is extremely complicated. For one, the quotient $\mathscr{C}(S)/\mathrm{Mod}(S)$ is finite, since two simple closed curves γ_1 and γ_2 are in the same mapping class group orbit if and only if $S\backslash\gamma_1$ and $S\backslash\gamma_2$ are homeomorphic to each other, as follows easily from the classification of surfaces. Thus, the action of $\mathrm{Mod}(S)$ on $\mathscr{C}(S)$ is highly transitive. This is in spite of the fact that $\mathscr{C}(S)$ is locally infinite, as mentioned above: if $\mathscr{C}(S)$ has at least one edge, then each vertex of $\mathscr{C}(S)$ has infinite degree. Thus, the action of $\mathrm{Mod}(S)$ on $\mathscr{C}(S)$ is far from properly discontinuous.

Note that proper discontinuity (as we have defined it at least) is perhaps not the best property to require from the action, since $\mathscr{C}(S)$ is not locally compact (by virtue of being a locally infinite graph). A better notion which is meaningful for actions on spaces like $\mathscr{C}(S)$ is *properness*. If G is a group generated by a finite set S, then G can be viewed as a metric space by declaring g and h to have distance one if $g = h \cdot s$ for some $s \in S$, and in general defining the distance between g and h to be the least n such that $g = h \cdot s_1 \cdots s_n$ for elements $\{s_1, \ldots, s_n\} \subset S$. The reader may recognize this as the graph metric on the (right) Cayley graph of G with respect to S. If G acts on a metric space X, the action is *proper* if (roughly) for all $x \in X$, the orbit map

$G \to X$ given by $g \mapsto g \cdot x$ is a proper map of metric spaces. The first example considered in this article, i.e. the translation action of $\mathbb{Z}$ on $\mathbb{R}$, is a proper action. Note that we can build another action of $\mathbb{Z}$ on $\mathbb{R}$, where a generator of $\mathbb{Z}$ acts by multiplication by 2. This action of $\mathbb{Z}$ on $\mathbb{R}$ is not proper. Returning to the situation at hand, since vertices of $\mathscr{C}(S)$ have infinite stabilizers in $\mathrm{Mod}(S)$, the action of $\mathrm{Mod}(S)$ on $\mathscr{C}(S)$ is not proper.

One way to see this is to observe the following: let $\gamma \subset S$ be an essential, nonperipheral, simple closed curve as in Figure 4. The surface $S \setminus \gamma$ is a surface with boundary, albeit simpler (in the sense that the Euler characteristic is strictly larger).

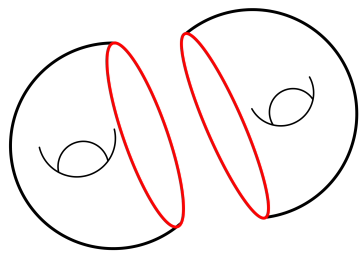

FIGURE 4. By considering an essential, nonperipheral, simple closed curve in red on this surface of genus 2, we can see that the action of $\mathrm{Mod}(S)$ on $\mathscr{C}(S)$ cannot be proper. Fortunately, it satisfies a weaker property: it is acylindrical.

The surface $S \setminus \gamma$ generally admits many homotopically nontrivial homeomorphisms which act by the identity near γ, which therefore extend to homeomorphisms of S which fix γ. Moreover, one can build the *Dehn twist* about γ, which is given by cutting S open along γ and regluing with a full twist. These (homotopy classes of) homeomorphisms taken together form an infinite subgroup of $\mathrm{Mod}(S)$ which fixes the vertex γ of $\mathscr{C}(S)$, whence it is clear that the action of $\mathrm{Mod}(S)$ on $\mathscr{C}(S)$ cannot be proper.

How badly behaved is the action of $\mathrm{Mod}(S)$ on $\mathscr{C}(S)$? Can something be said about it which is not a general statement about isometric group actions on graphs? It turns out that yes, indeed one can. The action is *acylindrical*, which is in some sense the next best thing after properness.

Acylindrical actions were first generally defined by Bowditch in 2008. Let G be a group acting by isometries on a path–metric space X. The action of G on X is *acylindrical* if for all $r \geqslant 0$, there exist constants $R, N \geqslant 0$ such that for any pair $a, b \in X$ with $d(a, b) \geqslant R$, we have

$$|\{g \in G \mid d(g \cdot a, a) \leqslant r \text{ and } d(g \cdot b, b) \leqslant r\}| \leqslant N.$$

The set of elements

$$\{g \in G \mid d(g \cdot a, a) \leqslant r\}$$

is not quite the stabilizer of a, but rather the *r–quasi–stabilizer* of a. Acylindricity can be summed up as saying that "for all r, simultaneous r–quasi–stabilizers of sufficiently distant points are uniformly small". In more informal terms, an acylindrical action is "uniformly proper on sufficiently distant pairs". Dropping the uniformity condition (i.e. replacing the uniform constant N by a requirement that the relevant subset of G is finite), one gets the closely related notion of a *weakly properly discontinuous* action. This latter notion appears in a 2002 paper of M. Bestvina and K. Fujiwara.

Observe that, like many concepts in geometric group theory and coarse geometry, acylindricity is blind to phenomena on a bounded scale. For instance, a group action on a bounded metric space is always acylindrical: just let R be greater than the diameter of X.

Bowditch proved the following fundamental result:

Theorem 1. *The action of* $\mathrm{Mod}(S)$ *on* $\mathscr{C}(S)$ *is acylindrical.*

The usefulness of acylindricity is perhaps not immediately clear. The most productive setting for studying acylindricity is in the case where X is a *hyperbolic graph*. This means that X is a graph equipped with the graph metric, and the graph metric is *(Gromov) hyperbolic*. That is to say, there is a constant $\delta \geqslant 0$ such that for any triple $x, y, z \in V(X)$ of vertices and geodesic segments $[x, y], [y, z], [x, z] \subset X$, we have

$$[x, z] \subset N_\delta([x, y] \cup [y, z]).$$

In other words, every geodesic triangle is δ–thin in the sense that a δ–neighborhood of two sides contains the third side (see Figure 5).

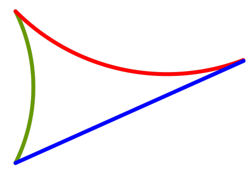

Figure 5. A path–metric space (such as a connected graph) is hyperbolic if there exists a $\delta \geqslant 0$ such that for every geodesic triangle, a δ–neighborhood of two sides contains the third side.

Note that the definition of hyperbolicity makes sense for any path metric space, and indeed this is the definition of a *hyperbolic (metric) space* which is not necessarily a graph. It is highly non–obvious though true that $\mathscr{C}(S)$ is a hyperbolic graph, by a deep result of H. Masur and Y. Minsky from 1999.

In the case of an action of a group G on a hyperbolic graph X, acylindricity of the G–action gives a tractable geometric shadow of G in X, given by considering the orbit of an arbitrary vertex $v \in V(X)$. To make sense of this notion, we define the *translation length* of an element $g \in G$, a definition which makes sense for any isometric action of G on X. We write

$$\tau(g) = \lim_{n\to\infty} \frac{d(g^n \cdot x, x)}{n},$$

a limit which always exists and which is independent of the choice of x.

The translation length of g is either positive or zero. In the former case, the element g is called *loxodromic*. An example of a loxodromic isometry is a homothetic expansion of the upper half–space model for hyperbolic space. One way that $\tau(g)$ can be zero is if some (or indeed every) g–orbit has finite diameter, in which case g is called *elliptic*. An example of this latter case is rotation of the Poincaré disk model for hyperbolic space. General actions can have elements such that $\{g^n \cdot x\}_{n\in\mathbb{Z}}$ is unbounded but where $d(g^n \cdot x, x)$ grows strictly sublinearly as a function of n, in which case g is *parabolic*. Translation by one is an example of a parabolic isometry of the upper half–space model of hyperbolic space. Parabolic isometries can imbue group actions with significant complexity, as in the case of lattices acting on symmetric spaces. In many higher rank situations, parabolic elements can generate lattices which can often be shown to never admit interesting acylindrical actions.

For acylindrical actions on hyperbolic graphs, Bowditch proved the following general fact which simplifies the picture somewhat:

Theorem 2. *Let G be a group acting acylindrically on a hyperbolic graph X. Then every nontrivial $g \in G$ is either loxodromic or elliptic. Moreover, there is a constant $\epsilon > 0$ depending only on the acylindricity and hyperbolicity constants such that if g is loxodromic then $\tau(g) \geqslant \epsilon$.*

In terms of terminology, acylindrical actions are either *elementary* or *non–elementary*. An action is elementary if it is purely elliptic or if there is (essentially) only one cyclic subgroup consisting of loxodromic elements. Non–elementary acylindrical actions are the only interesting ones. As might be expected, the $\mathrm{Mod}(S)$ action on $\mathscr{C}(S)$ is non–elementary.

Returning to the $\mathrm{Mod}(S)$ action on $\mathscr{C}(S)$, it is possible to show that loxodromic elements are exactly the mapping classes such that no power fixes a simple closed curve, as was done by Masur and Minsky. Such mapping classes are called *pseudo-Anosov*, and are arguably the most interesting mapping classes. Thurston's beautiful 1988 article in the *Bulletin of the American Mathematical Society* provides an accessible introduction. The literature on the coarse geometry of subgroup of $\mathrm{Mod}(S)$ which consist entirely of pseudo-Anosov elements (i.e. *purely pseudo-Anosov subgroups*) is vast, and is the domain of *convex cocompact subgroups* of $\mathrm{Mod}(S)$. Convex cocompact subgroups, which we will not define precisely here, are of central importance of the geometry of the moduli space of curves, hyperbolic group extensions, and the algebraic and geometric structure of mapping class groups. Note that it is not obvious *a priori* that there exist non–cyclic purely pseudo–Anosov subgroups of $\mathrm{Mod}(S)$. They can be produced directly with some work, but the acylindricity of the $\mathrm{Mod}(S)$ action on $\mathscr{C}(S)$ again intercedes to furnish a profusion of them, by the following recent result of F. Dahmani, V. Guirardel, and D. Osin:

Theorem 3 (Dahmani–Guirardel–Osin (2017))**.** *Let G be a group acting acylindrically on a hyperbolic space X. Then there exists a natural number $N > 0$ such that for every loxodromic $g \in G$, the normal closure $\langle\langle g^N \rangle\rangle$ is free and purely loxodromic.*

In recent years, there has been an explosion of results by many authors on acylindrical actions of various groups on hyperbolic spaces. In addition to mapping class groups, examples of groups admitting non–elementary acylindrical actions on hyperbolic spaces include non–elementary hyperbolic groups, groups which are non–elementary hyperbolic relative to proper subgroups, outer automorphism groups of free groups, Cremona groups, non–virtually nilpotent groups acting properly on a hyperbolic space of uniformly bounded geometry, right-angled Artin groups, and compact 3–manifold groups which are not Seifert fibered. A lot of these examples have been organized and their properties developed recently by the work of J. Behrstock, M. Hagen, and A. Sisto, to form the class of hierarchically hyperbolic groups.

Acylindricity has been an extremely productive and pervasive concept in geometric group theory, and has led to fast paced and dramatic advances. Undoubtedly, it will continue to do so for some time.

References

1. Brian H. Bowditch, *Tight geodesics in the curve complex*, Invent. Math. **171** (2008), no. 2, 281–300. MR 2367021 (2008m:57040)