The phase diagram of polar condensates
Taking the square root of a vortex

Austen Lamacraft [with Andrew James]
arXiv:1009.0043

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KITP, UCSB
1 Magnetism in Bose condensates
2 Order parameters and topology in polar condensates
3 Domain walls, disclinations, and the phase diagram in 2D
4 Generalized XY model: view from field theory
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4 Generalized XY model: view from field theory
Magnetic Bose condensates

BEC $\Rightarrow$ magnetism for bosons with spin!

Condensate wavefunction $\langle \phi(r) \rangle \neq 0$ is a spinor and must choose a ‘direction’ in spin space.

Example: spin-polarized Hydrogen [Siggia & Ruckenstein, 1980]

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\phi = \left(\frac{\phi_1}{2} - \frac{\phi_{-1}}{2}\right) = e^{i\psi} \left(e^{-i\phi/2} \cos \theta/2 e^{i\phi/2} \sin \theta/2\right)
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All states (of fixed norm) obtained by rotation of reference state.

Does magnetism $\Rightarrow$ BEC? or can ordering happen sequentially as temperature lowered?
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Higher spin gives additional possibilities

Consider the spin-1 state

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\phi = \begin{pmatrix} \phi_1 \\ \phi_0 \\ \phi_{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
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Not hard to show that

\[
\phi \dagger (m \cdot S(1)) \phi = 0
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(S(1) the spin-1 matrices)

Yet evidently there is still an axis (headless?)

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Contact interactions in a spin-1 gas

Total spin 2
Contact interactions in a spin-1 gas

Total spin 0

- $m_F = 0$
- $m_F = 1$
- $m_F = -1$

Collision
Contact interactions in a spin-1 gas

\[ H_{\text{int}} = \sum_{i < j} \delta(r_i - r_j) [g_0 P_0 + g_2 P_2] \]

\[ = \sum_{i < j} \delta(r_i - r_j) [c_0 + c_2 S_i \cdot S_j] \]

\[ c_0 = \frac{(g_0 + 2g_2)}{3} \quad c_2 = \frac{(g_2 - g_0)}{3} \]
Mean field ground states

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Energy of state \( \Psi_{m_1 \cdots m_N}(r_1, \ldots, r_N) = \phi_{m_1}(r_1) \cdots \phi_{m_N}(r_N) \) involves

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- \( c_2 < 0 \) (e.g. \(^{87}\text{Rb}\))
  - \( \phi^\dagger S \phi \) is maximized

- \( c_2 > 0 \) (e.g. \(^{23}\text{Na}\))
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Spin-1 states: Cartesian representation

\[ \phi = a + ib \]

\[ \left( S_i^{(1)} \right)_{jk} = -i \epsilon_{ijk} \]

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- Polar condensate
The Bose Ferromagnet

Stamper-Kurn group, Berkeley
Mean field ground state written $\phi = ne^{i\theta}$

There is an evident redundancy: $(n, \theta)$ and $(-n, \theta + \pi)$ are the same
Mean field ground state written \( \phi = n e^{i\theta} \). There is an evident redundancy: \((n, \theta)\) and \((-n, \theta + \pi)\) are the same.
Order parameter manifold for polar condensates

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Disclinations in nematic liquid crystals

Principles of condensed matter physics

P. M. Chaikin & T. C. Lubensky

Austen Lamacraft (University of Virginia)
Consequences for Kosterlitz–Thouless transition

Circulation quantum is *halved*

\[ \oint \mathbf{v} \cdot d\mathbf{l} = \frac{\hbar}{m} \oint \nabla \theta = \frac{\hbar}{m} \pi n = \frac{h}{2m} n, \quad n \in \mathbb{Z} \]
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Consider free energy of a single half vortex / disclination

\[ E = \frac{n_s}{2m} \int d\mathbf{r} \mathbf{v}^2 = \frac{\pi n_s \hbar^2}{4m} \ln \left( \frac{L}{\xi} \right) \]

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\[ E = n_s \int d\mathbf{r} v^2 - \pi n_s \hbar^2 \ln (L) \]

Possible Experiments on Two-dimensional Nematics

*BY P. G. DE Gennes*

Physique du Solide, Faculté des Sciences, 91 Orsay

Received 9th July, 1971

Vanishes for

\[ n_s = \frac{8}{\pi} \frac{m k_B T}{\hbar^2} \]
Jump is 4× bigger than usual! (Korshunov, 1985)

\[ \Delta n_{KT/2} = 4 \Delta n_{KT} = \frac{8}{\pi} \frac{mk_B T_c}{\pi \hbar^2} \]
Topological Defects and the Superfluid Transition of the $s = 1$ Spinor Condensate in Two Dimensions

Subroto Mukerjee,\textsuperscript{1,2} Cenke Xu,\textsuperscript{1} and J.E. Moore\textsuperscript{1,2}

![Diagram](https://example.com/diagram.png)

**KT transition mediated by half-vortices / disclinations**

PRL 97, 120406 (2006)
What about the $n$ degrees of freedom?

A simple model is

$$H = -\sum_{ij} \phi^\dagger_i \phi_j + \text{c.c} = -2t \sum_{ij} n_i \cdot n_j \cos (\theta_i - \theta_j)$$

Taking the continuum limit...

$$H \to a^2 - d t \int d^3 r \left[ (\nabla \theta)^2 + (\nabla n)^2 \right]$$

Identifying $(n, \theta)$ and $(-n, \theta + \pi)$ ties half vortices to disclinations.

Beneath KT transition, half vortices are absent and one can treat the $n$ degrees of freedom as Heisenberg spins.

Mermin–Wagner theorem says they do not order at finite temperature.
What about the $n$ degrees of freedom?

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Within the spin-1 multiplet, the Zeeman energy is

\[ H_{Z,m} = pm + qm^2 \]

\( p \propto B \text{ linear and } q \propto B^2/A_{HF} \text{ quadratic Zeeman effects} \)
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The Zeeman effect

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- At large \( q \ m = 0 \) state only occupied: ordering via usual KT transition
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**Basic problem:**

How does the phase diagram evolve with \( q \) from the \( \frac{1}{2} \)KT transition mediated by half vortices to the usual KT transition?
Consequences of quadratic Zeeman effect

\[ H_{QZ} = q \sum_i \phi_i^\dagger S_z^2 \phi_i = q \sum_i (1 - n_{z,i}^2) \]
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\[ q > 0 \text{ favors alignment of } \mathbf{n} \text{ in } z\text{-direction (easy axis)} \]

\[ H = -2t \sum_{ij} \mathbf{n}_i \cdot \mathbf{n}_j \cos (\theta_i - \theta_j) - q \sum_i n_{z,i}^2 \]
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Large \( q \) fixes \( \mathbf{n} \): regular KT transition
Conjectured phase diagram

Temperature

Spin disordered Superfluid

Ising

Superfluid

Quadratic Zeeman (Domain wall energy)
Monte Carlo simulation

Include hopping of singlet pairs \( \phi \cdot \phi = \cos(2\theta) \)

\[
H = - \sum_{ij} [2t \mathbf{n}_i \cdot \mathbf{n}_j \cos(\theta_i - \theta_j) + u \cos(2 [\theta_i - \theta_j])] - q \sum_i n_{z,i}^2
\]
Intermediate phase with singlet pair quasi-long range order

(data at $u = 1$)
The phase diagram of polar condensates
Binder cumulant

\[ \Phi = \sum_i \phi_i \]
\[ U_4 = \frac{\langle (\Phi^\dagger \Phi)^2 \rangle}{\langle \Phi^\dagger \Phi \rangle^2} \]

q=1

![Graph showing the relationship between Binder cumulant and temperature for different system sizes L=8, 16, 24, 32.](image-url)
Ising scaling for the lower transition

\[ U_4 = f \left( \frac{L}{\xi} \right) = f \left( L \left| \frac{T - T_c}{T_c} \right|^{\nu} \right) \]
Ising scaling for the lower transition

Binder cumulant

\[ L = 8, 16, 24, 32, 40 \]

Consistent with \( \nu = 1 \)
Ising scaling for the lower transition

Consistent with $\nu = 1$

Austen Lamacraft (University of Virginia)
What about finite magnetization?

At finite magnetization have *spin-flop* transition: \( n \) flops into \( x - y \) plane

Analogous to antiferromagnet
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**Diagram:**

Zero temperature Heisenberg fixed point
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4. Generalized XY model: view from field theory
Generalized XY model

\[ H_{\text{gen}} = - \sum_{\langle ij \rangle} \left( \Delta \cos(\theta_i - \theta_j) + (1 - \Delta) \cos(2\theta_i - 2\theta_j) \right) \]

Korshunov (1985), Grinstein & Lee (1985)
Field theoretic view: how can domain walls end?\footnote{Thanks to Paul Fendley for discussions}
Recall *disorder operator* $\mu$ in Ising model
Recall *disorder operator* $\mu$ in Ising model

Half vortex insertion tied to disorder operator

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1 Thanks to Paul Fendley for discussions
The phase diagram of polar condensates from $\mu \cos(\phi/2)$ perturbation.

- Temperature axis.
- $1/2 KT$ and $KT$ lines.
- Spin disordered superfluid and superfluid regions.
- Ising line for $\langle \mu \rangle \neq 0$.
- Quadratic Zeeman (Domain wall energy) parameter.
\( \frac{1}{2} K T \) occurs when \( \cos(\phi/2) \) becomes relevant if \( \langle \mu \rangle \neq 0 \)
Phase diagram from $\mu \cos(\phi/2)$ perturbation

- $\frac{1}{2}KT$ occurs when $\cos(\phi/2)$ becomes relevant if $\langle \mu \rangle \neq 0$
- Along Ising line: continuous transition until $\mu \cos(\phi/2)$ relevant
Scaling with $\mu \cos(\phi/2)$ perturbation

$$H_{\text{vortex}} = \lambda \mu \cos \phi/2$$

Three couplings to keep track of

1. $\lambda$, half vortex fugacity
2. $t_I$, energy operator for Ising
3. Stiffness $K$ of superfluid
Scaling with $\mu \cos(\phi/2)$ perturbation

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Three couplings to keep track of

1. $\lambda$, half vortex fugacity
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3. Stiffness $K$ of superfluid

\[
\frac{d\lambda}{d\ell} = \left(2 - \frac{1}{8} - \frac{\pi K}{4}\right) \lambda + \frac{1}{2} \lambda t_I
\]

\[
\frac{dt_I}{d\ell} = t_I + \frac{\lambda^2}{2}
\]

\[
\frac{dK}{d\ell} = \lambda^2
\]
RG flow for $K > 15/2\pi$

Temperature

$<\mu>$ nonzero

Spin disordered Superfluid

$1/2 K T$

Ising

Ising fp