1 Introduction

I am an algebraic topologist. My main interests are the theory of topological and spectral operads, their Koszul duality, Goodwillie calculus, and, more recently, the interaction of these topics with genuine equivariant homotopy theory.

The following paragraphs give a rough overview of these topics.

Operads were introduced by May in [15]. Informally, given a suitable category $\mathcal{C}$, an operad $\mathcal{O}$ in $\mathcal{C}$ is a sequence of objects $\{\mathcal{O}(n)\}_{n\geq 0}$ along with structure maps that make $\mathcal{O}(n)$ behave as a “set”/“space” of $n$-ary operations. The main point of the theory is to study the associated category of algebras $\text{Alg}_\mathcal{O}(\mathcal{C})$, i.e. objects $X \in \mathcal{C}$ together with algebraic operations “indexed” by $\mathcal{O}$.

Koszul duality is a notion of duality for operads introduced by Ginzburg and Kapranov in [7] for the algebraic setting and later adapted to the topological and spectral settings by Ching in [2]. Given an operad $\mathcal{O}$ this duality forms a new operad $K\mathcal{O}$ and in good cases one has that $K(K\mathcal{O})$ is then $\mathcal{O}$ again (or suitably equivalent) and that the categories $\text{Alg}_\mathcal{O}$ and $\text{Alg}_{K\mathcal{O}}$ are equivalent.

Goodwillie calculus is a technique developed by Goodwillie in [8], [9] and [10] to study functors between the categories of topological spaces and spectra by approximating them by so-called “polynomial functors”. In fact, it was clear from Goodwillie’s original arguments that the theory should apply to more general model categories (categories where one can define homotopical notions), and part of my thesis work ([16]) was to make this precise.

Finally, genuine homotopy theory deals with the correct notion of homotopy when in the presence of the action of a group $G$. For example, for a $G$-equivariant map $f:X \to Y$ between spaces with $G$-actions to be a “genuine equivariant homotopy equivalence”, $f$ should induce “non-equivariant equivalences” $f^H:X^H \to Y^H$ between fixed points for all subgroups $H \leq G$.

2 Goodwillie calculus

Suppose one is interested in functors $F: \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are either the category of pointed topological spaces $\text{Top}_*$ or its stabilization, spectra $\Sigma^\infty \text{Sp}$. In [10], Goodwillie constructs an associated tower of functors

$$F \to \cdots \mathcal{P}_n F \to \mathcal{P}_{n-1} F \to \cdots \mathcal{P}_0 F$$

(2.1)

where the functor $\mathcal{P}_n F$ is characterized as being the universal $n$-excisive (or $n$-polynomial) approximation of the functor $F$. 

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Further, Goodwillie proves a remarkable characterization of the layers $D_n F$ (i.e. the fiber functors $D_n F \to P_n F \to P_{n-1} F$). Explicitly, in the harder case of functors from spaces to spaces (for the other cases, remove $\Sigma^\infty$ (resp. $\Omega^\infty$) from the formula if the source (resp. target) category is spectra), those layers have the form

$$
\Omega^\infty((\partial_n F \land \Sigma^\infty X)_{h \Sigma_n})
$$

where the $\partial_n F$ are spectra with $\Sigma_n$-actions.

### 3 Past work

The goal of my thesis was to study Goodwillie calculus applied to the category $\text{Alg}_{\mathcal{O}}(\text{Sp}^\Sigma)$. Due to the lack of a general setup for Goodwillie calculus in the literature at the time, the first step was to provide one. This was the goal of [16], the main results of which are the following generalizations of Goodwillie’s results.

**Theorem 3.1.** Let $\mathcal{C}, \mathcal{D}$ be cofibrantly generated model categories, and $F: \mathcal{C} \to \mathcal{D}$ be a homotopy functor. Assume further that $\mathcal{C}$ is either pointed and simplicial or that cofibrations in $\mathcal{C}$ are categorical injections.

Then there exists a universal $n$-excisive approximation $P_n F$ to $F$.

**Theorem 3.2.** Assume $\mathcal{C}, \mathcal{D}$ are pointed simplicial model categories, and $F: \mathcal{C} \to \mathcal{D}$ is a simplicial $n$-homogeneous functor. Assume further that: spectra categories $\text{Sp}(\mathcal{C}), \text{Sp}(\mathcal{D})$ can be defined, where weak equivalences of spectra are detected by the $\Omega^\infty-i$ functors; in $\mathcal{D}$ finite homotopy limits commute with countable directed homotopy colimits.

Then $F$ can be factored through the spectra categories, i.e., $F \sim \Omega^\infty \circ \bar{F} \circ \Sigma^\infty$, where $\bar{F}: \text{Sp}(\mathcal{C}) \to \text{Sp}(\mathcal{D})$ is itself $n$-homogeneous.

Applying the Goodwillie calculus set-up above to $\text{Alg}_{\mathcal{O}}(\text{Sp}^\Sigma)$ I showed the following in [17].

**Theorem 3.3.** Let $\mathcal{O}$ be a spectral operad. Then there is a zigzag of Quillen equivalences between $\text{Sp}(\text{Alg}_{\mathcal{O}})$ and $\mathcal{O}(1)^{-\text{Mod}}$. Further, this identifies $\Sigma^\infty$ with Topological André-Quillen homology and $\Omega^\infty$ with the trivial algebra functor.

**Theorem 3.4.** The Goodwillie tower of the identity for $\text{Alg}_{\mathcal{O}}$ is given by the (left derived) truncation functors $O_{\Sigma_n} \circ O (\cdot)$. Further, the $n$-th derivative is $O(n)$ itself with its canonical $(O(1), O(1)^{\infty})$-bimodule structure.

Additionally, [17] also features the following interesting result.

**Theorem 3.5.** Let $\text{Alg}_{\mathcal{O}} \overset{F}{\to} \text{Alg}_{\mathcal{O}'}$ be a (finitary) $n$-excisive functor.

Then there is a unique up to equivalence $n$-excisive functor $\text{Alg}_{\mathcal{O}_{\Sigma_n}} \overset{\bar{F}}{\to} \text{Alg}_{\mathcal{O}'_{\Sigma_n}}$ such that one has a factorization up to homotopy

$$
\begin{align*}
\text{Alg}_{\mathcal{O}} & \overset{F}{\to} \text{Alg}_{\mathcal{O}'} \\
\text{Alg}_{\mathcal{O}_{\Sigma_n}} & \overset{\bar{F}}{\to} \text{Alg}_{\mathcal{O}'_{\Sigma_n}}
\end{align*}
$$
4 Recent work

4.1 $\Sigma$-cofibrancy of smash powers of positive spectra

One of the key technical tools of the work in [17] was the positive model category structure on symmetric spectra $\text{Sp}^\Sigma$, which has been long known to be convenient when studying algebras over an operad in spectra (e.g. [20], [4], [11]). Informally speaking, this is because for $X$ a positive cofibrant spectrum the $n$-fold smash power $X^{\wedge n}$ behaves as if it was build out of free $\Sigma_n$-cells, or, to put it more formally, as if it was $\Sigma_n$-cofibrant. However, the fact that this claim is false “on the nose” represented a serious obstacle to establishing useful cofibrancy conditions for certain key constructions in $\text{Alg}_O$. Solving this problem was the main goal of [18], where I introduced a notion of lax $\Sigma_n$-cofibrancy for $n$-fold powers of positive spectra which I then proved to have the same key properties as genuine $\Sigma_n$-cofibrancy. Here are the main results (redacted for readability).

**Theorem 4.1.** Let $f: A \to B$ be a positive $S$ cofibration in $\text{Sp}^\Sigma$. Then $f^{\wedge n} \cdot Q_{n-1}^n(f) \to B^{\wedge n}$ is a lax $\Sigma_n$-cofibration in $(\text{Sp}^\Sigma)^{\Sigma_n}$. Further, if $A$ is positive $S$ cofibrant then $Q_{n-1}^n(f)$ (resp. $f^{\wedge n}: A^{\wedge n} \to B^{\wedge n}$) is lax $\Sigma_n$-cofibrant (resp. lax $\Sigma_n$-cofibration between cofibrant objects).

**Theorem 4.2.** Consider the bifunctor $(\text{Sp}^\Sigma)^G \times (\text{Sp}^\Sigma)^G \to \text{Sp}^\Sigma$, where the first copy of $(\text{Sp}^\Sigma)^G$ is regarded as equipped with the lax $G$-projective stable model structure and the second $(\text{Sp}^\Sigma)^G$ and the target $\text{Sp}^\Sigma$ are equipped with the respective $S$ stable model structures.

Then $\wedge_G$ is a left Quillen bifunctor.

Theorems 4.1 and 4.2 are then used to prove the main result of [18]: that, under mild cofibrancy conditions, “operadic pushout products”

$$M \circ_O N \bigvee_{M \circ_O N} M \circ_O N \xrightarrow{f_1 \circ_O f_2} M \circ_O N.$$

are cofibrations, trivial if $f_1$ or $f_2$ are.

While technical, this is quite a powerful result, which easily implies all of the following (most of which strengthen previous results in [11], [12]).

**Theorem 4.3.** For $O$ any operad in $\text{Sp}^\Sigma$ there is a projective positive $S$ model structure on $\text{Alg}_O$. Further, $\hat{O} \circ_O (-): \text{Alg}_\hat{O} \rightleftarrows \text{Alg}_O: \text{fgt}$ is a Quillen equivalence when $O \to \hat{O}$ is a stable equivalence.

**Theorem 4.4.** Let $O$ be an operad in $\text{Sp}^\Sigma$ which is level cofibrant. Then the forgetful functor $\text{fgt}: \text{Alg}_O \to \text{Sp}^\Sigma$ sends cofibrations between cofibrant objects to cofibrations.

**Theorem 4.5.** For an operad $O$, right $O$-module $M$ and left $O$-module $N$ satisfying mild cofibrancy hypothesis, the bar construction $B_0(M, O, N) = M \circ O^{\wedge n} \circ N$ is Reedy cofibrant.

**Theorem 4.6.** If $A$ is cofibrant in $\text{Alg}_O$, the functor $\text{Mod}_{O}^{r}(-) \circ_A \text{Sp}^\Sigma$ preserves homotopy fiber sequences.
4.2 Operad bimodules and André-Quillen filtrations

In joint work with Nick Kuhn in [14], we study the filtration $\mathcal{O}_{\geq n} \circ \mathcal{O}(-)$ dual to the Goodwillie tower in Theorem 3.4. A key observation is that by iterating the filtration (i.e. applying the filtration to itself!) the operad structure leads to extra maps between the filtration levels. Our main results are as follows (where $\mathcal{O}_{\leq n} \circ \mathcal{O} J$ is shortened as $J^n$ for suggestiveness).

**Theorem 4.7.** Let $I, J \in \text{Alg}_\mathcal{O}(R\text{-Mod})$, and let $f : I \to J^d$ be a morphism in $\text{Alg}_\mathcal{O}(R\text{-Mod})$. Then $f$ induces compatible $\mathcal{O}$-algebra maps $f_n : I^n \to J^{dn}$ for all $n$, and the assignment $f \mapsto f_n$ is functorial and preserves weak equivalences.

We say that a map $f \in [I, J]_{\text{Alg}}$ has AQ-filtration $s$ if $f$ factors in $\text{ho}(\text{Alg}_\mathcal{O}(R))$ as the composition of $s$ maps

$$I = I(0) \xrightarrow{f(1)} I(1) \xrightarrow{f(2)} I(2) \to \cdots \to I(s-1) \xrightarrow{f(s)} I(s) = J$$

such that $TQ(f(i))$ is null for each $i$.

**Theorem 4.8.** Let $f \in [I, J]_{\text{Alg}}$ have AQ-filtration $s$. Then there exists $\tilde{f} \in [I, J^{2^s}]_{\text{Alg}}$ such that

$$I \xrightarrow{f} J \xrightarrow{\tilde{f}} J$$

commutes in $\text{ho}(\text{Alg}_\mathcal{O}(R))$.

These results are essential for follow up work of Nick Kuhn, where they are used to derive “exponential convergence” results for some spectral sequences.

Technically speaking, the work in [14] relies heavily on the work in [18], since in order to iterate the functors $\mathcal{O}_{\geq n} \circ \mathcal{O}(-)$ in a homotopically meaningful way one needs to understand associated cofibrancy properties.

4.3 Graph stable equivalences of equivariant operads

It is well known that a map $\mathcal{O} \to \tilde{\mathcal{O}}$ of operads in $\text{Sp}^{\Sigma}$ induces a (Quillen) equivalence between the corresponding categories of algebras iff $\mathcal{O}(n) \to \tilde{\mathcal{O}}(n)$ is an underlying weak equivalence of spectra for each $n$.

One key insight of Blumberg and Hill in [1] is that when dealing with genuine equivariant spectra, however, the story is more delicate. Explicitly, they show that for certain special operads in spaces, which they call $N_\infty$-operads, a map $\mathcal{O} \to \tilde{\mathcal{O}}$ induces a Quillen equivalence of algebras iff each $\mathcal{O}(n)^W \to \tilde{\mathcal{O}}(n)^W$ is an underlying equivalence when $W$ is a so-called “$G$-graph-subgroup” of $G \times \Sigma_n$.

In joint work with Markus Hausmann in [13], we follow their lead by introducing a new notion of stable equivalence on operads on $G$-spectra, which we call $G$-graph stable equivalences, and proving the following.

**Theorem 4.9.** For $\mathcal{O}$ any operad in $(\text{Sp}^{\Sigma})^G$ there is a projective positive $S$ $G$-stable model structure on $\text{Alg}_{\mathcal{O}}$. Further, $\mathcal{O} \circ \mathcal{O}(-) : \text{Alg}_{\mathcal{O}} \rightleftarrows \text{Alg}_{\tilde{\mathcal{O}}} : \text{fgt}$ is a Quillen equivalence if and only if $\mathcal{O} \to \tilde{\mathcal{O}}$ is a $G$-graph stable equivalence.
4.4 Excisiveness of truncated operadic functors

One of the key steps of the proof of Theorem 3.4 in my thesis was the following.

**Proposition 4.10.** For any \(m\)-truncated right \(O\)-module \(M\) the associated functor \(M \circ O(-) : \text{Alg}_O(\Sigma \mathcal{G}) \to \Sigma \mathcal{G}\) is \(m\)-excisive.

The original proof of this result is both long and indirect, requiring an understanding of both the stabilization of \(\text{Alg}_O\) and of multilinear functors between those. In [19], by refining certain well known filtrations of diagrams of algebras over colored operads, I provide a direct simple proof of Proposition 4.10.

5 Current and future work

There are several natural directions in which I want to continue my previous work, which I now list.

5.1 Model categories of equivariant operads

Following [1] and [13], which roughly speaking identify the “correct” notion of weak equivalence between equivariant operads in spaces or in spectra, I am currently working on building suitable model categories of operads with those “correct” weak equivalences. Further, as foreshadowed by the several notions of \(N_\infty\) operad in [1], one should actually expect there to be several such (localized) model structures, with varying classes of cofibrations and weak equivalences, such that the \(N_\infty\) operads are the cofibrant replacements of the commutative operad in each model structure.

The study of these differing notions of cofibration was another motivation for the colored operad filtrations introduced in [19] (since operads are themselves algebras over a certain colored operad).

5.2 Equivariant Goodwillie calculus on algebras over equivariant operads

In recent work ([3]), Dotto develops a theory of equivariant Goodwillie calculus. A key feature of that work is that, rather than just a tower of functors as in (2.1), \(G\)-equivariant calculus produces a tree indexed by finite \(G\)-sets \(F\) and containing inside it a so called “genuine tower” indexed by the \(G\)-sets \(n \cdot G\).

This is very much reminiscent of the key insight of [1] showing that the right notion of weak equivalence for \(G\)-operads needs to account for graph subgroups, and we expect the two theories to be closely related.

We hence list a few natural conjectures extending some of our results about Goodwillie calculus in \(\text{Alg}_O\) to the equivariant case.

Theorem 3.4 should extend directly to the genuine Goodwillie tower of an operad in genuine \(G\)-spectra with the notion of weak equivalence between genuine derivatives given by the \(G\)-graph stable equivalences of [13].

Similarly, Theorem 3.5 should generalize directly when dealing with full \(n \cdot G\)-excisiveness. More interestingly, we conjecture that a similar result should hold for \(F\)-excisiveness for a more general \(G\)-set \(F\), although in this case the story should be substantially more subtle: we expect that “\(F\)-truncating an operad
such models for $E_n$ and $E_m$ are suitably cofibrant. However, not much is known in general about $O \otimes \bar{O}$ or, even worse, tensor products of three or more operads, since the definition of the tensor product involves a tricky quotient.

One way to better understand the homotopical properties of $\otimes$ would be to produce filtrations of $O$ and $\bar{O}$ when $O, \bar{O}$ are suitably cofibrant operads. In fact we conjecture the following: denoting by $Op$ the category of operads and by $Op_{O} \downarrow$ the category of operads receiving a map from $O$, it should be possible to make the adjunction

$$O \otimes (-) : Op \to Op_{O} \downarrow; (-)^O$$

into a Quillen adjunction by giving $Op_{O} \downarrow$ the induction projective model structure. The key challenge to proving such a result is to build suitable filtrations of pushouts involving the functor $O \otimes (-)$. This is similar to the technical work in [19], which in particular specializes to give similar filtrations on $Op$ itself, and I hope that the finer degree of control of those filtrations (compared with related filtrations elsewhere in the literature) will ease the task of building the desired $Op_{O} \downarrow$ filtrations.

Ultimately, I hope such filtrations will help in answering the following two questions. It has long been conjectured that the little $n$-cubes operads $E_n$ are self-Koszul dual (up to a shift), the most compelling evidence probably being Fresse’s result [6] at the chain complex level. Since the conjecture is obvious for $E_1$ and $E_n \simeq E_1 \otimes \cdots \otimes E_1$, a better understanding of $\otimes$ should provide a different line of attack for this conjecture.

The $N_\infty$ operads of [1] encode differing flavors of commutativity in the equivariant context. It then becomes natural to ask what can be said about $O \otimes \bar{O}$ when $O, \bar{O}$ are both $N_\infty$, the most natural conjecture being that $O \otimes \bar{O}$ should again be $N_\infty$ of with its "$N_\infty$-type" the maximum of the types of $O$ and $\bar{O}$.

References

\[\begin{align*}
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\end{align*}\]