I am an algebraic topologist. My main interests are the theory of topological and spectral operads, their Koszul duality, Goodwillie calculus, and, more recently, the interaction of these topics with genuine equivariant homotopy theory.

1 Main current research project

My current main project concerns the study of the interaction between the theory of operads and genuine equivariant homotopy theory.

Briefly, an operad (introduced by May in [13]) consists of a sequence $O(n)$ of sets/spaces of \textquotedblleft{}$n$-ary operations\textquotedblright{} together with $\Sigma_n$-actions and suitable compositions. The main point of operad theory is then the study of the algebras over a fixed operad $O$, which are objects $X$ (in some appropriate monoidal category $(\mathcal{C}, \otimes)$) together with $n$-ary operations suitably indexed by $O(n)$.

On the other hand, equivariant homotopy theory deals with the correct notion of homotopy when in the presence of the action of a group $G$. For example, a $G$-equivariant map $f:X \to Y$ between spaces with $G$-actions is considered a \textquotedblleft{}genuine equivariant homotopy equivalence\textquotedblright{} only if $f$ induces \textquotedblleft{}non-equivariant equivalences\textquotedblright{} $f^H:X^H \to Y^H$ between fixed points for all subgroups $H \leq G$.

Work of Hill, Hopkins and Ravenel on the Kervaire invariant problem has revealed the importance of norms (given a $G$-object $X$ and $G$-set $A$, the associated norm is the tensor product $\otimes_A X$ together with an appropriate mixed $G$-action) and norm maps (i.e. maps between norms). Furthermore, follow up work of Blumberg and Hill in [1] has shown that (i) norm maps can be encoded in terms of $G$-equivariant operads by looking at certain fixed points $O(n)\Gamma$ for special subgroups $\Gamma \leq G \times \Sigma_n$; (ii) general $G$-equivariant operads contain only some types of norm maps; (iii) each $G$-equivariant operad $O$ has associated families of $H$-sets (for $H$ ranging over subgroups $H \leq G$) which encode the norm maps present in $O$; (iv) these families of $H$-sets satisfy a number of novel and non-obvious closure conditions. Moreover, [1] calls families satisfying such closure conditions \textit{indexing systems} and makes the key (and non-trivial) conjecture that all indexing systems can be realized by some operad.

My current project stemmed from an attempt to find a conceptual understanding for the (somewhat opaque) closure conditions for indexing systems. In doing so, and jointly with Peter Bonventre, we discovered a theory of $G$-trees (a non-obvious generalization of the trees of Cisinski-Moerdijk-Weiss), which (i) provide a compact way to think about norm maps and their closure properties; (ii) suggest alternate models for equivariant operads.
As a simple first example, we note that given subgroups $G \geq H_1 \geq H_2 \geq H_3$, it is obvious from the theory of $G$-trees (cf. [18, §3]) that operadic composition restricts to maps

$$\mathcal{O}(H_1/H_2)^{H_1} \times \mathcal{O}(H_2/H_3)^{H_2} \to \mathcal{O}(H_1/H_3)^{H_1},$$

and such maps are responsible for the most subtle of the closure conditions for indexing systems, dubbed self-induction. However, we note that though in hindsight the proofs in [1] do rely on the existence of the maps (1.1), the existence of such maps is not clear from the perspective therein.

### 1.1 Past work

Moerdijk and Weiss in [14] and Cisinski and Moerdijk in the follow up paper [2] develop a suitable category $\Omega$ of trees and, writing $d\text{Set} = \text{Set}^{\Omega^{op}}$, build a model structure on $d\text{Set}$ whose fibrant objects are “operads up to homotopy” called $\infty$-operads. Furthermore, follow up work of Cisinski and Moerdijk in [3], [4] establishes a Quillen equivalence

$$W : d\text{Set} \rightleftarrows \text{sOp : } hcN,$$

where $\text{sOp}$ is the category of (colored) simplicial operads.

In [18], I generalized the work in [14] and [2] to obtain the following.

**Theorem 1.3.** There is a model structure on $d\text{Set}^G$ for which the fibrant objects are the $G$-$\infty$-operads.

Furthermore, there are variant model structures associated to each indexing system.

Here the notion of $G$-$\infty$-operads is defined using the category $\Omega_G$ of $G$-trees I discovered jointly with Peter Bonventre. In particular, we note that intuitively $G$-$\infty$-operads are “operads with norm maps up to homotopy” and that, for this reason, Theorem 1.3 can not be obtained by simply applying “formal genuine model structure” constructions (as found in, for example, [20]) to the original non-equivariant Cisinski-Moerdijk result in [2]. Indeed, such “formal genuine model structures” would only lead to “operads without norm maps up to homotopy”.

### 1.2 Current work

A classical result of Elmendorf states that there is a Quillen equivalence $\text{Top}^G \simeq \text{Top}^{O_G^{op}}$. Here $O_G$ is the $G$-orbit category, formed by the $G$-sets $G/H$ for each $H \leq G$. $\text{Top}^G$ is given its “genuine model structure”, where weak equivalences are detected by looking at all fixed points, and $\text{Top}^{O_G^{op}}$ is given its “projective model structure”, with weak equivalences detected by looking at each level of the presheaf.

Elmendorf’s proof is fairly robust: reasonable conditions on a model category $\mathcal{C}$ allow for analogous Quillen equivalences $\mathcal{C}^G \simeq \mathcal{C}^{O_G^{op}}$. Indeed, this allows for equivalences $\text{sCat}^G \simeq \text{sCat}^{O_G^{op}}$ and $\text{sOp}^G \simeq \text{sOp}^{O_G^{op}}$, where $\text{sCat}$, $\text{sOp}$ denote the categories of simplicial categories and of simplicial (colored) operads. However, in the operad case $\text{sOp}^G$ such an equivalence only works if studying operads without norm maps.
To produce the more interesting analogue of Elmendorf’s theorem for operads with norm maps, one instead needs to replace the category $s\text{Op}^{\mathsf{G}}_G$ with a more complex one. Informally, the flaw of $s\text{Op}^{\mathsf{G}}_G$ is that its presheaf direction and its operadic direction are required to simply commute, and this fails to capture the more interesting aspects of the interaction between the two, namely the existence of maps as in (1.1).

In a nearly finished joint paper with Peter Bonventre, we define a category $s\text{Op}_G$ of what we call genuine equivariant operads and which, informally, can be thought of as objects with maps as in (1.1), satisfying strict associativity, but with restriction maps enforced only in a presheaf way (by analogy with $\mathsf{Top}^{\mathsf{G}}_G$). We then prove the following.

**Theorem 1.4.** The projective model structure (with weak equivalences detected at each level) on $s\text{Op}_G$ exists.

**Theorem 1.5.** There is a Quillen equivalence
\[ s\text{Op}_G \rightleftarrows s\text{Op}^{\mathsf{G}}_G \]
where the category $s\text{Op}_G$ of genuine equivariant operads is given the projective model structure and the category $s\text{Op}^{\mathsf{G}}_G$ of (regular) $\mathsf{G}$-equivariant operads is given its “genuine with norms” model structure.

As a consequence, we deduce the Blumberg-Hill realization conjecture. Briefly, this follows since indexing systems are obviously realized by a genuine equivariant operad in $s\text{Op}_G$.

**Corollary 1.6.** Any indexing system is realized by a $\mathsf{G}$-equivariant operad.

### 1.3 Future work

#### 1.3.1 Generalize Cisinski-Moerdijk-Weiss

One of the main goals of my project is to complete the generalization of the Cisinski-Moerdijk-Weiss results to the equivariant setup by adapting the work in [3] and [4] and obtaining the following.

**Conjecture 1.7.** There is a Quillen equivalence
\[ W^\mathsf{G}:\mathsf{dSet}^\mathsf{G} \rightleftarrows s\text{Op}^{\mathsf{G}}_G:hcN^\mathsf{G}. \] (1.8)

where $s\text{Op}^{\mathsf{G}}_G$ is given its “genuine with norms” model structure.

While the full proof of this result is still work in progress, we list some already established positive results in this direction: (i) the right adjoint $hcN^\mathsf{G}$ sends “locally $\mathsf{G}$-graph fibrant” $\mathsf{G}$-operads to $\mathsf{G}$-$\infty$-operads; (ii) in both the case of locally $\mathsf{G}$-graph fibrant $\mathsf{G}$-operads and the case of $\mathsf{G}$-$\infty$-operads the traditional “homotopy operad” construction can be used to build a “genuine equivariant operad” (as in the previous section).

Furthermore, preliminary work in this direction suggests that most of the adaptations necessary to establish this result mirror the adaptations found in [18], indicating that proving this conjecture may be mostly straightforward.
1.3.2 Boardman-Vogt tensor product of $N_\infty$-operads

One of the main highlights of [1] is the introduction of the notion of $N_\infty$-operads, which are equivariant operads encoding “different degrees of equivariant commutativity”, in a way somewhat reminiscent to the classical non-equivariant $E_n$-operads. In fact, similarly to the classical result that $O \otimes O'$ is an $E_{n+m}$-operad when $O$ is $E_n$ and $O'$ is $E_m$, it is reasonable for formulate a similar conjecture for $N_\infty$-operads.

Conjecture 1.9. For indexing systems $\mathcal{F}$, $\mathcal{F}'$ and $O$ a $N_\infty$-operad and $O'$ a $N_{\mathcal{F}'}$-operad then $O \otimes O'$ is a $N_{\mathcal{F} \vee \mathcal{F}'}$-operad where $\mathcal{F} \vee \mathcal{F}'$ denotes the indexing system generated by $\mathcal{F}$ and $\mathcal{F}'$.

Briefly, this conjecture amounts to showing that graph subgroup fixed points $\hat{\hat{O}} \otimes \hat{\hat{O}}'$ are appropriately either empty or contractible in a way depending on $\mathcal{F}$, $\mathcal{F}'$, $\Gamma$. The main challenge is posed by the Boardman-Vogt tensor product $\otimes$, which is a hard operation to study. Nonetheless, using the $G$-tree language, we have proven that such fixed points are suitably empty or connected, a non obvious claim. The proof of the full result is the subject of current work.

2 Other projects

2.1 Goodwillie calculus for algebras over an operad

The main goal of my thesis work ([15],[16]) was the study of Goodwillie calculus on the category $\text{Alg}_O$ of algebras over a spectral operad $O$.

The main results in [15] were a generalization of Goodwillie’s original results in [7], [8] and [9] from the setup of functors $\mathcal{C} \to \mathcal{D}$ between either the categories of pointed spaces or spectra to more general categories, as follows.

Theorem 2.1. Let $\mathcal{C}, \mathcal{D}$ be cofibrantly generated model categories, and $F: \mathcal{C} \to \mathcal{D}$ be a homotopy functor. Assume further that $\mathcal{C}$ is either pointed and simplicial or that cofibrations in $\mathcal{C}$ are categorical injections.

Then there exists a universal $n$-excisive approximation $P_nF$ to $F$.

Theorem 2.2. Assume $\mathcal{C}, \mathcal{D}$ are pointed simplicial model categories, and $F: \mathcal{C} \to \mathcal{D}$ is a simplicial $n$-homogeneous functor. Assume further that spectra categories $\text{Sp}(\mathcal{C})$, $\text{Sp}(\mathcal{D})$ can be defined, where weak equivalences of spectra are detected by the $\Omega^\infty_-$ functors; in $\mathcal{D}$ finite homotopy limits commute with countable directed homotopy colimits.

Then $F$ can be factored through the spectra categories, i.e., $F \simeq \Omega^\infty \circ F \circ \Sigma^\infty$, where $F: \text{Sp}(\mathcal{C}) \to \text{Sp}(\mathcal{D})$ is itself $n$-homogeneous.

Applying the Goodwillie calculus set-up above to $\text{Alg}_O(\text{Sp}^\Sigma)$ I showed the following in [16].

Theorem 2.3. Let $O$ be a spectral operad. Then there is a zigzag of Quillen equivalences between $\text{Sp}(\text{Alg}_O)$ and $O(1)-\text{Mod}$. Further, this identifies $\Sigma^\infty$ with Topological André-Quillen homology and $\Omega^\infty$ with the trivial algebra functor.

Theorem 2.4. The Goodwillie tower of the identity for $\text{Alg}_O$ is given by the (left derived) truncation functors $O_{cn} \circ (-)$. Further, the $n$-th derivative is $O(n)$ itself with its canonical $(O(1), O(1)^n)$-bimodule structure.
There exists a non trivial extension of Goodwillie calculus to the equivariant setting developed by Dotto in [5]. I conjecture that this theory should interact with the genuine equivariant operads of mine and Peter Bonventre’s to produce a suitable analogue of Theorem 2.4.

2.2 Operad bimodules and André-Quillen filtrations

In joint work with Nick Kuhn in [12], we study the filtration $O_{\Sigma n} \circ \mathcal{O} (-)$ dual to the Goodwillie tower in Theorem 2.4. A key observation is that by iterating the filtration (i.e. applying the filtration to itself!) the operad structure leads to extra maps between the filtration levels. Our main results are as follows (where $O_{\Sigma n} \circ \mathcal{O} J$ is shortened as $J^n$ for suggestiveness).

**Theorem 2.5.** Let $I, J \in \text{Alg}_O(R - \text{Mod})$, and let $f : I \rightarrow J^d$ be a morphism in $\text{Alg}_O(R - \text{Mod})$. Then $f$ induces compatible $O$–algebra maps $f_n : I^n \rightarrow J^{dn}$ for all $n$, and the assignment $f \mapsto f_n$ is functorial and preserves weak equivalences.

We say that a map $f \in [I, J]_{\text{Alg}}$ has AQ-filtration $s$ if $f$ factors in $\text{ho}(\text{Alg}_O(R))$ as the composition of $s$ maps

$$I = I(0) \xrightarrow{f(1)} I(1) \xrightarrow{f(2)} I(2) \rightarrow \cdots \rightarrow I(s - 1) \xrightarrow{f(s)} I(s) = J$$

such that $TQ(f(i))$ is null for each $i$.

**Theorem 2.6.** Let $f \in [I, J]_{\text{Alg}}$ have AQ-filtration $s$. Then there exists $\tilde{f} \in [I, J^{2^s}]_{\text{Alg}}$ such that

$$\xymatrix{ I \ar[r]^f \ar[d] & J \ar[d] \\ I \ar[r] & J^{2^s} }$$

commutes in $\text{ho}(\text{Alg}_O(R))$.

These results are essential for follow up work of Nick Kuhn, where they are used to derive “exponential convergence” results for some spectral sequences.

Technically speaking, the work in [12] relies heavily on my work in [17], since in order to iterate the functors $O_{\Sigma n} \circ \mathcal{O} (-)$ in a homotopically meaningful way one needs to understand associated cofibrancy properties established therein.

In current follow up joint work with Nick Kuhn we are studying the closely related filtrations of functors between algebra categories associated to the skeletal filtrations of right $\mathcal{O}$-modules.

2.3 $\Sigma$-cofibrancy of smash powers of positive spectra

The work in both [16] and [12] crucially relied on the positive model category structure on symmetric spectra $\text{Sp}^\Sigma$, which has long been known to be convenient when studying algebras over an operad in spectra (e.g. [19], [6], [10]). Informally, this convenience comes from the fact that if $X$ is a positive cofibrant spectrum then the $n$-fold smash power $X^{\wedge n}$ is “almost $\Sigma_n$-cofibrant”. However, this informal statement is not literally true, and the closest approximate formulations of such a result found elsewhere in the literature were not strong enough.
for our purposes. Solving this problem was the main goal of [17], where I introduced a notion of lax $\Sigma_n$-cofibrancy for $n$-fold powers of positive spectra which I then proved to have the same key properties as genuine $\Sigma_n$-cofibrancy.

This was used to prove the main result of [17]: that, under mild cofibrancy conditions, "operadic pushout products"

\[ M \circ O N \bigvee_{M \circ O N} \tilde{M} \circ O \tilde{N} \xrightarrow{f_1 \circ O f_2} \tilde{M} \circ O \tilde{N}. \]

are cofibrations, trivial if $f_1$ or $f_2$ are.

While technical, this is quite a powerful result, which easily implies all of the following (most of which strengthen previous results in [10], [11]), which were necessary in [16] and [12].

**Theorem 2.7.** For $O$ any operad in $Sp^\Sigma$ there is a projective positive $S$ model structure on $Alg_O$. Further, $O \circ_C (-): Alg_O \xrightarrow{\simeq} Alg_{O}: fgt$ is a Quillen equivalence when $O \to \tilde{O}$ is a stable equivalence.

**Theorem 2.8.** Let $O$ be an operad in $Sp^\Sigma$ which is level cofibrant. Then the forgetful functor $fgt: Alg_O \to Sp^\Sigma$ sends cofibrations between cofibrant objects to cofibrations between cofibrant objects.

**Theorem 2.9.** For an operad $O$, right $O$-module $M$ and left $O$-module $N$ satisfying mild cofibrancy hypothesis, the bar construction $B_n(M, O, N) = M \circ O^{\otimes n} \circ N$ is Reedy cofibrant.

**Theorem 2.10.** If $A$ is cofibrant in $Alg_O$, the functor $Mod_O \xrightarrow{(\cdot) \circ A} Sp^\Sigma$ preserves homotopy fiber sequences.

**References**


