SOME CONVERGENCE ARGUMENTS FOR MATRIX GROUP-VALUED SDE SOLUTIONS

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This note is a collection of standard convergence results in a matrix group setting which are necessary to resolve certain convergence issues on Lie groups in Section 3 of the paper [5]. For all notation, please see this reference.

The following standard proposition will be used in the sequel; see for example Driver [1].

Proposition 1. *Suppose $p \in [2,\infty)$, α_t is a predictable $\mathbb{R}^d$–valued process, A_t is a predictable $\mathrm{Hom}(\mathfrak{g}_0, \mathbb{R}^d)$–valued process, and*

$$Y_t := \int_0^t A_\tau \, d\vec{b}_\tau + \int_0^t \alpha_\tau \, d\tau = \int_0^t A_\tau X_i \, db^i_\tau + \int_0^t \alpha_\tau \, d\tau,$$

where $\{b^1, \ldots, b^k\}$ are k independent real Brownian motions. Then

$$\mathbb{E} \sup_{\tau \le t} |Y_\tau|^p \le C_p \left\{ \mathbb{E} \left(\int_0^t |A_\tau|^2 \, d\tau \right)^{p/2} + \mathbb{E} \left(\int_0^t |\alpha_\tau| \, d\tau \right)^p \right\},$$

where

$$|A|^2 = \mathrm{tr}(AA^*) = \sum_{i=1}^n (AA^*)_{ii} = \sum_{i,j} A_{ij} A_{ij} = \mathrm{tr}(A^*A).$$

Notation 2. Let δ_n denote constants such that $\lim_{n\to\infty} \delta_n = 0$. Also, write $f \lesssim g$, if there is a positive constant C so that $f \le Cg$.

Let $M = \mathrm{End}(\mathfrak{g})$ and $A_i = \mathrm{ad}_{X_i} \in M$, for $i = 1, \ldots, k$. Further, let $B := A_i \, b^i = \mathrm{ad}_{X_i} \, b^i = \mathrm{ad}_{\vec{b}}$, where $\{b^i\}_{i=1}^k$ is a set of independent real-valued Brownian motions, and let $H := A_i \, h^i = \mathrm{ad}_{X_i} \, h^i$, where $h = (h^1, \ldots, h^k)$ is a fixed element of $\mathscr{H}$.

Proposition 3. *Let $W : [0,1] \times \mathscr{W} \to M$ denote the solution to the Stratonovich differential equations*

$$dW = W \circ dB = W \, dB + \frac{1}{2} \sum_{i=1}^k W A_i^2 \, dt, \text{ with } W_0 = I, \tag{0.1}$$

and let $W^s : [0,1]^2 \times \mathscr{W} \to M$ denote the solution to the equation

$$dW^s = W^s \left(\circ dB + s dH \right), \text{ with } W^s_0 = I. \tag{0.2}$$

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Then

$$\lim_{s\downarrow 0}\mathbb{E}\sup_{\tau\le 1}|W^s_\tau - W_\tau|^p = 0,$$

for all $p\in(1,\infty)$.

Proof. Writing Equation (0.2) in Itô form gives

$$dW^s = W^s dB + \frac{1}{2}\sum_{i=1}^k W^s A_i^2 dt + sW^s dH.$$

Then by Proposition 1, for any $s\in[0,1]$,

$$\begin{aligned}\mathbb{E}\sup_{\tau\le 1}|W^s_\tau|^p &\lesssim 1 + \mathbb{E}\left(\int_0^t |W^s_\tau|^2\,d\tau\right)^{p/2} + \mathbb{E}\left(\int_0^t |W^s_\tau|\,d\tau\right)^p \\ &\lesssim 1 + \mathbb{E}\int_0^t |W^s_\tau|^p\,d\tau,\end{aligned}$$

for all $t\in[0,1]$. An application of Gronwall's inequality then shows that

$$\mathbb{E}\sup_{\tau\le 1}|W^s_\tau|^p \le Ce^C,$$

where these constants are independent of s; that is, there exists some finite constant C_p such that

$$\sup_{s\in[0,1]}\left(\mathbb{E}\sup_{\tau\le 1}|W^s_\tau|^p\right)\le C_p,$$

for all $p\in(1,\infty)$.

Now let $\varepsilon^s := W^s - W$. Then from Equation (0.1), ε^s satisfies

$$\begin{aligned}d\varepsilon^s &= (W^s - W)dB + \frac{1}{2}\sum_{i=1}^k (W^s - W)A_i^2 dt + sW^s dH \\ &= \varepsilon^s dB + \frac{1}{2}\sum_{i=1}^k \varepsilon^s A_i^2 dt + sW^s dH.\end{aligned}$$

Then applying Proposition 1 gives

$$\mathbb{E}\sup_{\tau\le 1}|\varepsilon^s_\tau|^p \lesssim \int_0^t |\varepsilon^s|^p d\tau + \delta_s,$$

for all $t\in[0,1]$, where

$$\delta_s = \sum_{i=1}^k \int_0^t s^p\left|W^s \dot{h}^i\right|^p d\tau \to 0,$$

as $s\downarrow 0$, by the dominated convergence theorem. Thus, by Gronwall's inequality,

$$\mathbb{E}\sup_{\tau\le 1}|W^s_\tau - W_\tau|^p = \mathbb{E}\sup_{\tau\le 1}|\varepsilon^s_\tau|^p \le \delta_s e^C \to 0,$$

as $s\downarrow 0$, for all $p\in(1,\infty)$. ∎

Proposition 4. *Let W be the solution to Equation (0.1). Then $W_t \in \mathcal{D}^\infty(\mathrm{End}(\mathfrak{g}))$, for all $t \in [0,1]$, and $\partial_h W : [0,1] \times \mathscr{W} \to M$ solves the equation*

$$\partial_h W_t = \left(\int_0^t W_\tau \dot{H} W_\tau^{-1}\, d\tau \right) W_t. \tag{0.3}$$

Furthermore, if W^s is the solution to Equation (0.2), then

$$\lim_{s \downarrow 0} \mathbb{E} \sup_{\tau \le 1} \left| \frac{W_\tau^s - W_\tau}{s} - \partial_h W_\tau \right|^p = 0,$$

for all $p \in (1, \infty)$.

Proof. Note that $W = \mathrm{Ad}_\xi$ satisfies the Stratonovich stochastic differential equation

$$d\,\mathrm{Ad}_\xi = \mathrm{Ad}_\xi \circ \mathrm{ad}_{db} = \mathrm{Ad}_\xi\, \mathrm{ad}_{X_i} \circ db^i,$$

a linear differential equation with smooth coefficients. Then by Theorem V-10.1 of Ikeda and Watanabe [4], $W_t = \mathrm{Ad}_{\xi_t} \in \mathcal{D}^\infty(\mathrm{End}(\mathfrak{g}))$ componentwise with respect to some basis.

Now let $\Psi : [0,1] \times \mathscr{W} \to M$ denote the solution to the equation

$$d\Psi = \Psi W \circ dB + W dH = \Psi W dB + \frac{1}{2} \sum_{i=1}^k \Psi A_i^2 dt + W dH,$$

with $\Psi_0 = 0$. Let $\varepsilon^s := \left(\frac{W^s - W}{s} - \Psi \right)$. Then

$$\begin{aligned} d\varepsilon^s &= \left(\frac{W^s - W}{s} - \Psi \right) dB + \frac{1}{2} \sum_{i=1}^k \left(\frac{W^s - W}{s} - \Psi \right) A_i^2 dt + (W^s - W) dH \\ &= \varepsilon^s dB + \frac{1}{2} \sum_{i=1}^k \varepsilon^s A_i^2 dt + (W^s - W) dH. \end{aligned}$$

By Proposition 1, this implies that

$$\mathbb{E} \sup_{\tau \le t} |\varepsilon_\tau^s|^p \lesssim \int_0^t |\varepsilon^s|^p d\tau + \delta_s,$$

for all $t \in [0,1]$, where

$$\delta_s = \sum_{i=1}^k \int_0^t \left| (W^s - W) \dot{h}^i \right|^p d\tau \to 0,$$

as $s \downarrow 0$, by Proposition 3 and the dominated convergence theorem. An application of Gronwall's inequality then gives

$$\mathbb{E} \sup_{\tau \le 1} \left| \frac{W_\tau^s - W_\tau}{s} - \Psi_\tau \right|^p = \mathbb{E} \sup_{\tau \le 1} |\varepsilon_\tau^s|^p \le \delta_s e^C \to 0,$$

as $s \downarrow 0$, for all $p \in (1, \infty)$.

By Theorem VIII.2B of Elworthy [2], there exists a modification of W_t^s so that the mapping $s \mapsto W_t^s$ is smooth. Let $F \in \mathcal{S}$ be a smooth cylinder function on $\mathscr{W}$. By the above convergence,

$$\frac{d}{ds}\bigg|_0 \mathbb{E}\left[W_t^s F\right] = \mathbb{E}[\Psi_t F].$$

Consider also

$$\begin{aligned}\frac{d}{ds}\bigg|_0 \mathbb{E}\left[W_t^s F\right] &= \int \frac{d}{ds}\bigg|_0 W_t(b+sh)F(b)\,d\mu(b)\\ &= \int W_t(b)\frac{d}{ds}\bigg|_0 F(b-sh)\,d\mu(b-sh)\\ &= \int W_t(b)\left[-\partial_h F(b) + \left(\int_0^1 \dot{h}_s\cdot db_s\right)F(b)\right]\,d\mu(b) = \mathbb{E}[W_t\partial_h^* F];\end{aligned}$$

where the third equality follows from differentiating the shifted measure, and the final equality follows from a standard integration by parts (see for example Theorems 8.1.1 and 8.22 of Hsu [3]. This then implies that $\mathbb{E}[\Psi_t F] = \mathbb{E}[W_t\partial_h^* F]$, and so $\partial_h W_t = \Psi_t$. Thus, $\partial_h W$ satisfies the differential equation

$$d(\partial_h W) = \partial_h W \circ dB + W dH,$$

with $\partial_h W_0 = 0$. Equation (0.3) then follows from an application of Duhamel's principle. ∎

Proposition 5. *Let W be the solution to Equation (0.1), and let $\overline{W} : [0,1]\times\mathscr{W} \to \mathrm{End}(\mathfrak{g})$ be given by*

$$\overline{W}_t := \int_0^t W_\tau\,d\tau.$$

Then $\overline{W} \in \mathcal{D}^\infty(\mathscr{H}(\mathrm{End}(\mathfrak{g})))$ (see Remark 6).

Remark 6. Since $\mathrm{End}(\mathfrak{g}) \cong \mathbb{R}^N$ for some N, $\mathscr{H}(\mathrm{End}(\mathfrak{g})) \cong \mathscr{H}(\mathbb{R}^N)$.

Proof. Let $V = W^{-1} : [0,1]\times\mathscr{W} \to \mathrm{End}(\mathfrak{g})$. By differentiating the identity $W_t W_t^{-1} = I$, one may verify that V satisfies the differential equation

$$dV = -\circ dB\; V = -A_i V \circ db^i, \text{ with } V_0 = I.$$

Let $V_t^s : [0,1]^2\times\mathscr{W} \to \mathrm{End}(\mathfrak{g})$ denote the solution to the equation

$$dV_t^s = -(\circ dB + s dH)V_t^s = -A_i V_t^s \circ db_t^i - sA_i V_t^s \dot{h}_t^i dt, \text{ with } V_0^s = I. \tag{0.4}$$

By the same arguments as in Propositions 3 and 4,

$$\lim_{s\downarrow 0}\mathbb{E}\sup_{\tau\le 1}|V_\tau^s - V_\tau|^p = 0, \tag{0.5}$$

$$\partial_h V_t = -V_t\int_0^t V_\tau^{-1}\dot{H}V_\tau\,d\tau = -V_t\int_0^t W_\tau\dot{H}V_\tau\,d\tau, \tag{0.6}$$

and

$$\lim_{s\downarrow 0}\mathbb{E}\sup_{\tau\le 1}\left|\frac{V^s_\tau - V_\tau}{s} - \partial_h V_\tau\right|^p = 0, \tag{0.7}$$

for all $p\in(1,\infty)$. From the proof of Proposition 3, there exists a finite constant C_p such that

$$\sup_{s\in[0,1]}\left(\sup_{\tau\le 1}\mathbb{E}|W^s_\tau|^p\right)\le C_p,$$

for all $p\in(1,\infty)$. By a similar proof,

$$\sup_{s\in[0,1]}\left(\sup_{\tau\le 1}\mathbb{E}|V^s_\tau|^p\right)\le C_p,$$

for all $p\in(1,\infty)$.

Now let $W_{s,t} := V_s W_t$. Note that $W^{-1}_{s,t} = W_{t,s}$, and $W_{s,t}W_{t,u} = W_{s,u}$. By the above bounds on W and V, there exist finite constants C_p such that

$$\sup_{\tau_1,\tau_2\le t}\mathbb{E}|W_{\tau_1,\tau_2}|^p\le C_p, \tag{0.8}$$

for all $p\in(1,\infty)$. Using this notation and Equation (0.3),

$$\begin{aligned}\partial_h W_t &= \left(\int_0^t W_\tau A_i W_\tau^{-1}\dot h^i_\tau\, d\tau\right) W_t\\ &= \int_{\{0\le\tau\le t\}} W_\tau A_i W_{\tau,t}\dot h^i_\tau\, d\tau,\end{aligned}\tag{0.9}$$

and so, for $t_1\le t$ and $\alpha_1 = 1\ldots,k$,

$$[D_{t_1}W_t]^{\alpha_1} = \int_{\{0\le\tau\le t_1\}} W_\tau A_{\alpha_1} W_{\tau,t}\, d\tau, \tag{0.10}$$

where, for $F\in\mathrm{End}(\mathfrak{g})$, $F=\sum_{i=1}^k F^i\otimes A_i$. For $t_1>t$, $D_{t_1}W_t = 0$.

Let $r\in\mathbb{N}$. For $\{t_1,\ldots,t_r\}\subset[0,1]$ and multi-index $\alpha=(\alpha_1,\ldots,\alpha_r)\subset\{1,\ldots,k\}^r$, let $D^{r,\alpha}_{t_r,\ldots,t_1}W_t$ denote the α^{th} component of $D^r_{t_r,\ldots,t_1}W_t$; that is,

$$D^{r,\alpha}_{t_r,\ldots,t_1}W_t := [D_{t_r}[\cdots[D_{t_2}[D_{t_1}W_t]^{\alpha_1}]^{\alpha_2}\cdots]^{\alpha_r}.$$

Now show that the following properties hold for any integer $r\ge 1$:

(P1) For any $p\in(1,\infty)$ and multi-index α, $W_t\in\mathcal{D}^{r,p}(\mathrm{End}(\mathfrak{g}))$ and

$$\sup_{\{t_1,\ldots,t_r\}\in[0,1]}\mathbb{E}|D^{r,\alpha}_{t_r,\ldots,t_1}W_t|^p<\infty.$$

(P2) Let $T:=\min\{t_1,\ldots,t_r\}$. If $T\le t$, then the r^{th} derivative of W_t satisfies the linear differential equation

$$D^{r,\alpha}_{t_r,\ldots,t_1}W_t = \int_{[0,T]^r} W_{T_1}A_{J_1}W_{T_1,T_2}A_{J_2}\cdots W_{T_{r-1},T_r}A_{J_r}W_{T_r,t}\, d\tau_r\cdots d\tau_1,$$

where T_i denotes the i^{th} smallest element of the set $\{\tau_1,\ldots,\tau_r\}$, and J_i denotes the index corresponding to T_i (that is, $J_i := \sum_{l=1}^r 1_{\{T_i=\tau_l\}}\alpha_l$). If $T>t$, then

$D^r_{t_1,\dots,t_r} W_t = 0$.

By Equation (0.10), the above holds for $r = 1$. Now assume that these properties hold up to and including order r. Note that Equations (0.6) and (0.9) imply that

$$\begin{aligned}\partial_h W_{s,t} &= (\partial_h V_s) W_t + V_s(\partial_h W_t) \\ &= -V_s \left(\int_0^s W_\tau A_i V_\tau \dot{h}^i_\tau \, d\tau \right) W_t + V_s \left(\int_0^t W_\tau A_i V_\tau \dot{h}^i_\tau \, d\tau \right) W_t \\ &= \int_{\{s \le \tau \le t\}} W_{s,\tau} A_i W_{\tau,t} \dot{h}^i_\tau \, d\tau.\end{aligned}$$

Let W^s denote the solution to Equation (0.2), and V^s denote the solution to (0.4). Then, for $W^s_{\tau_1,\tau_2} = V^s_{\tau_1} W^s_{\tau_2}$,

$$\begin{aligned}&\left| \frac{W^s_{\tau_1,\tau_2} - W_{\tau_1,\tau_2}}{s} - \partial_h W_{\tau_1,\tau_2} \right| \\ &\qquad = \left| V^s_{\tau_1} \frac{W^s_{\tau_2} - W_{\tau_2}}{s} - V_{\tau_1}(\partial_h W_{\tau_2}) + \frac{V^s_{\tau_1} - V_{\tau_1}}{s} W_{\tau_2} - (\partial_h V_{\tau_1}) W_{\tau_2} \right| \\ &\qquad \le \left| V^s_{\tau_1} - V_{\tau_1} \right| \left| \frac{W^s_{\tau_2} - W_{\tau_2}}{s} \right| + |V_{\tau_1}| \left| \frac{W^s_{\tau_2} - W_{\tau_2}}{s} - (\partial_h W_{\tau_2}) \right| \\ &\qquad\quad + \left| \frac{V^s_{\tau_1} - V_{\tau_1}}{s} - (\partial_h V_{\tau_1}) \right| |W_{\tau_2}|,\end{aligned}$$

and thus Proposition 4 and Equations (0.5) and (0.7) imply that

$$\lim_{s \downarrow 0} \sup_{\tau_1,\tau_2 \le t} \mathbb{E} \left| \frac{W^s_{\tau_1,\tau_2} - W_{\tau_1,\tau_2}}{s} - \partial_h W_{\tau_1,\tau_2} \right|^p = 0.$$

So for $a_1, a_2, b_1, b_2 \in [0,1]$,

$$\begin{aligned}&\lim_{s \downarrow 0} \mathbb{E} \left| \int_{a_2}^{b_2} \int_{a_1}^{b_1} \frac{W^s_{\tau_1,\tau_2} - W_{\tau_1,\tau_2}}{s} \, d\tau_1 \, d\tau_2 - \int_{a_2}^{b_2} \int_{a_1}^{b_1} \partial_h W_{\tau_1,\tau_2} \, d\tau_1 \, d\tau_2 \right|^p \\ &\qquad \le \lim_{s \downarrow 0} \mathbb{E} \int_{a_2}^{b_2} \int_{a_1}^{b_1} \left| \frac{W^s_{\tau_1,\tau_2} - W_{\tau_1,\tau_2}}{s} - \partial_h W_{\tau_1,\tau_2} \right|^p \, d\tau_1 \, d\tau_2 \\ &\qquad = \mathbb{E} \int_{a_2}^{b_2} \int_{a_1}^{b_1} \lim_{s \downarrow 0} \left| \frac{W^s_{\tau_1,\tau_2} - W_{\tau_1,\tau_2}}{s} - \partial_h W_{\tau_1,\tau_2} \right|^p \, d\tau_1 \, d\tau_2 = 0,\end{aligned}$$

by dominated convergence. Thus,

$$\partial_h \left(\int_{a_2}^{b_2} \int_{a_1}^{b_1} W_{\tau_1,\tau_2} \, d\tau_1 \, d\tau_2 \right) = \int_{a_2}^{b_2} \int_{a_1}^{b_1} (\partial_h W_{\tau_1,\tau_2}) \, d\tau_1 \, d\tau_2.$$

Performing similar estimates shows that

$$
\begin{aligned}
&\partial_h D^{r,\alpha}_{t_r,\dots,t_1} \\
&\quad= \int_{[0,T]^r} \Big[(\partial_h W_{T_1}) A_{J_1} W_{T_1,T_2} A_{J_2} \cdots W_{T_{r-1},T_r} A_{J_r} W_{T_r,t} \\
&\qquad + W_{T_1} A_{J_1} (\partial_h W_{T_1,T_2}) A_{J_2} \cdots W_{T_{r-1},T_r} A_{J_r} W_{T_r,t} \\
&\qquad + \cdots + W_{T_1} A_{J_1} W_{T_1,T_2} A_{J_2} \cdots (\partial_h W_{T_{r-1},T_r}) A_{J_r} W_{T_r,t} \\
&\qquad + W_{T_1} A_{J_1} W_{T_1,T_2} A_{J_2} \cdots W_{T_{r-1},T_r} A_{J_r} (\partial_h W_{T_r,t}) \Big] \, d\tau_r \cdots d\tau_1 \\
&\quad= \int_{[0,T]^r} \Big[\int_{\{0 \le \tau_{r+1} \le T_1\}} W_{\tau_{r+1}} A_{\alpha_{r+1}} W_{\tau_{r+1},T_1} A_{J_1} W_{T_1,T_2} A_{J_2} \cdots W_{T_{r-1},T_r} A_{J_r} W_{T_r,t} \\
&\qquad + \int_{\{T_1 \le \tau_{r+1} \le T_2\}} W_{T_1} A_{J_1} W_{T_1,\tau_{r+1}} A_{\alpha_{r+1}} W_{\tau_{r+1},T_2} A_{J_2} \cdots W_{T_{r-1},T_r} A_{J_r} W_{T_r,t} \\
&\quad + \cdots + \int_{\{T_{r-1} \le \tau_{r+1} \le T_r\}} W_{T_1} A_{J_1} W_{T_1,T_2} A_{J_2} \cdots W_{T_{r-1},\tau_{r+1}} A_{\alpha_{r+1}} W_{\tau_{r+1},T_r} A_{J_r} W_{T_r,t} \\
&\qquad + \int_{\{T_r \le \tau_{r+1} \le t\}} W_{T_1} A_{J_1} W_{T_1,T_2} A_{J_2} \cdots W_{T_{r-1},T_r} A_{J_r} W_{T_r,\tau_{r+1}} A_{\alpha_{r+1}} W_{\tau_{r+1},t} \Big] \\
&\qquad\quad \times \dot{h}^{\alpha_{r+1}}_{\tau_{r+1}} \, d\tau_{r+1} \, d\tau_r \cdots d\tau_1,
\end{aligned}
$$

which implies exactly that (P2) holds for $r+1$, since

$$\{0 \le \tau_{r+1} \le T_1\}, \{T_1 \le \tau_{r+1} \le T_2\}, \cdots, \{T_{r-1} \le \tau_{r+1} \le T_r\}, \{T_r \le \tau_{r+1} \le t\}$$

partitions the set $[0,t]$. Clearly, this also implies that, for all $t_{r+1} \in [0,1]$ and $\alpha_{r+1} = 1, \dots, k$, $[D_{t_{r+1}} D^{r,\alpha}_{t_r,\dots,t_1}]^{\alpha_{r+1}} \in L^p(\mu)$ for all $p \in (1,\infty)$, by Equation (0.8), and so $W_t \in \mathcal{D}^{r+1,p}(\mathrm{End}(\mathfrak{g}))$.

Now, for $\overline{W}_t = \int_0^t W_\tau \, d\tau$, the above arguments imply that

$$\partial_h \overline{W}_t = \int_0^t \partial_h W_\tau \, d\tau = \int_0^t \int_{\{0 \le \tau_1 \le \tau\}} W_{\tau_1} A_{\alpha_1} W_{\tau_1,\tau} \dot{h}^{\alpha_1}_{\tau_1} \, d\tau_1 \, d\tau$$

and, for $t_1 \le t$ and $\alpha_1 = 1, \dots, k$,

$$[D_{t_1} \overline{W}_t]^{\alpha_1} = \int_0^t \int_{\{0 \le \tau_1 \le \tau \wedge t_1\}} W_{\tau_1} A_{\alpha_1} W_{\tau_1,\tau} \, d\tau_1 \, d\tau.$$

Let $r \in \mathbb{N}$. For any $\{t_1, \dots, t_r\} \subset [0,1]$ and multindex $\alpha = (\alpha_1, \dots, \alpha_r)$, let $D^{r,\alpha}_{t_1,\dots,t_r} \overline{W}_t$ be the α^{th} component of $D^r_{t_1,\dots,t_r} \overline{W}_t$; that is,

$$D^{r,\alpha}_{t_r,\dots,t_1} \overline{W}_t := [D_{t_r} [\cdots [D_{t_2} [D_{t_1} \overline{W}_t]^{\alpha_1}]^{\alpha_2}] \cdots]^{\alpha_r}.$$

Then working as above,

$$D^{r,\alpha}_{t_r,\dots,t_1}\overline{W}_t = \int_0^t \int_{[0,T\wedge\tau]^r} W_{T_1}A_{J_1}W_{T_1,T_2}A_{J_2}\cdots W_{T_{r-1},T_r}A_{J_r}W_{T_r,\tau}\,d\tau_r\cdots d\tau_1\,d\tau$$
$$= \int_0^t D^{r,\alpha}_{t_r,\dots,t_1}W_\tau\,d\tau,$$

and since W_τ satisfies (P1) for all $\tau\in[0,t]$, $\overline{W}\in\mathcal{D}^{r,p}(\mathscr{H}(\mathrm{End}(\mathfrak{g})))$, for all $r\in\mathbb{N}$ and $p\in(1,\infty)$. ■

Proposition 7. *Let $\psi\in C_c^\infty(M)$ such that $\psi=1$ near I and $\psi(x)=0$ if $|x|\geq 2$, where $|\cdot|$ is the distance from I with respect to any metric on M. For any $n\in\mathbb{N}$, define $\psi_n(x):=\psi(x/n)$, and, for any $A\in M$, define $\langle\psi'(x),A\rangle := \frac{d}{dt}\big|_0\psi(x+tA)$. Let W denote the solution to Equation (0.1), and let $W^n:[0,1]\times\mathscr{W}\to M$ denote the solution to the Stratonovich differential equation*

$$dW^n = \psi_n(W^n)W^n\circ dB,\ \text{with } W_0^n = I. \tag{0.11}$$

Then

$$\lim_{n\to\infty}\mathbb{E}\sup_{\tau\leq 1}|W^n_\tau - W_\tau|^p = 0,$$

for all $p\in(1,\infty)$.

Remark 8. Notice that $\psi_n'(x) = n^{-1}\psi'(x/n)$, and therefore

$$|\psi_n'(x)|\,|x| \leq n^{-1}C2n = 2C,$$

where C is a bound on ψ'. Similarly,

$$|\psi_n''(x)|\,|x|^2\leq C,$$

where C is determined by a bound on ψ''. These bounds will be used repeatedly in the sequel without further mention.

Proof. Equation (0.11) in Itô form is

$$\begin{aligned} dW^n &= \psi_n(W^n)W^n dB \\ &\qquad + \frac{1}{2}\left[\langle\psi_n'(W^n),\psi_n(W^n)W^n dB\rangle\,W^n + \psi_n(W^n)\psi_n(W^n)W^n\,dB\right]dB \\ &= \psi_n(W^n)W^n dB \\ &\qquad + \frac{1}{2}\sum_{i=1}^k\left[\psi_n(W^n)\,\langle\psi_n'(W^n),W^nA_i\rangle\,W^nA_i + \psi_n^2(W^n)W^nA_i^2\right]dt. \end{aligned} \tag{0.12}$$

By Proposition 1, Equation (0.12) implies that

$$\mathbb{E}\sup_{\tau\leq t}|W^n_\tau|^p \lesssim 1 + \mathbb{E}\left(\int_0^t |W^n|^2\,d\tau\right)^{p/2} + \mathbb{E}\left(\int_0^t|W^n|\,d\tau\right)^p$$
$$\lesssim 1 + \mathbb{E}\int_0^t |W^n|^p\ d\tau,$$

for all $t \in [0,1]$. An application of Gronwall's inequality then shows that

$$\mathbb{E} \sup_{\tau \le 1} |W_\tau^n|^p \le C e^C,$$

where these constants are independent of n. Thus, there exists some finite constants C_p so that

$$\sup_{n \in \mathbb{N}} \left(\mathbb{E} \sup_{\tau \le 1} |W_\tau^n|^p \right) \le C_p,$$

for all $p \in (1, \infty)$.

Now let $\varepsilon^n := W^n - W$ (so $W^n = W + \varepsilon^n$). Then

$$\begin{aligned} d\varepsilon^n &= [\psi_n(W^n) W^n - W] \, dB \\ &\quad + \frac{1}{2} \sum_{i=1}^k \left[\psi_n(W^n) \left\langle \psi_n'(W^n), W^n A_i \right\rangle W^n A_i + \left(\psi_n^2(W^n) W^n - W \right) A_i^2 \right] dt. \end{aligned} \tag{0.13}$$

Bound $\mathbb{E}|\varepsilon^n|^p$ by applying Proposition 1 to each term in Equation (0.13). For the first term,

$$\begin{aligned} \mathbb{E}\left| \int_0^t [\psi_n(W^n) W^n - W] \, dB \right|^p &= \mathbb{E}\left| \int_0^t \left[\psi_n(W^n)(\varepsilon^n + W) - W \right] \, dB \right|^p \\ &\lesssim \mathbb{E} \int_0^t |\psi_n(W^n)\varepsilon^n + (\psi_n(W^n) - 1)W|^p \, d\tau \\ &\lesssim \mathbb{E} \int_0^t |\psi_n(W^n)|^p |\varepsilon^n|^p \, d\tau + \mathbb{E} \int_0^t |\psi_n(W^n) - 1|^p |W|^p \, d\tau. \end{aligned}$$

Similarly, for the second term,

$$\begin{aligned} &\mathbb{E}\left| \int_0^t \psi_n(W^n) \langle \psi_n'(W^n), W^n A_i \rangle W^n A_i \, d\tau \right|^p \\ &\qquad = \mathbb{E}\left| \int_0^t \psi_n(W^n) \langle \psi_n'(W^n), W^n A_i \rangle (\varepsilon^n + W) A_i \, d\tau \right|^p \\ &\qquad \lesssim \mathbb{E} \int_0^t \left[|\psi_n(W^n)| |\psi_n'(W^n)| |W^n| |\varepsilon^n + W| \right]^p d\tau \\ &\qquad \lesssim \mathbb{E} \int_0^t \left[|\psi_n'(W^n)| |W^n| \right]^p |\varepsilon^n|^p \, d\tau + \mathbb{E} \int_0^t \left[|\psi_n'(W^n)| |W^n| |W| \right]^p \, d\tau. \end{aligned}$$

And finally,

$$\begin{aligned} \mathbb{E}\left| \int_0^t [\psi_n^2(W^n) W^n - W] A_i^2 \, d\tau \right|^p &= \mathbb{E}\left| \int_0^t [\psi_n^2(W^n)(\varepsilon^n + W) - W] A_i^2 \, d\tau \right|^p \\ &\lesssim \mathbb{E} \int_0^t \left| \psi_n^2(W^n)(\varepsilon^n + W) - W \right|^p \, d\tau \\ &\lesssim \mathbb{E} \int_0^t |\psi_n^2(W^n)|^p |\varepsilon^n|^p \, d\tau + \mathbb{E} \int_0^t |\psi_n^2(W^n) - 1|^p |W|^p \, d\tau. \end{aligned}$$

Bringing all of this together together gives

$$\mathbb{E}\sup_{\tau\leq t}|\varepsilon^n_\tau|^p \leq C\mathbb{E}\int_0^t |\varepsilon^n|^p\,d\tau + \delta_n,$$

for all $t\in[0,1]$, where (up to constant multiple)

$$\delta_n = \mathbb{E}\int_0^t |\psi_n(W^n)-1|^p|W|^p\,d\tau + \mathbb{E}\int_0^t \left[|\psi_n'(W^n)||W^n||W|\right]^p\,d\tau + \mathbb{E}\int_0^t |\psi_n^2(W^n)-1|^p|W|^p\,d\tau.$$

Since $|\psi_n'(W^n)||W^n|$ and $|W|$ remain bounded, the dominated convergence theorem implies that $\lim_{n\to\infty}\delta_n=0$. Thus, by Gronwall again,

$$\mathbb{E}\sup_{\tau\leq 1}|W^n_\tau - W_\tau|^p = \mathbb{E}\sup_{\tau\leq 1}|\varepsilon^n_\tau|^p \leq \delta_n e^C \to 0,$$

as $n\to\infty$. ■

Proposition 9. *Let W be the solution to Equation (0.1), and let W^n be the solution to Equation (0.11). Then $W^n\in\mathrm{Dom}(\partial_h)$ and*

$$\lim_{n\to\infty}\mathbb{E}\sup_{\tau\leq 1}|\partial_h W^n_\tau - \partial_h W_\tau|^p = 0,$$

for all $p\in(1,\infty)$.

Proof. As in Proposition 4, $\partial_h W^n$ satisfies the Itô equation

$$d(\partial_h W^n) = \left[\langle\psi_n'(W^n),\partial_h W^n\rangle W^n + \psi_n(W^n)(\partial_h W^n)\right]dB + \psi_n(W^n)W^n dH + \frac{1}{2}\sum_{i=1}^k \begin{bmatrix} \langle\psi_n'(W^n),\partial_h W^n\rangle\langle\psi_n'(W^n),W^nA_i\rangle W^nA_i \\ +\psi_n(W^n)\langle\psi_n''(W^n),\partial_h W^n\otimes W^nA_i\rangle W^nA_i \\ +\psi_n(W^n)\langle\psi_n'(W^n),(\partial_h W^n)A_i\rangle W^nA_i \\ +\psi_n(W^n)\langle\psi_n'(W^n),W^nA_i\rangle(\partial_h W^n)A_i \\ +2\psi_n(W^n)\langle\psi_n'(W^n),\partial_h W^n\rangle W^nA_i^2 + \psi_n^2(W^n)(\partial_h W^n)A_i^2 \end{bmatrix} dt.$$

Recall also that

$$d(\partial_h W) = (\partial_h W)dB + W dH + \frac{1}{2}(\partial_h W)A_i^2\,dt.$$

Let $\varepsilon^n := \partial_h W^n - \partial_h W$. Then

$$d\varepsilon^n = \left[\langle\psi_n'(W^n),\partial_h W^n\rangle W^n + (\psi_n(W^n)(\partial_h W^n) - (\partial_h W))\right]dB + \left[\psi_n(W^n)W^n - W\right]dH + \frac{1}{2}\sum_{i=1}^k \begin{bmatrix} \langle\psi_n'(W^n),\partial_h W^n\rangle\langle\psi_n'(W^n),W^nA_i\rangle W^nA_i \\ +\psi_n(W^n)\langle\psi_n''(W^n),\partial_h W^n\otimes W^nA_i\rangle W^nA_i \\ +\psi_n(W^n)\langle\psi_n'(W^n),(\partial_h W^n)A_i\rangle W^nA_i \\ +\psi_n(W^n)\langle\psi_n'(W^n),W^nA_i\rangle(\partial_h W^n)A_i \\ +2\psi_n(W^n)\langle\psi_n'(W^n),\partial_h W^n\rangle W^nA_i^2 + \left[\psi_n^2(W^n)\partial_h W^n - \partial_h W\right]A_i^2 \end{bmatrix} dt.$$

That is,

$$
\begin{aligned}
(0.14)\quad d\varepsilon^n = {} & \langle \psi_n'(W^n), \partial_h W^n\rangle \, W^n dB \\
& + [\psi_n(W^n)(\partial_h W^n) - (\partial_h W)]dB + [\psi_n(W^n)W^n - W]dH \\
& + \frac{1}{2}\sum_{i=1}^{k} \left[\Gamma_n(\partial_h W^n) + [\psi_n^2(W^n)\partial_h W^n - \partial_h W]A_i^2\right] dt,
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma_n(A) = \sum_{i=1}^{k} [& \langle \psi_n'(W^n), A\rangle \, \langle \psi_n'(W^n), W^n A_i\rangle \, W^n A_i \\
& + \psi_n(W^n)\, \langle \psi_n''(W^n), A \otimes W^n A_i\rangle \, W^n A_i + \psi_n(W^n)\, \langle \psi_n'(W^n), AA_i\rangle \, W^n A_i \\
& + \psi_n(W^n)\, \langle \psi_n'(W^n), W^n A_i\rangle \, AA_i + 2\psi_n(W^n)\, \langle \psi_n'(W^n), A\rangle \, W^n A_i^2]
\end{aligned}
$$

satisfies the bound

$$
(0.15)\qquad |\Gamma_n| \lesssim |\psi_n''(W^n)||W^n|^2 + |\psi_n'(W^n)||W^n| + |\psi_n'(W^n)|^2|W^n|^2.
$$

Work through (0.14) term by term to bound $\mathbb{E}|\varepsilon^n|^p$, again using Proposition (1): For the first term in the sum,

$$
\begin{aligned}
\mathbb{E}\left|\int_0^t \langle \psi_n'(W^n), \partial_h W^n\rangle \, W^n \, dB\right|^p & \lesssim \mathbb{E}\int_0^t |\langle \psi_n'(W^n), \varepsilon^n + \partial_h W\rangle \, W^n|^p \, d\tau \\
& \lesssim \mathbb{E}\int_0^t |\varepsilon^n|^p d\tau + \mathbb{E}\int_0^t [|\psi_n'(W^n)||\partial_h W||W^n|]^p \, d\tau.
\end{aligned}
$$

Considering the second term,

$$
\begin{aligned}
\mathbb{E}\left|\int_0^t [\psi_n(W^n)\partial_h W^n - \partial_h W] \, dB\right|^p & \lesssim \mathbb{E}\int_0^t |\psi_n(W^n)(\varepsilon^n + \partial_h W) - \partial_h W|^p \, d\tau \\
& \lesssim \mathbb{E}\int_0^t |\varepsilon^n|^p d\tau + \mathbb{E}\int_0^t |\psi_n(W^n) - 1|^p |\partial_h W|^p \, d\tau.
\end{aligned}
$$

For the third term, note that

$$
\mathbb{E}\left|\int_0^t [\psi_n(W^n)W^n - W] \, dH\right|^p = C\|H\|_{\mathscr{H}}^p \mathbb{E}\sup_{\tau\le 1} |\psi_n(W_\tau^n)W_\tau^n - W_\tau|^p
$$

and

$$
\begin{aligned}
\mathbb{E}\sup_{\tau\le 1} |\psi_n(W_\tau^n)W_\tau^n - W_\tau|^p & = \mathbb{E}\sup_{\tau\le 1} |\psi_n(W_t^n)(W_t + (W_t^n - W_t)) - W_t|^p \\
& \le \mathbb{E}\sup_{\tau\le 1} (|\psi_n(W_\tau^n) - 1|^p |W_\tau|^p + |\psi_n(W_\tau^n)|^n |W_\tau^n - W_\tau|^p) \to 0,
\end{aligned}
$$

as $n \to \infty$, by Proposition 7 and dominated convergence. Using the bound in (0.15) on the fourth term,

$$\begin{aligned}\mathbb{E}\left|\int_0^t \Gamma_n(\partial_h W^n)\, d\tau\right|^p &= \mathbb{E}\left|\int_0^t \Gamma_n(\varepsilon^n + \partial_h W)\, d\tau\right|^p \\ &\lesssim \mathbb{E}\int_0^t |\varepsilon^n|^p\, d\tau \\ &\quad + \mathbb{E}\int_0^t \left(|\psi_n''(W^n)||W^n|^2 + |\psi_n'(W^n)||W^n| + |\psi_n'(W^n)|^2\, |W^n|^2\right)^p |\partial_h W|^p\, d\tau.\end{aligned}$$

Finally, for the last term,

$$\begin{aligned}\mathbb{E}\left|\int_0^t \left[\psi_n^2(W^n)\partial_h W^n - \partial_h W\right] A_i^2\, d\tau\right|^p &\lesssim \mathbb{E}\int_0^t \left|\psi_n^2(W^n)(\varepsilon^n + \partial_h W) - \partial_h W\right|^p\, d\tau \\ &\lesssim \mathbb{E}\int_0^t |\varepsilon^n|^p d\tau + \mathbb{E}\int_0^t |\psi_n^2(W^n) - 1|^p |\partial_h W|^p\, d\tau.\end{aligned}$$

Putting this all together shows

$$\mathbb{E}\sup_{\tau\le t}|\varepsilon_t^n|^p \le C\mathbb{E}\int_0^t |\varepsilon^n|^p d\tau + \delta_n,$$

for all $t \in [0,1]$, where

$$\begin{aligned}\delta_n &= \mathbb{E}\int_0^t |\psi_n'(W^n)|^p |\partial_h W|^p |W^n|^p d\tau + \mathbb{E}\int_0^t |\psi_n(W^n) - 1|^p |\partial_h W|^p d\tau \\ &\quad + \mathbb{E}\left|\int_0^t \left[\psi_n(W^n)W^n - W\right] dH\right|^p \\ &\quad + \mathbb{E}\int_0^t \left(|\psi_n''(W^n)||W^n|^2 + |\psi_n'(W^n)||W^n| + |\psi_n'(W^n)|^2\, |W^n|^2\right)^p |\partial_h W|^p\, d\tau \\ &\quad + \mathbb{E}\int_0^t |\psi_n^2(W^n) - 1|^p |\partial_h W|^p\, d\tau.\end{aligned}$$

Again using Remark 8, $\lim_{n\to\infty}\delta_n = 0$, by the dominated convergence theorem. Thus, another application of Gronwall's inequality shows that

$$\mathbb{E}\sup_{\tau\le 1}|\partial_h W_\tau^n - \partial_h W_\tau|^p = \mathbb{E}\sup_{\tau\le 1}|\varepsilon_\tau^n|^p \le \delta_n e^C \to 0,$$

as $n \to \infty$. ∎

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