SMOOTHNESS OF HEAT KERNEL MEASURES ON INFINITE-DIMENSIONAL HEISENBERG-LIKE GROUPS

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Abstract. We study measures associated to Brownian motions on infinite-dimensional Heisenberg-like groups. In particular, we prove that the associated path space measure and heat kernel measure satisfy a strong definition of smoothness.

Contents

1. Introduction

Recall that a measure μ on $\mathbb{R}^n$ is *smooth* if μ is absolutely continuous with respect to Lebesgue measure and the associated density is a smooth function on $\mathbb{R}^n$. If one wishes to generalize this notion of smoothness of measure to an infinite-dimensional space, one immediately encounters complications due to the lack of an infinite-dimensional Lebesgue measure. Thus, we consider the following more intrinsic definition of smoothness for a measure on $\mathbb{R}^n$: for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0, 1, 2, \ldots\}^n$, there exists a function $z_\alpha \in C^\infty(\mathbb{R}^n) \cap L^{\infty-}(\mu)$ such that

$$\int_{\mathbb{R}^n} \partial^\alpha f \, d\mu = \int_{\mathbb{R}^n} f z_\alpha \, d\mu, \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n),$$

where $L^{\infty-} := \cap_{p\geq 1} L^p$ and $\partial^\alpha = \prod_{i=1}^n \partial_i^{\alpha_i}$. It turns out that this definition of smoothness is in fact equivalent to our first understanding, and it is obviously better suited to adapt to infinite dimensions and the absence of a canonical reference measure.

In the present paper we adapt the above definition to give a direct proof of the smoothness of elliptic heat kernel measures on infinite-dimensional Heisenberg-like

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groups. In particular, let G be an infinite-dimensional Heisenberg-like group, $\mathfrak{g}_{CM}$ be its Cameron-Martin Lie subalgebra, and $\{\xi_t\}_{t\geq 0}$ be a Brownian motion on G (see Sections 2.2 and 2.3 for definitions). Then we have the following theorem.

Theorem 1.1. *Fix $T > 0$, and let $m \in \mathbb{N}$ and $h_1, \ldots, h_m \in \mathfrak{g}_{CM}$. Then there exist $\tilde{z}, \hat{z} \in L^{\infty-}$ depending on $h_1, \ldots, h_m$ such that, for any suitably nice function f on G,*

$$\mathbb{E}\left[(\tilde{h}_1 \cdots \tilde{h}_m f)(\xi_T)\right] = \mathbb{E}[f(\xi_T)\tilde{z}] \quad \text{and} \quad \mathbb{E}\left[(\hat{h}_1 \cdots \hat{h}_m f)(\xi_T)\right] = \mathbb{E}[f(\xi_T)\hat{z}],$$

where $\tilde{h}$ and $\hat{h}$ are the left and right invariant vector fields, respectively, associated to $h \in \mathfrak{g}_{CM}$.

This result is proved by first establishing smoothness results for the induced measure on the associated path space. In particular, let $\mathcal{W}_T(G)$ denote continuous path space on G and $\mathcal{H}_T(\mathfrak{g}_{CM})$ denote the space of absolutely continuous paths on $\mathfrak{g}_{CM}$ with finite energy (see Notation 3.1 for definitions). Then we prove the following theorem.

Theorem 1.2. *Let $m \in \mathbb{N}$ and $\mathbf{h}_1, \ldots, \mathbf{h}_m \in \mathcal{H}_T(\mathfrak{g}_{CM})$. Then there exists $\hat{Z} \in L^{\infty-}$ depending on $\mathbf{h}_1, \ldots, \mathbf{h}_m$ such that, for any suitably nice function F on $\mathcal{W}_T(G)$,*

$$\mathbb{E}\left[(\hat{\mathbf{h}}_1 \cdots \hat{\mathbf{h}}_m F)(\xi)\right] = \mathbb{E}[F(\xi)\hat{Z}],$$

where $\hat{\mathbf{h}}$ is the right invariant vector field associated to $\mathbf{h} \in \mathcal{H}_T(\mathfrak{g}_{CM})$.

Theorem 1.2 is stated more precisely and proved in Theorem 3.14; Theorem 1.1 is the content of Theorem 4.4 and Corollary 4.6. Note that these theorems give a strong satisfaction of smoothness for measures in infinite dimensions. Typically, it is not possible to verify that a measure on an infinite-dimensional space is smooth in this way and much weaker interpretations must be made; see for example [3, 7, 13, 14].

The organization of the paper is as follows. In Section 2.1, we recall the requisite results for abstract Wiener spaces and take this opportunity to review the canonical example of a smooth measure in infinite dimensions. Sections 2.2 and 2.3 give the constructions of infinite-dimensional Heisenberg-like groups and Brownian motions on these groups, first studied in [8]. In Section 3, we recall the quasi-invariance and first-order integration by parts results proved in [8] for the path space measure, and, building on these results, give the integration by parts formulae that prove Theorem 1.2. In Section 4, we show how these path space results immediately give integration by parts formulae for heat kernel measures on the group.

Finally, let us here mention some references to other quasi-invariance and integration by parts results for measures in infinite-dimensional curved settings; see [1, 2, 4, 6, 10, 11] and their references.

2. Infinite-dimensional Heisenberg-like groups

2.1. **Abstract Wiener spaces.** In this section, we recall the construction of abstract Wiener spaces. For proofs of these results, see for example Section 2 of [8]. We conclude the section with the canonical Wiener space example. One may see [5, 12] for more on abstract Wiener spaces as well as some more examples.

Suppose that W is a real separable Banach space with Borel σ-algebra $\mathcal{B}_W$.

Definition 2.1. *A measure μ on $(W, \mathcal{B}_W)$ is called a (mean zero, non-degenerate)* Gaussian measure *provided that its characteristic functional is given by*

$$\hat{\mu}(u) := \int_W e^{iu(w)} d\mu(w) = e^{-\frac{1}{2}q(u,u)}, \qquad \text{for all } u \in W^*,$$

for some $q = q_\mu : W^* \times W^* \to \mathbb{R}$ a symmetric, positive definite quadratic form. That is, q is a real inner product on W^*.

Recall that Fernique's theorem implies that there exists δ_0 such that for all $\delta < \delta_0$

$$\int_W e^{\delta \|w\|_W^2} \, d\mu(w) < \infty; \tag{2.1}$$

see for example Theorem 2.8.5 of [5] or Theorem 3.1 of [12]. Among other things, this implies that, for any $p \in [1, \infty)$

$$C_p := \int_W \|w\|_W^p \, d\mu(w) < \infty. \tag{2.2}$$

Theorem 2.2. *Let μ be a Gaussian measure on a real separable Banach space W, and, for $w \in W$, let*

$$\|w\|_H := \sup_{u \in W^* \setminus \{0\}} \frac{|u(w)|}{\sqrt{q(u,u)}}.$$

Define the Cameron-Martin subspace $H \subset W$ *by*

$$H := \{h \in W : \|h\|_H < \infty\}.$$

Then

1. *H is a dense subspace of W.*
2. *There exists a unique inner product $\langle \cdot, \cdot \rangle_H$ on H such that $\|h\|_H^2 = \langle h, h \rangle_H$ for all $h \in H$, and H is a separable Hilbert space with respect to this inner product.*
3. *For any $h \in H$, $\|h\|_W \le \sqrt{C_2} \|h\|_H$, where C_2 is as in* (2.2).

A triple (W, H, μ) where W is a separable Banach space with Gaussian measure μ and H is as defined in Theorem 2.2 will be called an *abstract Wiener space*. The Hilbert space H is called the *Cameron-Martin space*.

Now let $i : H \to W$ be the inclusion map, and let $i^* : W^* \to H^*$ denote its transpose, that is, $i^* \ell = \ell \circ i$ for all $\ell \in W^*$. Also, let

$$H_* := \{h \in H : \langle \cdot, h \rangle_H \in \mathrm{Range}(i^*) \subset H^*\}.$$

That is, for $h \in H$, $h \in H_*$ if and only if $\langle \cdot, h \rangle_H \in H^*$ extends to a continuous linear functional on W, which we will continue to denote by $\langle \cdot, h \rangle_H$. Because H is a dense subspace of W, i^* is injective. Because i is injective, i^* has a dense range. Since $H \ni h \mapsto \langle \cdot, h \rangle_H \in H^*$ is a linear isometric isomorphism, it follows that the map $H_* \ni h \mapsto \langle \cdot, h \rangle_H \in W^*$ is also a linear isomorphism and H_* is a dense subspace of H. In fact, for $h \in H_*$, $\|\langle \cdot, h \rangle_H\|_{L^2(W,\mu)} = \|h\|_H$ and so there exists a unique continuous linear extension $I : H \to L^2(W, \mu)$ which is an isometry into $L^2(W, \mu)$.

At this point, we may also recall the Cameron-Martin theorem. For any $h \in H$,

$$\begin{aligned} \int_W f(w+h) \, d\mu(w) &= \int_W f(w) \, d\mu(w-h) =: \int_W f(w) \, d\mu^h(w) \\ &= \int_W f(w) \exp\left(I(h)(w) - \frac{1}{2}\|h\|_H^2 \right) d\mu(w); \end{aligned}$$

see for example Theorem 1.2 of Chapter II of [12]. Moreover, for any $h \in W \setminus H$, $\mu^h \perp \mu$.

It will also be useful later to recall that associated to any abstract Wiener space is a class of canonical projections. Suppose that $P : H \to H$ is a finite rank orthogonal projection such that $PH \subset H_*$. Let $\{e_j\}_{j=1}^n$ be an orthonormal basis for PH. Then we may extend P to a (unique) continuous operator from $W \to H$ (still denoted by P) by letting

$$Pw := \sum_{j=1}^{n} \langle w, e_j \rangle_H e_j \tag{2.3}$$

for all $w \in W$.

Notation 2.3. *Let* $\mathrm{Proj}(W)$ *denote the collection of finite rank projections on* W *such that* $PW \subset H_*$ *and* $P|_H : H \to H$ *is an orthogonal projection (that is,* P *has the form given in equation* (2.3)*).*

We complete this section by including the canonical example of Wiener measure on path space.

Example 2.4 (Canonical Wiener space)**.** *When* $W = W(\mathbb{R}^n)$ *is the space of continuous paths* $w : [0, 1] \to \mathbb{R}^n$ *with* $w(0) = 0$, $H = H(\mathbb{R}^n)$ *is the Cameron-Martin subspace of finite-energy paths, and* μ *is Wiener measure, the Cameron-Martin formula for* $h \in H(\mathbb{R}^n)$ *is*

$$\begin{aligned} \frac{d\mu^h}{d\mu}(w) &= \exp\left(\int_0^1 \langle \dot{h}(s), dw(s) \rangle - \frac{1}{2} \int_0^1 |\dot{h}(s)|^2 \, ds \right) \\ &=: \exp\left(\langle h, w \rangle - \frac{1}{2} \|h\|^2_{H(\mathbb{R}^n)} \right). \end{aligned}$$

In the following way, this yields integration by parts formulae. Let $\{e_i\}_{i=1}^\infty$ *be an orthonormal basis of* $H(\mathbb{R}^n)$, *and let* ∂_i *denote the derivative in the direction* e_i. *Also, for any multi-index* $\alpha = (\alpha_1, \alpha_2, \ldots) \in \{0, 1, 2, \ldots\}^{\mathbb{N}}$ *with* $|\alpha| = \sum_{i=1}^\infty \alpha_i < \infty$, *let* $\partial^\alpha = \prod_{i=1}^\infty \partial_i^{\alpha_i}$. *Then*

$$\int_{W(\mathbb{R}^n)} (\partial^\alpha f)(w) \, d\mu(w) = \int_{W(\mathbb{R}^n)} f(w) H_\alpha(w) \, d\mu(w)$$

where

$$H_\alpha(w) = \prod_{i=1}^\infty H_{\alpha_i}(\langle e_i, w \rangle)$$

and H_k *are the usual Hermite polynomials.*

More generally, if we consider $\{\mu_t\}_{t \geq 0}$ *the heat kernel sequence on* W *based at* 0, *then for any* $t > 0$

$$\frac{d\mu_t^h}{d\mu_t}(w) = \exp\left(\frac{1}{t} \langle h, w \rangle - \frac{1}{2t} \|h\|^2_{H(\mathbb{R}^n)} \right)$$

and

$$\int_{W(\mathbb{R}^n)} (\partial^\alpha f)(w) \, d\mu_t(w) = \int_{W(\mathbb{R}^n)} f(w) H_\alpha(w; t) \, d\mu_t(w)$$

where

$$H_\alpha(w; t) = \prod_{i=1}^\infty t^{|\alpha|} H_{\alpha_i}(\langle e_i, w \rangle; t)$$

with

$$H_k(x;t) = (-1)^k e^{x^2/2t} \left(\frac{d^k}{dx^k} e^{-x^2/2t} \right).$$

The collection $\{H_k\}_{k=0}^\infty$ forms an orthogonal basis of the space of functions on $\mathbb{R}$ which are square-integrable with respect to the centered normal distribution with variance t.

2.2. **Infinite-dimensional Heisenberg-like groups.** We revisit the definition of the infinite-dimensional Heisenberg-like groups that were first considered in [8]. First we fix the following notation for the sequel.

Notation 2.5. *Let (W, H, μ) be a real abstract Wiener space as in Section 2.1. Let $\mathbf{C}$ be a real vector space with inner product $\langle \cdot, \cdot \rangle_{\mathbf{C}}$ and $\dim(\mathbf{C}) =: N < \infty$. Let $\omega : W \times W \to \mathbf{C}$ be a continuous skew-symmetric bilinear form on W.*

Definition 2.6. *Let $\mathfrak{g}$ denote $W \times \mathbf{C}$ when thought of as a Lie algebra with the Lie bracket given by*

$$[(X_1, V_1), (X_2, V_2)] := (0, \omega(X_1, X_2)). \tag{2.4}$$

Via the Baker-Campbell-Hausdorff-Dynkin formula and the nilpotence of the given bracket, we may equip $W \times \mathbf{C}$ with the group multiplication given by

$$g_1 g_2 := g_1 + g_2 + \frac{1}{2}[g_1, g_2].$$

For $g_i = (w_i, c_i)$, this may be written equivalently as

$$(w_1, c_1) \cdot (w_2, c_2) = \left(w_1 + w_2, c_1 + c_2 + \frac{1}{2}\omega(w_1, w_2) \right). \tag{2.5}$$

We will denote $W \times \mathbf{C}$ by G when thought of as a group, and we will call G constructed in this way a Heisenberg-like group.

It is easy to verify that, given this bracket and multiplication, $\mathfrak{g}$ is indeed a Lie algebra and G is a group with $g^{-1} = -g$ and identity $e = (0, 0)$.

Notation 2.7. *Let $\mathfrak{g}_{CM}$ denote $H \times \mathbf{C}$ when thought of as a Lie subalgebra of $\mathfrak{g}$, and we will refer to $\mathfrak{g}_{CM}$ as the* Cameron-Martin subalgebra *of $\mathfrak{g}$. Similarly, let G_{CM} denote $H \times \mathbf{C}$ when thought of as a subgroup of G, and we will refer to G_{CM} as the* Cameron-Martin subgroup *of G.*

The space $\mathfrak{g} = G = W \times \mathbf{C}$ is a Banach space with the norm

$$\|(w, c)\|_{\mathfrak{g}} := \|w\|_W + \|c\|_{\mathbf{C}},$$

and $\mathfrak{g}_{CM} = G_{CM} = H \times \mathbf{C}$ is a Hilbert space with respect to the inner product

$$\langle (A, a), (B, b) \rangle_{\mathfrak{g}_{CM}} := \langle A, B \rangle_H + \langle a, b \rangle_{\mathbf{C}}.$$

The associated Hilbertian norm on $\mathfrak{g}_{CM}$ is given by

$$\|(A, a)\|_{\mathfrak{g}_{CM}} := \sqrt{\|A\|_H^2 + \|a\|_{\mathbf{C}}^2}.$$

One may easily see that G and G_{CM} are topological groups with respect to the topologies induced by the norms, see for example Lemma 3.3 of [8].

Before proceeding, let us give the basic motivating examples for the construction of these infinite-dimensional Heisenberg-like groups. In what follows, if X is a complex vector space, let X_{Re} denote X thought of as a real vector space. If $(H, \langle \cdot, \cdot \rangle_H)$

is a complex Hilbert space, let $\langle\cdot,\cdot\rangle_{H_{\mathrm{Re}}} := \mathrm{Re}\langle\cdot,\cdot\rangle_H$, in which case $(H_{\mathrm{Re}}, \langle\cdot,\cdot\rangle_{H_{\mathrm{Re}}})$ becomes a real Hilbert space.

Example 2.8 (Finite-dimensional Heisenberg group)**.** *Let $W = H = (\mathbb{R}^n)_{\mathrm{Re}} \cong \mathbb{R}^{2n}$ and μ be standard Gaussian measure on $\mathbb{R}^{2n}$. Then (W, H, μ) is an abstract Wiener space. Let $\mathbf{C} = \mathbb{R}$ and $\omega(w, z) := \mathrm{Im}\langle w, z\rangle$, where $\langle w, z\rangle = w \cdot \bar{z}$ is the usual inner product on $\mathbb{R}^n$. Then $G = \mathbb{R}^{2n} \times \mathbb{R}$ equipped with a group operation as defined in (2.5) is a finite-dimensional Heisenberg group.*

Example 2.9 (Heisenberg group of a symplectic vector space)**.** *Let $(K, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and Q be a strictly positive trace class operator on K. For $h, k \in K$, let $\langle h, k\rangle_Q := \langle h, Qk\rangle$ and $\|h\|_Q := \sqrt{\langle h, h\rangle_Q}$, and let $(K_Q, \langle\cdot,\cdot\rangle_Q)$ denote the Hilbert space completion of $(K, \|\cdot\|_Q)$. Then $W = (K_Q)_{\mathrm{Re}}$ and $H = K_{\mathrm{Re}}$ determines an abstract Wiener space (see, for example, exercise 17 on p.59 of* [12]*). Letting $\mathbf{C} = \mathbb{R}$ and*

$$\omega(w, z) := \mathrm{Im}\langle w, z\rangle_Q,$$

then $G = (K_Q)_{\mathrm{Re}} \times \mathbb{R}$ equipped with a group operation as defined in (2.5) is an infinite-dimensional Heisenberg-like group.

2.3. **Brownian motion on G.** We define Brownian motion on G and collect various of its properties. The primary references for this section are Sections 4 of [8] and [9]. Any statements made here without proof are proved in these references.

Let $\{B_t, B_t^0\}_{t\geq 0}$ be a Brownian motion on $\mathfrak{g}$ with variance determined by

$$\mathbb{E}\left[\langle (B_s, B_s^0), (A, a)\rangle_{\mathfrak{g}_{CM}} \langle (B_t, B_t^0), (C, c)\rangle_{\mathfrak{g}_{CM}}\right] = \langle (A, a), (C, c)\rangle_{\mathfrak{g}_{CM}} \min(s, t),$$

for all $s, t \geq 0$, $A, C \in H_*$, and $a, c \in \mathbf{C}$. The following is Proposition 4.1 of [8].

Proposition 2.10. *For $P \in \mathrm{Proj}(W)$ as in Notation 2.3, let $\{M_t^P\}_{t\geq 0}$ denote the continuous L^2-martingale on $\mathbf{C}$ defined by*

$$M_t^P = \int_0^t \omega(PB_s, dPB_s).$$

In particular, if $\{P_n\}_{n=1}^\infty \subset \mathrm{Proj}(W)$ is an increasing sequence of projections and $M_t^n := M_t^{P_n}$, then there exists an L^2-martingale $\{M_t\}_{t\geq 0}$ in $\mathbf{C}$ such that, for all $p \in [1, \infty)$ and $T > 0$,

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\leq t\leq T} \|M_t^n - M_t\|_{\mathbf{C}}^p\right] = 0,$$

and $\{M_t\}_{t\geq 0}$ is independent of the sequence of projections. Also, for all $p \in [1, \infty)$ and $T > 0$, $\{M_t\}_{t\geq 0}$ satisfies

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \|M_t\|_{\mathbf{C}}^p\right] < \infty. \tag{2.6}$$

As $\{M_t\}$ is independent of the defining sequence of projections, we will denote the limiting process by

$$M_t = \int_0^t \omega(B_s, dB_s).$$

Remark 2.11. *In fact, Driver and Gordina prove in* [8] *the following bound which is much stronger than the one given in* (2.6)*: for any $T > 0$, there exists $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$*

$$\mathbb{E}\left[e^{\lambda\|M_T\|_{\mathbf{C}}}\right] < \infty.$$

Definition 2.12. *The continuous G-valued process given by*

$$\xi_t = \left(B_t, B_t^0 + \frac{1}{2}M_t\right) = \left(B_t, B_t^0 + \frac{1}{2}\int_0^t \omega(B_s, dB_s)\right) \tag{2.7}$$

is a Brownian motion *on G. For $T > 0$, let $\nu_T = \mathrm{Law}(\xi_T)$ denote the* heat kernel measure at time T *on G.*

We include the following proposition (see [9, Proposition 4.6]) which states that, as the name suggests, the Cameron-Martin subgroup is a subspace of heat kernel measure 0.

Proposition 2.13. *For all $T > 0$, $\nu_T(G_{CM}) = 0$.*

Notation 2.14. *For $g \in G$, let $\ell_g, r_g : G \to G$ denote left and right multiplication by g, respectively. As G is a vector space, to each $g \in G$ we can associate the tangent space T_gG to G at g, which is naturally isomorphic to G.*

Let $f : G \to \mathbb{R}$ be a Fréchet smooth function on G. For $g \in G$ and $h, k \in \mathfrak{g}$, let

$$f'(g)h := \partial_h f(g) = \frac{d}{d\varepsilon}\bigg|_0 f(g + \varepsilon h) \quad \text{and} \quad f''(g)(h \otimes k) := \partial_h \partial_k f(g).$$

Also, let

$$\tilde{h}(g) := \ell_{g*}h = \frac{d}{d\varepsilon}\bigg|_0 g \cdot \sigma(\varepsilon),$$

where σ is any smooth curve in G such that $\sigma(0) = e$ and $\dot{\sigma}(0) = h$ (for example, $\sigma(\varepsilon) = \varepsilon h$). So $\tilde{h}$ is the unique left invariant vector field on G such that $\tilde{h}(e) = h$, and we view $\tilde{h}$ as a first order differential operator acting on smooth functions by

$$(\tilde{h}f)(g) = \frac{d}{d\varepsilon}\bigg|_0 f(g \cdot \sigma(\varepsilon)).$$

Similarly, we define

$$\hat{h}(g) := r_{g*}h = \frac{d}{d\varepsilon}\bigg|_0 \sigma(\varepsilon) \cdot g$$

to be the unique right invariant vector field associated to h.

It is proved in Proposition 3.7 of [8] that the Lie algebra structure on $\mathfrak{g}$ induced by the left invariant vector fields on G is the same as the Lie algebra structure defined in equation (2.4).

Definition 2.15. *A function $f : G \to \mathbb{R}$ is a* (smooth) cylinder function *if it may be written as $f = F \circ \pi_P$, for some $P \in \mathrm{Proj}(W)$ and (smooth) $F : PH \times \mathbf{C} \to \mathbb{R}$.*

The following proposition is proved in Proposition 3.29 and Theorem 4.4 of [8].

Proposition 2.16. *Let $\{e_j\}_{j=1}^\infty$ and $\{v_\ell\}_{\ell=1}^N$ be orthonormal bases for H and $\mathbf{C}$, respectively. Then, for any smooth cylinder function $f : G \to \mathbb{R}$,*

$$Lf(x) := \sum_{j=1}^\infty \left[\widetilde{(e_j, 0)}^2 f\right](x) + \sum_{\ell=1}^N \left[\widetilde{(0, v_\ell)}^2 f\right](x)$$

is well-defined, that is, the above sum is convergent and independent of basis. We will call L the Laplacian, and $\frac{1}{2}L$ is the generator for $\{\xi_t\}_{t\geq 0}$, so that, for any smooth cylinder function $f : G \to \mathbb{R}$,

$$f(\xi_t) - \frac{1}{2}\int_0^t Lf(\xi_s)\,ds$$

is a local martingale.

Definition 2.17. *Given a normed space X and a function $F : X \to \mathbb{R}$, we say F is* polynomially bounded *if there exist constants $K, M < \infty$ such that*

$$|F(x)| \le K\,(1 + \|x\|_X)^M$$

for all $x \in X$.

The following proposition is Corollary 4.5 of [8].

Corollary 2.18. *Let $f = F \circ \pi_P$ be a cylinder function on G such that $F \in C^2(PH \times \mathbf{C})$ and F, F', F'' are polynomially bounded. Then*

$$\mathbb{E}[f(\xi_T)] = f(e) + \frac{1}{2}\int_0^T \mathbb{E}[(Lf)(\xi_t)]\,dt.$$

That is,

$$\nu_T(f) := \int f\,d\nu_T = f(e) + \frac{1}{2}\int_0^T \nu_t(Lf)\,dt$$

is a weak solution to the heat equation

$$\partial_t \nu_t = \frac{1}{2} L\nu_t, \qquad \textit{with } \lim_{t\downarrow 0} \nu_t = \delta_e.$$

The following proposition is proved in Corollary 4.9 of [8].

Proposition 2.19. *For any $T > 0$, the heat kernel measure ν_T is invariant under the inversion map $g \mapsto g^{-1}$; that is,*

$$\mathbb{E}[f(\xi_T)] = \int_G f(g)\,d\nu_T(g) = \int_G f(g^{-1})\,d\nu_T(g) = \mathbb{E}[f(\xi_T^{-1})].$$

3. The path space measure

In this section, we prove that $\nu = \mathrm{Law}(\xi)$ satisfies its own strong smoothness properties.

Notation 3.1. *Fix $T > 0$. Given a Banach space X, let*

$$\mathcal{W}_T(X) := \{x : [0,T] \to X : x \textit{ continuous and } x(0) = 0\}$$

equipped with the sup norm topology. Also, for a Hilbert space K, let $\mathcal{H}_T(K)$ denote the absolutely continuous paths in $\mathcal{W}_T(K)$ with finite energy. In particular, for $X = G$ let

$$\|\mathbf{g}\|_{\mathcal{W}_T(G)} := \sup_{0\le t\le T} \|\mathbf{g}(t)\|_{\mathfrak{g}} = \sup_{0\le t\le T} (\|\mathbf{w}(t)\|_W + \|\mathbf{c}(t)\|_{\mathbf{C}})$$

for all $\mathbf{g} = (\mathbf{w}, \mathbf{c}) \in \mathcal{W}_T(G)$, and for $K = \mathfrak{g}_{CM}$ let

$$\|\mathbf{h}\|^2_{\mathcal{H}_T(\mathfrak{g}_{CM})} := \int_0^T \|\dot{\mathbf{h}}(t)\|^2_{\mathfrak{g}_{CM}}\,dt = \int_0^T \left(\|\dot{\mathbf{A}}(t)\|_H^2 + \|\dot{\mathbf{a}}(t)\|^2_{\mathbf{C}}\right)\,dt$$

for all $\mathbf{h} = (\mathbf{A}, \mathbf{a}) \in \mathcal{H}_T(\mathfrak{g}_{CM})$.

Remark 3.2. *Recall that, for $\{B_t\}_{t\ge 0}$ Brownian motion on W, $\mathrm{Law}(B)$ is a Gaussian measure on the separable Banach space $\mathcal{W}_T(W)$. Thus, by Fernique's theorem (see equation (2.1)), there exists $\delta_0 > 0$ such that for all $\delta < \delta_0$*

$$\mathbb{E}\left[\exp(\delta\|B\|^2_{\mathcal{W}_T(W)})\right] < \infty.$$

Combining this with equation (2.6) implies that, if F is a polynomially bounded function on $\mathcal{W}_T(G)$, then $F(\xi) \in L^p$ for all $p \in [1,\infty)$.

The following theorem is a slight generalization of Theorem 5.2 in [8], and we follow the proof of that result.

Theorem 3.3. *Let $\mathbf{h} = (\mathbf{A}, \mathbf{a}) \in \mathcal{H}_T(\mathfrak{g}_{CM})$. If $F, Z : \mathcal{W}_T(G) \to [0,\infty]$ are measurable functions, then*

$$\mathbb{E}[F(\mathbf{h}\cdot\xi)Z(B,B^0)] = \mathbb{E}[F(\xi)Z(B-\mathbf{A}, B^0 - \mathbf{a} - u_{\mathbf{A}})J_{\mathbf{h}}], \tag{3.1}$$

where

$$u_{\mathbf{A}}(t) := \frac{1}{2}\int_0^t \omega(\mathbf{A}(s) - 2B_s, \dot{\mathbf{A}}(s))\,ds \in \mathcal{H}_T(\mathbf{C})$$

and $J_{\mathbf{h}} = J_{\mathbf{h}}(B, B^0)$ is given by

$$\begin{aligned} J_{\mathbf{h}} := \exp\Bigg\{ &\int_0^T \langle \dot{\mathbf{A}}(t), dB_t\rangle_H + \left\langle \dot{\mathbf{a}}(t) + \frac{1}{2}\omega(\mathbf{A}(t) - 2B_t, \dot{\mathbf{A}}(t)), dB_t^0 \right\rangle_{\mathbf{C}} \\ &- \frac{1}{2}\int_0^T \left(\|\dot{\mathbf{A}}(t)\|_H^2 + \left\| \dot{\mathbf{a}}(t) + \frac{1}{2}\omega(\mathbf{A}(t) - 2B_t, \dot{\mathbf{A}}(t)) \right\|_{\mathbf{C}}^2 \right) dt \Bigg\}. \end{aligned} \tag{3.2}$$

Moreover, equation (3.1) holds for all measurable $F, Z : \mathcal{W}_T(G) \to \mathbb{R}$ such that

$$\mathbb{E}|F(\mathbf{h}\cdot\xi)Z(B,B^0)| = \mathbb{E}|F(\xi)Z(B-\mathbf{A}, B^0 - \mathbf{a} - u_{\mathbf{A}})J_{\mathbf{h}}| < \infty.$$

Proof. First combining (2.5) and (2.7) gives

$$\begin{aligned} &\mathbb{E}[F(\mathbf{h}\cdot\xi)Z(B,B^0)] \\ &\qquad = \mathbb{E}\left[F\left(B + \mathbf{A}, B^0 + \mathbf{a} + \frac{1}{2}\int_0^\cdot \omega(B_s, dB_s) + \frac{1}{2}\omega(\mathbf{A}, B) \right) Z(B, B^0) \right]. \end{aligned}$$

Now translating $(B, B^0) \mapsto (B - \mathbf{A}, B^0 - \mathbf{a})$ and applying the standard Cameron-Martin theorem implies that

$$\begin{aligned} &\mathbb{E}[F(\mathbf{h}\cdot\xi)Z(B,B^0)] \\ &\quad = \mathbb{E}\Bigg[F\left(B, B^0 + \frac{1}{2}\int_0^\cdot \omega(B_s - \mathbf{A}(s), d(B_s - \mathbf{A}(s))) + \frac{1}{2}\omega(\mathbf{A}, B - \mathbf{A}) \right) \\ &\qquad\quad \times Z(B - \mathbf{A}, B^0 - \mathbf{a})\bar{J}_{\mathbf{h}}(B, B_0) \Bigg] \end{aligned}$$

where $\bar{J}_{\mathbf{h}} = \bar{J}_{\mathbf{h}}(B, B^0)$ is given by

$$\begin{aligned} \bar{J}_{\mathbf{h}} := \exp&\left(\int_0^T \langle \dot{\mathbf{A}}(t), dB_t\rangle_H - \frac{1}{2}\int_0^T \|\dot{\mathbf{A}}(t)\|_H^2\,dt \right) \\ &\qquad \exp\left(\int_0^T \langle \dot{\mathbf{a}}(t), dB_t^0\rangle_{\mathbf{C}} - \frac{1}{2}\int_0^T \|\dot{\mathbf{a}}(t)\|_{\mathbf{C}}^2\,dt \right). \end{aligned}$$

This may be rewritten as

$$\begin{aligned}\mathbb{E}[F(\mathbf{h}\cdot\xi)Z(B,B^0)]\\ &= \mathbb{E}\bigg[F\left(B, B^0 + \frac{1}{2}\int_0^{\cdot}\omega(B_s, dB_s) + \frac{1}{2}\int_0^{\cdot}\omega(\mathbf{A}(s) - 2B_s, \dot{\mathbf{A}}(s))\,ds\right)\\ &\qquad \times Z(B-\mathbf{A}, B^0 - \mathbf{a})\bar{J}_{\mathbf{h}}(B, B_0)\bigg].\end{aligned}$$

Freezing integration over B (that is, using Fubini) and translating again, this time $B_0 \mapsto B_0 - u_{\mathbf{A}}$ with

$$u_{\mathbf{A}}(t) := \frac{1}{2}\int_0^t \omega(\mathbf{A} - 2B, \dot{\mathbf{A}})\,ds \in \mathcal{H}_T(\mathbf{C}), \tag{3.3}$$

we may again apply the Cameron-Martin theorem to get that

$$\mathbb{E}[F(\mathbf{h}\cdot\xi)Z(B,B^0)] = \mathbb{E}\left[F(\xi)Z(B-\mathbf{A}, B^0 - \mathbf{a} - u_{\mathbf{A}})\bar{J}_{\mathbf{h}}(B, B^0 - u_{\mathbf{A}})\bar{J}_{(0,u_{\mathbf{A}})}\right].$$

Now one may simplify to show that

$$\bar{J}_{\mathbf{h}}(B, B^0 - u_{\mathbf{A}})\bar{J}_{(0,u_{\mathbf{A}})} = J_{\mathbf{h}},$$

where $J_{\mathbf{h}}$ is as defined in (3.2). □

Remark 3.4. *If we take $Z \equiv 1$ in the previous theorem, this is the statement that $\nu = \mathrm{Law}(\xi)$ is quasi-invariant under left translation by elements of $\mathcal{H}_T(\mathfrak{g}_{CM})$. As pointed out in [8], the above proof fails for right translation, as the requisite translating element in that case*

$$v_{\mathbf{A}}(t) = \frac{1}{2}\int_0^t \Big(\omega(\mathbf{A}(s), \dot{\mathbf{A}}(s)) - 2\omega(\mathbf{A}(s), dB_s)\Big)$$

(taking the role of $u_{\mathbf{A}}$) is not absolutely continuous and thus the Cameron-Martin theorem is no longer available.

We now have a few technical estimates and notations that will allow us to prove the desired integration by parts formulae in Theorem 3.14. The following result is a restatement of Proposition 5.4 of [8]. We include the proof here for the reader's convenience.

Proposition 3.5. *Let $p \in [1,\infty)$. Then there exists $\kappa = \kappa(p) > 0$ such that, for all $\mathbf{h} \in \mathcal{H}_T(\mathfrak{g}_{CM})$ such that $\|\mathbf{h}\|_{\mathcal{H}_T(\mathfrak{g}_{CM})} < \kappa$,*

$$\mathbb{E}[J_{\mathbf{h}}(B,B^0)^p] < \infty.$$

Proof. For the purpose of this proof, let $\mathbb{E}_{B^0}$ and $\mathbb{E}_B$ denote expectation relative to B^0 and B, respectively. We may write

$$\begin{aligned}J_{\mathbf{h}}(B,B^0)^p &= \exp\left\{p\int_0^T \left\langle \dot{\mathbf{a}}(t) + \frac{1}{2}\omega(\mathbf{A}(t) - 2B_t, \dot{\mathbf{A}}(t)), dB_t^0\right\rangle_{\mathbf{C}}\right\}\\ &\quad\times \exp\left\{p\int_0^T \langle \dot{\mathbf{A}}(t), dB_t\rangle_H - \frac{1}{2}p\int_0^T \|\dot{\mathbf{A}}(t)\|_H^2\,dt\right\}\\ &\quad\times \exp\left\{-\frac{1}{2}p\int_0^T \left\|\dot{\mathbf{a}}(t) + \frac{1}{2}\omega(\mathbf{A}(t) - 2B_t, \dot{\mathbf{A}}(t))\right\|_{\mathbf{C}}^2\,dt\right\}.\end{aligned}$$

Since

$$\mathbb{E}_{B^0}\left[\exp\left\{p\int_0^T\left\langle \dot{\mathbf{a}}(t)+\frac{1}{2}\omega(\mathbf{A}(t)-2B_t,\dot{\mathbf{A}}(t)),dB_t^0\right\rangle_{\mathbf{C}}\right\}\right]$$
$$=\exp\left\{\frac{1}{2}p^2\int_0^T\left\|\dot{\mathbf{a}}(t)+\frac{1}{2}\omega(\mathbf{A}(t)-2B_t,\dot{\mathbf{A}}(t))\right\|_{\mathbf{C}}^2dt\right\},$$

we may write $\mathbb{E}_{B^0}[J_{\mathbf{h}}(B,B^0)^p]=UV$, where

$$U:=\exp\left\{p\int_0^T\langle\dot{\mathbf{A}}(t),dB_t\rangle_H-\frac{1}{2}p\int_0^T\|\dot{\mathbf{A}}(t)\|_H^2\,dt\right\}$$

and

$$V:=\exp\left\{\frac{1}{2}(p^2-p)\int_0^T\left\|\dot{\mathbf{a}}(t)+\frac{1}{2}\omega(\mathbf{A}(t)-2B_t,\dot{\mathbf{A}}(t))\right\|_{\mathbf{C}}^2dt\right\}.$$

In particular, when $p=1$, this and Tonelli's theorem imply that

$$\mathbb{E}[J_{\mathbf{h}}(B,B^0)]=\mathbb{E}_B\mathbb{E}_{B^0}[J_{\mathbf{h}}(B,B^0)]=\mathbb{E}_B[U]=1.$$

When $p>1$, applying Tonelli again and the Cauchy-Schwarz inequality gives

$$\mathbb{E}[J_{\mathbf{h}}(B,B^0)^p]=\mathbb{E}_B[UV]\leq\left(\mathbb{E}_B[U^2]\right)^{1/2}\left(\mathbb{E}_B[V^2]\right)^{1/2}.$$

For the first factor, we have that

$$\mathbb{E}_B[U^2]=\exp\left(\frac{1}{2}(p^2-p)\int_0^T\|\dot{\mathbf{A}}(t)\|_H^2\,dt\right)$$
$$=\exp\left(\frac{1}{2}(p^2-p)\|\mathbf{A}\|_{\mathcal{H}_T(H)}^2\right)\leq\exp\left(\frac{1}{2}(p^2-p)\|\mathbf{h}\|_{\mathcal{H}_T(\mathfrak{g}_{CM})}^2\right)<\infty.$$

For the second factor, first note that

$$\left\|\dot{\mathbf{a}}(t)+\frac{1}{2}\omega(\mathbf{A}(t)-2B_t,\dot{\mathbf{A}}(t))\right\|_{\mathbf{C}}^2\leq2\|\dot{\mathbf{a}}(t)\|_{\mathbf{C}}^2+2\cdot\frac{1}{4}\|\omega(\mathbf{A}(\mathbf{t})-2B_t,\dot{\mathbf{A}}(t))\|_{\mathbf{C}}^2$$
$$\leq2\|\dot{\mathbf{a}}(t)\|_{\mathbf{C}}^2+\frac{1}{2}\|\omega\|_0^2\|\mathbf{A}(t)-2B_t\|_W^2\|\dot{\mathbf{A}}(t)\|_W^2$$
$$\leq2\|\dot{\mathbf{a}}(t)\|_{\mathbf{C}}^2+\|\omega\|_0^2\left(\|\mathbf{A}(t)\|_W^2+4\|B\|_{\mathcal{W}_T(W)}^2\right)\|\dot{\mathbf{A}}(t)\|_W^2.$$

Recall that, by Theorem 2.2, $\|\cdot\|_W\leq C_2\|\cdot\|_H$ for $C_2<\infty$ as given in (2.2). Combining this with the fact that

$$\|\mathbf{A}(t)\|_H\leq\int_0^T\|\dot{\mathbf{A}}(s)\|_H\,ds\leq\sqrt{T}\left(\int_0^T\|\dot{\mathbf{A}}(s)\|_H^2\,ds\right)^{1/2}\leq\sqrt{T}\|\mathbf{h}\|_{\mathcal{H}_T(\mathfrak{g}_{CM})},$$

implies that

$$V^2\leq\exp\left\{(p^2-p)\left(2\|\mathbf{h}\|_{\mathcal{H}_T(\mathfrak{g}_{CM})}^2+C_2^4\|\omega\|_0^2T\|\mathbf{h}\|_{\mathcal{H}_T(\mathfrak{g}_{CM})}^4\right)\right\}$$
$$\times\exp\left\{4(p^2-p)C_2^2\|\omega\|_0^2\|\mathbf{h}\|_{\mathcal{H}_T(\mathfrak{g}_{CM})}^2\|B\|_{\mathcal{W}_T(W)}^2\right\}.$$

So letting δ_0 be as in Remark 3.2, $\mathbb{E}_B[V^2]<\infty$ as long as

$$4(p^2-p)C_2^2\|\omega\|_0^2\|\mathbf{h}\|_{\mathcal{H}_T(\mathfrak{g}_{CM})}^2<\delta_0,$$

that is, for all $\|\mathbf{h}\|_{\mathcal{H}_T(\mathfrak{g}_{CM})} < \kappa := \sqrt{\delta_0/4(p^2-p)C_2^2\|\omega\|_0^2}$. □

In a similar way we may prove the following proposition.

Proposition 3.6. *Let $p \in [1,\infty)$ and $\mathbf{h} \in \mathcal{H}_T(\mathfrak{g}_{CM})$. Then there exists $\varepsilon_0 = \varepsilon_0(p) > 0$ such that*

$$\mathbb{E}\left[\sup_{|\varepsilon|\le\varepsilon_0}\left|\frac{d}{d\varepsilon}J_{\varepsilon\mathbf{h}}(B,B^0)\right|^p\right] < \infty.$$

Proof. Note that

$$J_{\varepsilon\mathbf{h}} = \exp\left(\varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \varepsilon^3\alpha_3 + \varepsilon^4\alpha_4\right)$$

where

$$\begin{aligned}
\alpha_1 = \alpha_1(\mathbf{h}) &= \int_0^T \langle \dot{\mathbf{A}}(t), dB_t\rangle_H + \langle \dot{\mathbf{a}}(t) - \omega(B_t, \dot{\mathbf{A}}(t)), dB_t^0\rangle_{\mathbf{C}} \\
\alpha_2 = \alpha_2(\mathbf{h}) &= -\frac{1}{2}\int_0^T \|\dot{\mathbf{A}}(t)\|_H^2\,dt + \frac{1}{2}\int_0^T \langle \omega(\mathbf{A}(t), \dot{\mathbf{A}}(t)), dB_t^0\rangle_{\mathbf{C}} \\
&\quad - \frac{1}{2}\int_0^T \|\dot{\mathbf{a}}(t) - \omega(B_t, \dot{\mathbf{A}}(t))\|_{\mathbf{C}}^2\,dt \\
\alpha_3 = \alpha_3(\mathbf{h}) &= -\frac{1}{2}\int_0^T \langle \dot{\mathbf{a}}(t) - \omega(B_t, \dot{\mathbf{A}}(t)), \omega(\mathbf{A}(t), \dot{\mathbf{A}}(t))\rangle_{\mathbf{C}}\,dt, \text{ and} \\
\alpha_4 = \alpha_4(\mathbf{h}) &= -\frac{1}{8}\int_0^T \|\omega(\mathbf{A}(t), \dot{\mathbf{A}}(t)\|_{\mathbf{C}}^2\,dt.
\end{aligned} \tag{3.4}$$

Thus,

$$\frac{d}{d\varepsilon}J_{\varepsilon\mathbf{h}} = J_{\varepsilon\mathbf{h}} \cdot (\alpha_1 + 2\varepsilon\alpha_2 + 3\varepsilon^2\alpha_3 + 4\varepsilon^3\alpha_4). \tag{3.5}$$

For fixed $p \in [1,\infty)$, we may choose $\varepsilon_0 = \varepsilon_0(p)$ sufficiently small that $\varepsilon < \varepsilon_0$ implies $\varepsilon\|\mathbf{h}\|_{\mathcal{H}_T(\mathfrak{g}_{CM})} < \kappa$, where κ is as given in Proposition 3.5, and so $\mathbb{E}[J_{\varepsilon\mathbf{h}}^p] < \infty$.

For the α_i's, note that $\int_0^T \langle \dot{\mathbf{A}}, dB\rangle_H$ and $\int_0^T \langle \omega(\mathbf{A}, \dot{\mathbf{A}}), dB^0\rangle_{\mathbf{C}}$ are Gaussian and hence have finite moments of all orders. Also,

$$\begin{aligned}
\int_0^T \|\dot{\mathbf{a}}(t) - \omega(B_t, \dot{\mathbf{A}}(t))\|_{\mathbf{C}}^2\,dt &\le 2\int_0^T \left(\|\dot{\mathbf{a}}(t)\|_{\mathbf{C}}^2 + \|\omega(B_t, \dot{\mathbf{A}}(t))\|_{\mathbf{C}}^2\right)\,dt \\
&\le 2\int_0^T \left(\|\dot{\mathbf{a}}(t)\|_{\mathbf{C}}^2 + \|\omega\|_0^2\|B\|_{\mathcal{W}_T(W)}^2\|\dot{\mathbf{A}}(t))\|_H^2\right)\,dt \\
&\le 2\left(\|\mathbf{h}\|_{\mathcal{H}_T(\mathfrak{g}_{CM})}^2 + \|\omega\|_0^2\|B\|_{\mathcal{W}_T(W)}^2\|\mathbf{h}\|_{\mathcal{H}_T(\mathfrak{g}_{CM})}^2\right) \\
&\le C\left(1 + \|B\|_{\mathcal{W}_T(W)}^2\right),
\end{aligned}$$

So by Fernique's Theorem (see Remark 3.2) this term is in L^p for all $p \in [1,\infty)$. Now if $N_t := \int_0^t \langle \dot{\mathbf{a}} - \omega(B, \dot{\mathbf{A}}), dB^0\rangle_{\mathbf{C}}$, then N is a martingale and $\langle N\rangle_T = \int_0^T \|\dot{\mathbf{a}} - \omega(B, \dot{\mathbf{A}})\|_{\mathbf{C}}^2\,dt$. So by the previous estimate, $\mathbb{E}[\langle N\rangle_T^{\frac{p}{2}}] < \infty$ for all $p \in [1,\infty)$ and hence by the Burkholder-Davis-Gundy inequalities, $\mathbb{E}|N_T|^p < \infty$. Finally, applying the Cauchy-Schwarz inequality and again the previous estimate implies that

$$\int_0^T |\langle \dot{\mathbf{a}}(t) - \omega(B_t, \dot{\mathbf{A}}(t)), \omega(\mathbf{A}(t), \dot{\mathbf{A}}(t))\rangle_{\mathbf{C}}|\,dt \le C\left(1 + \|B\|_{\mathcal{W}_T(W)}^2\right). \tag{3.6}$$

The remaining terms are deterministic and clearly finite. □

Notation 3.7. *For* $\mathbf{h}_i = (\mathbf{A}_i, \mathbf{a}_i) \in \mathcal{H}_T(\mathfrak{g}_{CM})$, *define*

$$\begin{aligned}
Z_i &:= Z_{\mathbf{h}_i}(B, B^0) \\
&:= \int_0^T \langle \dot{\mathbf{A}}_i(t), dB_t \rangle_H + \langle \dot{\mathbf{a}}_i(t) - \omega(B_t, \dot{\mathbf{A}}_i(t)), dB_t^0 \rangle_{\mathbf{C}} \\
Z_{ij} &:= Z_{\mathbf{h}_i,\mathbf{h}_j}(B, B^0) \\
&:= \int_0^T \langle \omega(\mathbf{A}_j(t), \dot{\mathbf{A}}_i(t)), dB_t^0 \rangle_{\mathbf{C}} \\
&\quad - \int_0^T \left[\langle \dot{\mathbf{A}}_i(t), \dot{\mathbf{A}}_j(t) \rangle_H + \langle \dot{\mathbf{a}}_i(t) - \omega(B_t, \dot{\mathbf{A}}_i(t)), \dot{\mathbf{a}}_j(t) - \omega(B_t, \dot{\mathbf{A}}_j(t)) \rangle_{\mathbf{C}} \right] dt \\
Z_{ijk} &:= Z_{\mathbf{h}_i,\mathbf{h}_j,\mathbf{h}_k}(B, B^0) \\
&:= -\int_0^T \Big[\langle \dot{\mathbf{a}}_i(t) + \omega(B_t, \dot{\mathbf{A}}_i(t)), \omega(\mathbf{A}_k(t), \dot{\mathbf{A}}_j(t)) \rangle_{\mathbf{C}} \\
&\quad + \langle \dot{\mathbf{a}}_j(t) + \omega(B_t, \dot{\mathbf{A}}_j(t)), \omega(\mathbf{A}_k(t), \dot{\mathbf{A}}_i(t)) \rangle_{\mathbf{C}} \\
&\quad + \langle \dot{\mathbf{a}}_k(t) + \omega(B_t, \dot{\mathbf{A}}_k(t)), \omega(\mathbf{A}_j(t), \dot{\mathbf{A}}_i(t)) \rangle_{\mathbf{C}} \Big] dt \\
Z_{ijkl} &:= Z_{\mathbf{h}_i,\dots,\mathbf{h}_l} \\
&:= -\int_0^T \Big[\langle \omega(\mathbf{A}_l(t), \dot{\mathbf{A}}_i(t)), \omega(\mathbf{A}_k(t), \dot{\mathbf{A}}_j(t)) \rangle_{\mathbf{C}} \\
&\quad + \langle \omega(\mathbf{A}_k(t), \dot{\mathbf{A}}_i(t)), \omega(\mathbf{A}_l(t), \dot{\mathbf{A}}_j(t)) \rangle_{\mathbf{C}} \\
&\quad + \langle \omega(\mathbf{A}_j(t), \dot{\mathbf{A}}_i(t)), \omega(\mathbf{A}_l(t), \dot{\mathbf{A}}_k(t)) \rangle_{\mathbf{C}} \Big] dt
\end{aligned}$$

The following lemma provides some motivation for Notation 3.7. In particular, these functions will comprise the factors appearing in the integration by parts formulae.

Lemma 3.8. *Let* $J_{\mathbf{h}}$ *be as given in equation (3.2) and* Z_i, Z_{ij}, Z_{ijk}, *and* Z_{ijkl} *be as in Notation 3.7. Then*

(*i*) $$Z_i = \frac{d}{d\varepsilon}\Big|_0 J_{\varepsilon \mathbf{h}_i}$$

(*ii*) $$Z_{ij} = \frac{d}{d\varepsilon}\Big|_0 Z_i(B - \varepsilon \mathbf{A}_j, B^0 - \varepsilon \mathbf{a}_j - u_{\varepsilon \mathbf{A}_j})$$

(*iii*) $$Z_{ijk} = \frac{d}{d\varepsilon}\Big|_0 Z_{ij}(B - \varepsilon \mathbf{A}_k, B^0 - \varepsilon \mathbf{a}_k - u_{\varepsilon \mathbf{A}_k})$$

(*iv*) $$Z_{ijkl} = \frac{d}{d\varepsilon}\Big|_0 Z_{ijk}(B - \varepsilon \mathbf{A}_l, B^0 - \varepsilon \mathbf{a}_l - u_{\varepsilon \mathbf{A}_l}).$$

Proof. The lemma follows from simple computations. For example, recall from equation (3.5) that

$$\left(\frac{d}{d\varepsilon} J_{\varepsilon \mathbf{h}}\right)\Big|_{\varepsilon=0} = \left(J_{\varepsilon \mathbf{h}} \cdot (\alpha_1 + 2\varepsilon\alpha_2 + 3\varepsilon^2\alpha_3 + 4\varepsilon^3\alpha_4)\right)\Big|_{\varepsilon=0} = \alpha_1,$$

where $\alpha_1 = \alpha_1(\mathbf{h})$ is given in (3.4). Taking $\mathbf{h} = \mathbf{h}_i$ and noting that $\alpha_1(\mathbf{h}_i) = Z_{\mathbf{h}_i} = Z_i$ completes the proof of (*i*).

Similarly, it may be checked that

$$Z_i(B-\varepsilon\mathbf{A}_j, B^0-\varepsilon\mathbf{a}_j-u_{\varepsilon\mathbf{A}_j}) = Z_i+\varepsilon Z_{ij}+\varepsilon^2\beta_2+\varepsilon^3\beta_3, \tag{3.7}$$

where

$$\beta_2 = -\int_0^T \left\{\frac{1}{2}\langle \dot{\mathbf{a}}_i(t)-\omega(B_t,\dot{\mathbf{A}}_i(t)), \omega(\mathbf{A}_j(t),\dot{\mathbf{A}}_j(t))\rangle_{\mathbf{C}} + \langle \dot{\mathbf{a}}_j(t)-\omega(B_t,\dot{\mathbf{A}}_j(t)), \omega(\mathbf{A}_j(t),\dot{\mathbf{A}}_i(t))\rangle_{\mathbf{C}}\right\} dt \tag{3.8}$$

and

$$\beta_3 = -\frac{1}{2}\int_0^T \langle \omega(\mathbf{A}_j(t),\dot{\mathbf{A}}_i(t)), \omega(\mathbf{A}_j(t),\dot{\mathbf{A}}_j(t))\rangle_{\mathbf{C}}\, dt, \tag{3.9}$$

thus satisfying (*ii*). The computations for (*iii*) and (*iv*) are analogous. □

Proposition 3.9. *For any $\mathbf{h}_i, \mathbf{h}_j, \mathbf{h}_k, \mathbf{h}_l \in \mathcal{H}_T(\mathfrak{g}_{CM})$, Z_i, Z_{ij}, Z_{ijk}, and Z_{ijkl} all satisfy the following: for all $p\in[1,\infty)$, $\mathbb{E}|Z|^p<\infty$.*

Proof. The integrability of $Z_i=\alpha_1(h_i)$ was already verified in the proof of Proposition 3.6. The terms in Z_{ij} and Z_{ijk} can be handled similarly as in that proof, and Z_{ijkl} is deterministic and clearly finite. □

In a similar way to Propositions 3.5 and 3.6 we may prove the following.

Proposition 3.10. *For any $\mathbf{h}_i, \mathbf{h}_j, \mathbf{h}_k \in \mathcal{H}_T(\mathfrak{g}_{CM})$, Z_i, Z_{ij}, and Z_{ijk} all satisfy the following: given any $p\in[1,\infty)$ and $\mathbf{h}=(\mathbf{A},\mathbf{a})\in\mathcal{H}_T(\mathfrak{g}_{CM})$,*

$$\mathbb{E}\left[\sup_{|\varepsilon|\le 1}\left|Z(B-\varepsilon\mathbf{A}, B^0-\varepsilon\mathbf{a}-u_{\varepsilon\mathbf{A}})\right|^p\right]<\infty$$

and

$$\mathbb{E}\left[\sup_{|\varepsilon|\le 1}\left|\frac{d}{d\varepsilon}Z(B-\varepsilon\mathbf{A}, B^0-\varepsilon\mathbf{a}-u_{\varepsilon\mathbf{A}})\right|^p\right]<\infty.$$

Proof. Recall from equation (3.7) that

$$Z_i(B-\varepsilon\mathbf{A}_j, B^0-\varepsilon\mathbf{a}_j-u_{\varepsilon\mathbf{A}_j}) = Z_i+Z_{ij}\varepsilon+\beta_2\varepsilon^2+\beta_3\varepsilon^3,$$

where β_2 and β_3 are as given in (3.8) and (3.9). The integrability of Z_i and Z_{ij} follows from Proposition 3.9, and thus one need only justify the integrability of β_2 (as β_3 is deterministic). This is easily done using the polynomial integrability of $\|B\|_{\mathcal{W}_T(W)}$ (compare with (3.6)). □

Notation 3.11. *For $m\in\mathbb{N}$, let*

$$\Lambda_m := \{\textit{partitions } \theta \textit{ of } \{1,\ldots,m\} : \theta=\{\gamma_1^\theta,\ldots,\gamma_{k_\theta}^\theta\} \textit{ with } \#\gamma_r^\theta\le 4 \textit{ for } r=1,\ldots,k_\theta\}.$$

For $\gamma=\{\ell_1,\ldots,\ell_n\}\in\theta\in\Lambda_m$, we will always assume that elements are listed in increasing order $\ell_1<\cdots<\ell_n$. (Note that $1\le n\le 4$.)

Notation 3.12. *For any $m \in \mathbb{N}$, $\gamma = \{\ell_1, \dots, \ell_n\} \in \theta \in \Lambda_m$, and $\mathbf{h}_1, \dots, \mathbf{h}_m \in \mathcal{H}_T(\mathfrak{g}_{CM})$ with $\mathbf{h}_k = (\mathbf{A}_k, \mathbf{a}_k)$, let $Z_\gamma := Z_{\ell_1 \cdots \ell_n}$ where the right hand side is as defined in Notation 3.7. Also let $\Phi_{\mathbf{h}_1,\dots,\mathbf{h}_m} = \Phi_{\mathbf{h}_1,\dots,\mathbf{h}_m}(B, B^0)$ be defined by*

$$\Phi_{\mathbf{h}_1,\dots,\mathbf{h}_m} := \sum_{\theta \in \Lambda_m} Z_{\gamma_1^\theta} \cdots Z_{\gamma_{k_\theta}^\theta}.$$

Further, for $\mathbf{h}_{m+1} \in \mathcal{H}_T(\mathfrak{g}_{CM})$, let

$$Z_{\gamma_j^\theta}^{\varepsilon \mathbf{h}_{m+1}} := Z_{\gamma_j^\theta}(B - \varepsilon \mathbf{A}_{m+1}, B^0 - \varepsilon \mathbf{a}_{m+1} - u_{\varepsilon \mathbf{A}_{m+1}}),$$

where $u_{\mathbf{A}}$ is as defined in (3.3), and

$$\begin{aligned} \Phi_{\mathbf{h}_1,\dots,\mathbf{h}_m}^{\varepsilon \mathbf{h}_{m+1}} &:= \Phi_{\mathbf{h}_1,\dots,\mathbf{h}_m}(B - \varepsilon \mathbf{A}_{m+1}, B^0 - \varepsilon \mathbf{a}_{m+1} - u_{\varepsilon \mathbf{A}_{m+1}}) \\ &= \sum_{\theta \in \Lambda_m} Z_{\gamma_1^\theta}^{\varepsilon \mathbf{h}_{m+1}} \cdots Z_{\gamma_{k_\theta}^\theta}^{\varepsilon \mathbf{h}_{m+1}}. \end{aligned}$$

Definition 3.13. *Given $\mathbf{h} \in \mathcal{H}_T(\mathfrak{g}_{CM})$, we say a function $F : \mathcal{W}_T(G) \to \mathbb{R}$* right **h**-differentiable *if*

$$(\hat{\mathbf{h}} F)(\mathbf{g}) := \frac{d}{d\varepsilon}\bigg|_0 F(\varepsilon \mathbf{h} \cdot \mathbf{g})$$

exists for all $\mathbf{g} \in \mathcal{W}_T(G)$. We will say that F is smooth *if $(\hat{\mathbf{h}}_1 \cdots \hat{\mathbf{h}}_m F)(\mathbf{g})$ exists for all $m \in \mathbb{N}$, $\mathbf{h}_1, \dots, \mathbf{h}_m \in \mathcal{H}_T(\mathfrak{g}_{CM})$, and $\mathbf{g} \in \mathcal{W}_T(G)$.*

Theorem 3.14. *Let $m \in \mathbb{N}$ and $\mathbf{h}_1, \dots, \mathbf{h}_m \in \mathcal{H}_T(\mathfrak{g}_{CM})$, and suppose that $F : \mathcal{W}_T(G) \to \mathbb{R}$ is a smooth function such that F and its right derivatives of all orders are polynomially bounded. Then*

$$\mathbb{E}\left[(\hat{\mathbf{h}}_1 \cdots \hat{\mathbf{h}}_m F)(\xi)\right] = \mathbb{E}\left[F(\xi) \Phi_{\mathbf{h}_1,\dots,\mathbf{h}_m}\right]$$

and $\mathbb{E}|\Phi_{\mathbf{h}_1,\dots,\mathbf{h}_m}|^p < \infty$ for all $p \in [1, \infty)$.

Proof. That $\Phi_{\mathbf{h}_1,\dots,\mathbf{h}_m} \in L^p$ for all $p \in [1, \infty)$ follows from the definition of Φ and Proposition 3.9, since $L^{\infty-}$ is closed under products. Given the integrability results of Propositions 3.5, 3.6, 3.9, and 3.10, verifying the integration by parts is now an exercise in the product rule. First note that, if $\hat{\mathbf{h}} F$ is polynomially bounded, then there exist $K, M < \infty$ such that

(3.10)
$$\begin{aligned} \sup_{|\varepsilon| \le 1} \left| \frac{d}{d\varepsilon} F(\varepsilon \mathbf{h} \cdot \xi) \right| &= \sup_{|\varepsilon| \le 1} \left| (\hat{\mathbf{h}} F)(\varepsilon \mathbf{h} \cdot \xi) \right| \\ &\le \sup_{|\varepsilon| \le 1} K \left(1 + \|\varepsilon \mathbf{h} \cdot \xi\|_{\mathcal{W}_T(\mathfrak{g})}\right)^M \le C(\mathbf{h}) \left(1 + \|\xi\|_{\mathcal{W}_T(\mathfrak{g})}\right)^M, \end{aligned}$$

where this last expression is integrable by Fernique's theorem and the moment estimates in (2.6).

Now consider the $m = 1$ case. This is the content of Corollary 5.6 of [8], but we include it here for completeness. By Theorem 3.3, we have that

$$\begin{aligned} \mathbb{E}\left[(\hat{\mathbf{h}}_1 F)(\xi)\right] &= \mathbb{E}\left[\frac{d}{d\varepsilon}\bigg|_0 F(\varepsilon \mathbf{h}_1 \cdot \xi)\right] = \frac{d}{d\varepsilon}\bigg|_0 \mathbb{E}\left[F(\varepsilon \mathbf{h}_1 \cdot \xi)\right] \\ &= \frac{d}{d\varepsilon}\bigg|_0 \mathbb{E}\left[F(\xi) J_{\varepsilon \mathbf{h}_1}\right] = \mathbb{E}\left[F(\xi) \frac{d}{d\varepsilon}\bigg|_0 J_{\varepsilon \mathbf{h}_1}\right], \end{aligned}$$

where the two interchanges of differentiation and integration are justified by (3.10) and Proposition 3.6, respectively. Then Lemma 3.8 implies that

$$\left.\frac{d}{d\varepsilon}\right|_0 J_{\varepsilon\mathbf{h}_1} = Z_{\mathbf{h}_1} = \Phi_{\mathbf{h}_1},$$

completing the proof for $m = 1$.

Now, assuming the formula for general m, we have that

$$\begin{aligned}\mathbb{E}\left[(\hat{\mathbf{h}}_1 \cdots \hat{\mathbf{h}}_{m+1} F)(\xi)\right] &= \mathbb{E}\left[(\hat{\mathbf{h}}_{m+1} F)(\xi)\Phi_{\mathbf{h}_1,\dots,\mathbf{h}_m}(B, B^0)\right] \\ &= \mathbb{E}\left[\left.\frac{d}{d\varepsilon}\right|_0 F(\varepsilon\mathbf{h}_{m+1}\cdot\xi)\Phi_{\mathbf{h}_1,\dots,\mathbf{h}_m}(B, B^0)\right] \\ &= \left.\frac{d}{d\varepsilon}\right|_0 \mathbb{E}\left[F(\varepsilon\mathbf{h}_{m+1}\cdot\xi)\Phi_{\mathbf{h}_1,\dots,\mathbf{h}_m}(B, B^0)\right]\end{aligned}$$

where again we justify the interchange of differentiation and integration by the estimate in (3.10) above. Now by Theorem 3.3

$$\begin{aligned}&\mathbb{E}[F(\varepsilon\mathbf{h}_{m+1}\cdot\xi)\Phi_{\mathbf{h}_1,\dots,\mathbf{h}_m}(B, B^0)] \\ &\qquad= \mathbb{E}\left[F(\xi)\Phi_{\mathbf{h}_1,\dots,\mathbf{h}_m}(B - \varepsilon\mathbf{A}_{m+1}, B^0 - \varepsilon\mathbf{a}_{m+1} - u_{\varepsilon\mathbf{A}_{m+1}})J_{\varepsilon\mathbf{h}_{m+1}}\right] \\ &\qquad= \mathbb{E}\left[F(\xi)\Phi^{\varepsilon\mathbf{h}_{m+1}}_{\mathbf{h}_1,\dots,\mathbf{h}_m} J_{\varepsilon\mathbf{h}_{m+1}}\right].\end{aligned}$$

Since

$$\begin{aligned}\frac{d}{d\varepsilon}\Phi^{\varepsilon\mathbf{h}_{m+1}}_{\mathbf{h}_1,\dots,\mathbf{h}_m} J_{\varepsilon\mathbf{h}_{m+1}} &= \sum_{\theta\in\Lambda_m}\sum_{j=1}^{k_\theta}\left(\left(\frac{d}{d\varepsilon}Z^{\varepsilon\mathbf{h}_{m+1}}_{\gamma_j^\theta}\right)\prod_{l\neq j} Z^{\varepsilon\mathbf{h}_{m+1}}_{\gamma_l^\theta}\right) J_{\varepsilon\mathbf{h}_{m+1}} \\ &\quad+ \left(\sum_{\theta\in\Lambda_m}\prod_{j=1}^{k_\theta} Z^{\varepsilon\mathbf{h}_{m+1}}_{\gamma_j^\theta}\right)\left(\frac{d}{d\varepsilon}J_{\varepsilon\mathbf{h}_{m+1}}\right),\end{aligned}$$

Propositions 3.5, 3.6, and 3.10 imply that, for all $p \in [1, \infty)$, there exists $\varepsilon_0 > 0$ such that

$$\mathbb{E}\left[\sup_{|\varepsilon|\le\varepsilon_0}\left|\frac{d}{d\varepsilon}\Phi^{\varepsilon\mathbf{h}_{m+1}}_{\mathbf{h}_1,\dots,\mathbf{h}_m} J_{\varepsilon\mathbf{h}_{m+1}}\right|^p\right] < \infty.$$

Thus,

$$\left.\frac{d}{d\varepsilon}\right|_0 \mathbb{E}\left[F(\xi)\Phi^{\varepsilon\mathbf{h}_{m+1}}_{\mathbf{h}_1,\dots,\mathbf{h}_m} J_{\varepsilon\mathbf{h}_{m+1}}\right] = \mathbb{E}\left[F(\xi)\left.\frac{d}{d\varepsilon}\right|_0 \Phi^{\varepsilon\mathbf{h}_{m+1}}_{\mathbf{h}_1,\dots,\mathbf{h}_m} J_{\varepsilon\mathbf{h}_{m+1}}\right].$$

By Lemma 3.8,

$$\left.\frac{d}{d\varepsilon}\right|_0 \Phi^{\varepsilon\mathbf{h}_{m+1}}_{\mathbf{h}_1,\dots,\mathbf{h}_m} = \sum_{\theta\in\Lambda_m}\left.\frac{d}{d\varepsilon}\right|_0 Z^{\varepsilon\mathbf{h}_{m+1}}_{\gamma_1^\theta}\cdots Z^{\varepsilon\mathbf{h}_{m+1}}_{\gamma_{k_\theta}^\theta} = \sum_{\theta\in\Lambda_m}\sum_{j=1}^{k_\theta} Z_{\gamma_j^\theta, m+1}\prod_{l\neq j} Z_{\gamma_l^\theta},$$

where, for $\gamma = \{\ell_1, \dots, \ell_n\}$,

$$Z_{\gamma,m+1} := \begin{cases} Z_{\gamma'} \text{ for } \gamma' = \{\ell_1, \dots, \ell_n, m+1\} & \text{if } n = 1, 2, 3 \\ 0 & \text{if } n = 4 \end{cases}.$$

Thus, we have that

$$\begin{aligned}\frac{d}{d\varepsilon}\bigg|_0 \Phi^{\varepsilon\mathbf{h}_{m+1}}_{\mathbf{h}_1,\dots,\mathbf{h}_m} J_{\varepsilon\mathbf{h}_{m+1}} &= \frac{d}{d\varepsilon}\bigg|_0 \Phi^{\varepsilon\mathbf{h}_{m+1}}_{\mathbf{h}_1,\dots,\mathbf{h}_m} + \Phi_{\mathbf{h}_1,\dots,\mathbf{h}_m}\frac{d}{d\varepsilon}\bigg|_0 J_{\varepsilon\mathbf{h}_{m+1}} \\ &= \sum_{\theta\in\Lambda_m}\sum_{\substack{j=1\\ \#\gamma_j^\theta\le 3}}^{k_\theta}\left(Z_{\gamma_j^\theta,m+1}\prod_{l\neq j} Z_{\gamma_l^\theta}\right) + \Phi_{\mathbf{h}_1,\dots,\mathbf{h}_m} Z_{m+1},\end{aligned}$$

and notice that each term in this sum is a partition of $\{1,\dots,m,m+1\}$. In particular, one may see that the final sum is over all of Λ_{m+1}, thus yielding the desired expression $\Phi_{h_1,\dots,h_m,h_{m+1}}$. □

Here we write out the first few iterations of the integration by parts formulae of Theorem 3.14.

$$\begin{aligned}\mathbb{E}\left[(\hat{\mathbf{h}}_1 F)(\xi)\right] &= \mathbb{E}\left[F(\xi)Z_1\right] \\ \mathbb{E}\left[(\hat{\mathbf{h}}_1\hat{\mathbf{h}}_2 F)(\xi)\right] &= \mathbb{E}[F(\xi)(Z_{12}+Z_1Z_2)] \\ \mathbb{E}\left[(\hat{\mathbf{h}}_1\hat{\mathbf{h}}_2\hat{\mathbf{h}}_3 F)(\xi)\right] &= \mathbb{E}[F(\xi)(Z_{123}+Z_{12}Z_3+Z_{13}Z_2+Z_1Z_{23}+Z_1Z_2Z_3)] \\ \mathbb{E}\left[(\hat{\mathbf{h}}_1\hat{\mathbf{h}}_2\hat{\mathbf{h}}_3\hat{\mathbf{h}}_4 F)(\xi)\right] &= \mathbb{E}[F(\xi)(Z_{1234}+Z_{123}Z_4+Z_{124}Z_3+Z_{12}Z_{34}+Z_{12}Z_3Z_4 \\ &\quad + Z_{134}Z_2+Z_{13}Z_{24}+Z_{13}Z_2Z_4+Z_{14}Z_{23}+Z_1Z_{234} \\ &\quad + Z_1Z_{23}Z_4+Z_{14}Z_2Z_3+Z_1Z_{24}Z_3+Z_1Z_2Z_{34}+Z_1Z_2Z_3Z_4)]\end{aligned}$$

Thus, $\Phi_{h_1}=Z_1$, $\Phi_{h_1,h_2}=Z_{12}+Z_1Z_2$, and so on.

4. Smooth heat kernel measures on G

The results for the path space measure in the previous section now allow us to prove smoothness results for the heat kernel measure on G. For example, the following quasi-invariance results for ν_T follow directly from the quasi-invariance of the path space measure ν under left translations. This is Theorem 6.1 and Corollary 6.2 of [8] and is not necessary for the sequel, but we include it here for completeness.

Theorem 4.1. *For $h=(A,a)\in G_{CM}$, let $\mathcal{J}_h=\mathcal{J}_h(B,B^0)$ be given by*

$$\mathcal{J}_h = \exp\left\{\frac{1}{T}\langle A,B_T\rangle_H - \frac{1}{2T^2}\|A\|_H^2 + \frac{1}{T}\int_0^T \langle a-\omega(B_t,A), dB_t^0\rangle_{\mathbf{C}} - \frac{1}{2T^2}\int_0^T \|a-\omega(B_t,A)\|^2_{\mathbf{C}}\,dt\right\}.$$

Then, for any $T>0$ and measurable function $f:G\to[0,\infty]$,

$$\mathbb{E}[f(h\cdot\xi_T)] = \mathbb{E}[f(\xi_T)\hat{\mathcal{J}}_h(\xi_T)] \tag{4.1}$$

and

$$\mathbb{E}[f(\xi_T\cdot h)] = \mathbb{E}[f(\xi_T)\tilde{\mathcal{J}}_h(\xi_T)] \tag{4.2}$$

where

$$\hat{\mathcal{J}}_h(\xi_T) = \mathbb{E}[\mathcal{J}_h(B,B^0)|\sigma(\xi_T)] \text{ a.s.} \quad \text{and} \quad \tilde{\mathcal{J}}_h(g)=\hat{\mathcal{J}}_{h^{-1}}(g^{-1}).$$

Moreover, (4.1) and (4.2) hold for any measurable $f:G\to\mathbb{R}$ where the above integrals make sense.

Proof. Consider Theorem 3.3 with $F : \mathcal{W}_T(G) \to [0,\infty]$ given by $F(\mathbf{g}) = f(\mathbf{g}(T))$ and $\mathbf{h}(t) = \frac{t}{T}(A,a) \in \mathcal{H}_T(\mathfrak{g}_{CM})$. Then we have that

$$\mathbb{E}[f(h\cdot\xi_T)] = \mathbb{E}[F(\mathbf{h}\cdot\xi)] = \mathbb{E}[F(\xi)J_{\mathbf{h}}] = \mathbb{E}[f(\xi_T)J_{\mathbf{h}}].$$

Using the anti-symmetry of ω, it is straightforward that $J_{\mathbf{h}} = \mathcal{J}_h$, thus completing the proof of (4.1).

To prove (4.2), let $u(g) := f(g^{-1})$. Then repeatedly applying Corollary 2.19 (the invariance of ν_T under inversion) gives

$$\begin{aligned}\mathbb{E}[f(\xi_T\cdot h)] &= \mathbb{E}[f(\xi_T^{-1}\cdot h)] = \mathbb{E}[f((h^{-1}\cdot\xi_T)^{-1})]\\ &= \mathbb{E}[u(h^{-1}\cdot\xi_T)] = \mathbb{E}[u(\xi_T)\hat{\mathcal{J}}_{h^{-1}}(\xi_T)]\\ &= \mathbb{E}[u(\xi_T^{-1})\hat{\mathcal{J}}_h(\xi_T^{-1})] = \mathbb{E}[f(\xi_T)\hat{\mathcal{J}}_h(\xi_T^{-1})].\end{aligned}$$

□

For $h \in G$, again let ℓ_h be left multiplication by h and similarly let r_h be right multiplication. Proposition 6.3 of [8] gives the following converse of Theorem 4.1 which we state here without proof.

Proposition 4.2. *Fix $T > 0$. If $h \in G\setminus G_{CM}$, then $\nu_T\circ\ell_h^{-1}$ and ν_T are singular, and similarly for $\nu_T\circ r_h^{-1}$.*

Now we move on to the proofs of the integration by parts formulae for ν_T, which again follow from the integration by parts results for ν.

Notation 4.3. *Fix $T > 0$. For $m \in \mathbb{N}$, and $h_1,\dots,h_m \in \mathfrak{g}_{CM}$, let $\mathbf{h}_i(t) := \frac{t}{T}h_i \in \mathcal{H}_T(\mathfrak{g}_{CM})$ and define $\Psi_{h_1,\dots,h_m} := \Phi_{\mathbf{h}_1,\dots,\mathbf{h}_m}$, where Φ is as in Notation 3.12.*

Clearly, the integrability of Φ proved in Theorem 3.14 implies that $\Psi_{h_1,\dots,h_m} \in L^p$ for all $p \in [1,\infty)$.

Theorem 4.4. *Fix $T > 0$. Let $m \in \mathbb{N}$, and $h_1,\dots,h_m \in \mathfrak{g}_{CM}$, and suppose that $f : G \to \mathbb{R}$ is a smooth function such that f and its right derivatives of all orders are polynomially bounded. Then*

$$\mathbb{E}\left[(\hat{h}_1\cdots\hat{h}_m f)(\xi_T)\right] = \mathbb{E}\left[f(\xi_T)\Psi_{h_1,\dots,h_m}\right].$$

Proof. This is a special case of Theorem 3.14. To see this, again let $F : \mathcal{W}_T(G) \to \mathbb{R}$ be given by $F(\mathbf{g}) = f(\mathbf{g}(T))$ and $\mathbf{h}_i(t) := \frac{t}{T}h_i \in \mathcal{H}_T(\mathfrak{g}_{CM})$. Now note that

$$\begin{aligned}\mathbb{E}\left[(\hat{h}_1\cdots\hat{h}_m f)(\xi_T)\right] &= \mathbb{E}\left[\frac{d}{d\varepsilon_1}\bigg|_0\cdots\frac{d}{d\varepsilon_m}\bigg|_0 f(\varepsilon_m h_m\cdot(\cdots(\varepsilon_1 h_1\cdot\xi_T))\right]\\ &= \mathbb{E}\left[\frac{d}{d\varepsilon_1}\bigg|_0\cdots\frac{d}{d\varepsilon_m}\bigg|_0 F(\varepsilon_m\mathbf{h}_m\cdot(\cdots(\varepsilon_1\mathbf{h}_1\cdot\xi))\right]\\ &= \mathbb{E}\left[(\hat{\mathbf{h}}_1\cdots\hat{\mathbf{h}}_m f)(\xi_T)\right] = \mathbb{E}\left[F(\xi)\Phi_{\mathbf{h}_1,\dots,\mathbf{h}_m}\right]\\ &= \mathbb{E}\left[f(\xi_T)\Psi_{h_1,\dots,h_m}\right].\end{aligned}$$

□

Remark 4.5. *Theorem 4.4 implies that, for all $h_1, \ldots, h_m \in \mathfrak{g}_{CM}$, there exists $\hat{\Psi}_{h_1,\ldots,h_m} \in L^{\infty-}(\nu_T)$ such that*

$$\begin{aligned}\int_G (\hat{h}_1 \cdots \hat{h}_m f)(g)\, d\nu_T(g) &= \mathbb{E}\left[(\hat{h}_1 \cdots \hat{h}_m f)(\xi_T)\right] \\ &= \mathbb{E}\left[f(\xi_T)\hat{\Psi}_{h_1,\ldots,h_m}(\xi_T)\right] = \int_G f(g)\hat{\Psi}_{h_1,\ldots,h_m}(g)\, d\nu_T(g).\end{aligned}$$

In particular,

$$\hat{\Psi}_{h_1,\ldots,h_m}(\xi_T) := \mathbb{E}[\Psi_{h_1,\ldots,h_m} \mid \sigma(\xi_T)] \text{ a.s.} \tag{4.3}$$

Corollary 4.6. *Under the hypotheses of Theorem 4.4,*

$$\mathbb{E}[(\tilde{h}_1 \cdots \tilde{h}_m f)(\xi_T)] = \mathbb{E}[f(\xi_T)\tilde{\Psi}_{h_1,\ldots,h_m}(\xi_T)],$$

where

$$\tilde{\Psi}_{h_1,\ldots,h_m}(g) := (-1)^m \hat{\Psi}_{h_1,\ldots,h_m}(g^{-1}).$$

and $\hat{\Psi}$ is as defined in (4.3).

Proof. Again, take $u(g) := f(g^{-1})$. We proceed by induction. The $m = 1$ case is proved in Corollary 6.5 of [8], but we include the proof here for the reader's convenience. Note first that, for any $g \in G$ and $h \in \mathfrak{g}_{CM}$,

$$(\tilde{h} f)(g) = \frac{d}{d\varepsilon}\bigg|_0 f(g \cdot \varepsilon h) = \frac{d}{d\varepsilon}\bigg|_0 u(-\varepsilon h \cdot g^{-1}) = -(\hat{h}u)(g^{-1}). \tag{4.4}$$

Thus, again making repeated use of Corollary 2.19, we have that

$$\begin{aligned}\mathbb{E}[(\tilde{h} f)(\xi_T)] &= -\mathbb{E}[(\hat{h}u)(\xi_T^{-1})] = -\mathbb{E}[(\hat{h}u)(\xi_T)] \\ &= -\mathbb{E}[u(\xi_T)\hat{\Psi}_h(\xi_T)] = -\mathbb{E}[f(\xi_T^{-1})\hat{\Psi}_h(\xi_T)] \\ &= -\mathbb{E}[f(\xi_T)\hat{\Psi}_h(\xi_T^{-1})],\end{aligned}$$

where we have applied Theorem 4.4 in the third equality. Now assuming the formula for m and again using equation (4.4), Corollary 2.19, and Theorem 4.4 gives

$$\begin{aligned}&\mathbb{E}\left[(\tilde{h}_1 \cdots \tilde{h}_{m+1} f)(\xi_T)\right] = (-1)^m \mathbb{E}\left[(\tilde{h}_{m+1} f)(\xi_T)\hat{\Psi}_{h_1,\ldots,h_m}(\xi_T^{-1})\right] \\ &\quad = (-1)^{m+1}\mathbb{E}\left[(\hat{h}_{m+1}u)(\xi_T^{-1})\hat{\Psi}_{h_1,\ldots,h_m}(\xi_T^{-1})\right] \\ &\quad = (-1)^{m+1}\mathbb{E}\left[(\hat{h}_{m+1}u)(\xi_T)\hat{\Psi}_{h_1,\ldots,h_m}(\xi_T)\right] = (-1)^{m+1}\mathbb{E}\left[(\hat{h}_{m+1}u)(\xi_T)\Psi_{h_1,\ldots,h_m}\right] \\ &\quad = (-1)^{m+1}\mathbb{E}\left[u(\xi_T)\Psi_{h_1,\ldots,h_{m+1}}\right] = (-1)^{m+1}\mathbb{E}\left[u(\xi_T)\hat{\Psi}_{h_1,\ldots,h_{m+1}}(\xi_T)\right] \\ &\quad = (-1)^{m+1}\mathbb{E}\left[f(\xi_T^{-1})\hat{\Psi}_{h_1,\ldots,h_{m+1}}(\xi_T)\right] = (-1)^{m+1}\mathbb{E}\left[f(\xi_T)\hat{\Psi}_{h_1,\ldots,h_{m+1}}(\xi_T^{-1})\right].\end{aligned}$$

□

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