HEAT KERNEL ANALYSIS ON SEMI-INFINITE LIE GROUPS

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Abstract. This paper studies Brownian motion and heat kernel measure on a class of infinite dimensional Lie groups. We prove a Cameron-Martin type quasi-invariance theorem for the heat kernel measure and give estimates on the L^p norms of the Radon-Nikodym derivatives. We also prove that a logarithmic Sobolev inequality holds in this setting.

Contents

1. Introduction

We define Brownian motion on a class of infinite dimensional Lie groups which we call *semi-infinite Lie groups.* We then prove a Cameron-Martin type quasi-invariance result for the associated heat kernel measure, as well as a logarithmic Sobolev inequality. A particular example of these semi-infinite Lie algebras was treated in [10], and we build on the methods used there.

We briefly describe here the main results and give an outline of the paper; see Sections 2 and 3 for definitions. Let (W, H, μ) be an abstract Wiener space and $\mathfrak{v}$ be a finite dimensional nilpotent Lie algebra equipped with an inner product.

2000 *Mathematics Subject Classification.* Primary 60J65 28D05; Secondary 58J65 22E65.
Key words and phrases. Heat kernel measure, infinite dimensional Lie group, quasi-invariance, logarithmic Sobolev inequality.

Let $\mathfrak{g} = W \oplus \mathfrak{v}$ be a nilpotent Lie algebra extension of W by $\mathfrak{v}$, and we will call $\mathfrak{g}_{CM} = H \oplus \mathfrak{v}$ the Cameron-Martin Lie subalgebra of $\mathfrak{g}$. Since $\mathfrak{g}$ is nilpotent, we may define an explicit group operation on $\mathfrak{g}$ via the Baker-Campbell-Hausdorff-Dynkin formula, and $W \oplus \mathfrak{v}$ equipped with this group operation will be denoted by G. Similarly, $G_{CM} = H \oplus \mathfrak{v}$ with the same group operation is called the Cameron-Martin subgroup of G, and we equip G_{CM} with the left invariant Riemannian metric which agrees with the inner product

$$\langle (A,a),(B,b)\rangle_{\mathfrak{g}_{CM}} = \langle A,B\rangle_H + \langle a,b\rangle_{\mathfrak{v}}$$

on $\mathfrak{g}_{CM} \cong T_{\mathbf{e}}G_{CM}$.

In Section 2, we set the notation and give some standard facts needed about abstract Wiener spaces and extensions of Lie algebras. In Section 3, we construct (nilpotent) *semi-infinite Lie algebras* and give some examples. We make some additional requirements so that the Lie bracket on $\mathfrak{g}$ is continuous, making $\mathfrak{g}$ into a Banach Lie algebra. In Section 3.2, this continuity gives bounded Hilbert-Schmidt norms for the Lie bracket, and, in Section 3.4, lower bounds on the Ricci curvature of G and a uniform lower bound on certain finite dimensional approximations of G.

In Section 4, we define Brownian motion on G as the solution to a stochastic differential equation with respect to a Wiener process on $\mathfrak{g}$. For a sketch of this construction, let B_t denote Brownian motion on $\mathfrak{g}$. Then, Brownian motion on G is the solution to the Stratonovich stochastic differential equation

$$\delta g_t = g_t\,\delta B_t := L_{g_t*}\delta B_t, \qquad \text{with } g_0 = \mathbf{e} = (0,0).$$

(Note that here and throughout this paper, δg_t and δB_t denote Stratonovich differentials.) For $t > 0$, let $\Delta_n(t)$ denote the simplex in $\mathbb{R}^n$ given by

$$\{s = (s_1, \cdots, s_n) \in \mathbb{R}^n : 0 < s_1 < s_2 < \cdots < s_n < t\}.$$

Let $\mathcal{S}_n$ denote the permutation group on $(1, \cdots, n)$, and, for each $\sigma \in \mathcal{S}_n$, let $e(\sigma)$ denote the number of "errors" in the ordering $(\sigma(1), \sigma(2), \cdots, \sigma(n))$, that is, $e(\sigma) = \#\{j < n : \sigma(j) > \sigma(j+1)\}$. Then the Brownian motion on G may be written as

$$g_t = \sum_{n=1}^{r-1} \sum_{\sigma\in\mathcal{S}_n} \left((-1)^{e(\sigma)} \Big/ n^2 \begin{bmatrix} n-1 \\ e(\sigma) \end{bmatrix} \right) \int_{\Delta_n(t)} [[\cdots[\delta B_{s_{\sigma(1)}}, \delta B_{s_{\sigma(2)}}], \cdots], \delta B_{s_{\sigma(n)}}],$$

where this sum is finite since $\mathfrak{g}$ is assumed to be nilpotent. In Section 4, we show that these stochastic integrals are well-defined and each may be expressed as a sum of iterated Itô integrals. We also show that g_t may be realized as a limit of Brownian motions living on the finite dimensional approximations to G. In particular, we show in Proposition 4.9 that this convergence holds in L^p, for all $p \in [1, \infty)$.

In Theorem 5.3, we apply the previous results and a theorem from [11] to prove that $\nu_t = \mathrm{Law}(g_t)$ is invariant under (right or left) translation by elements of G_{CM}. Moreover, this theorem gives good bounds on the L^p-norms of the Radon-Nikodym derivatives. These results are important for future applications to spaces of holomorphic functions on G, as in [12]. We also show in Theorem 5.7 that a logarithmic Sobolev inequality holds for polynomial cylinder functions on G.

For heat kernel analysis, quasi-invariance results, and logarithmic Sobolev inequalities in related infinite dimensional settings, see [1, 19].

Acknowledgement. The author thanks the anonymous referee for the careful reading and the valuable comments and suggestions to improve this paper.

2. Preliminaries

2.1. **Abstract Wiener spaces.** In this section, we summarize several well known properties of Gaussian measures and abstract Wiener spaces that are required for the sequel. For proofs of these results, see Section 2 of [10]. Also see [6, 21] for more on abstract Wiener spaces and some particular examples.

Suppose that W is a real separable Banach space and $\mathcal{B}_W$ is the Borel σ-algebra on W.

Definition 2.1. A measure μ on $(W, \mathcal{B}_W)$ is called a (mean zero, non-degenerate) *Gaussian measure* provided that its characteristic functional is given by

$$\hat{\mu}(u) := \int_W e^{iu(x)} d\mu(x) = e^{-\frac{1}{2}q(u,u)}, \qquad \text{for all } u \in W^*,$$

for $q = q_\mu : W^* \times W^* \to \mathbb{R}$ a symmetric, positive definite quadratic form. That is, q is a real inner product on W^*.

Theorem 2.2. *Let μ be a Gaussian measure on a real separable Banach space W. For $1 \le p < \infty$, let*

$$C_p := \int_W \|w\|_W^p \, d\mu(w). \tag{2.1}$$

For $w \in W$, let

$$\|w\|_H := \sup_{u \in W^* \setminus \{0\}} \frac{|u(w)|}{\sqrt{q(u,u)}}$$

and define the Cameron-Martin subspace *$H \subset W$ by*

$$H := \{h \in W : \|h\|_H < \infty\}.$$

Then

1. *For all $1 \le p < \infty$, $C_p < \infty$.*
2. *H is a dense subspace of W.*
3. *There exists a unique inner product $\langle \cdot, \cdot \rangle_H$ on H such that $\|h\|_H^2 = \langle h, h \rangle_H$ for all $h \in H$, and H is a separable Hilbert space with respect to this inner product.*
4. *For any $h \in H$, $\|h\|_W \le \sqrt{C_2}\|h\|_H$.*
5. *If $\{k_j\}_{j=1}^\infty$ is an orthonormal basis of H and φ is a bounded linear map from W to a real Hilbert space $\mathbf{C}$, then*

$$\|\varphi\|_{H^* \otimes \mathbf{C}}^2 := \sum_{j=1}^\infty \|\varphi(k_j)\|_{\mathbf{C}}^2 = \int_W \|\varphi(w)\|_{\mathbf{C}}^2 \, d\mu(w) < \infty. \tag{2.2}$$

A simple consequence of (2.2) is that

$$\|\varphi\|_{H^* \otimes \mathbf{C}}^2 \le \|\varphi\|_{W^* \otimes \mathbf{C}}^2 \int_W \|w\|_W^2 d\mu(w) = C_2 \|\varphi\|_{W^* \otimes \mathbf{C}}^2. \tag{2.3}$$

2.2. **Extensions of Lie algebras.** Suppose $\mathfrak{v}$ is a Lie algebra and $\mathrm{Der}(\mathfrak{v})$ is the set of derivations on $\mathfrak{v}$. That is, $\mathrm{Der}(\mathfrak{v})$ consists of all linear maps $\rho : \mathfrak{v} \to \mathfrak{v}$ satisfying Leibniz's rule:

$$\rho([X,Y]_\mathfrak{v}) = [\rho(X), Y]_\mathfrak{v} + [X, \rho(Y)]_\mathfrak{v}.$$

$\mathrm{Der}(\mathfrak{v})$ forms a Lie algebra with Lie bracket defined by the commutator:

$$[\rho_1, \rho_2] = \rho_1\rho_2 - \rho_2\rho_1, \qquad \text{for } \rho_1, \rho_2 \in \mathrm{Der}(\mathfrak{v}).$$

$\mathrm{Der}(\mathfrak{v})$ is a subset of linear maps on $\mathfrak{v}$, so if $\mathfrak{v}$ is a normed vector space, one may equip $\mathrm{Der}(\mathfrak{v})$ with the usual norm

$$\|\rho\|_0 = \sup\{\|\rho(X)\|_{\mathfrak{v}} : \|X\|_{\mathfrak{v}} = 1\}. \tag{2.4}$$

Now suppose that $\mathfrak{h}$ and $\mathfrak{v}$ are Lie algebras, and that there is a linear mapping

$$\alpha : \mathfrak{h} \to \mathrm{Der}(\mathfrak{v})$$

and a skew-symmetric bilinear mapping

$$\omega : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{v},$$

satisfying, for all $X, Y, Z \in \mathfrak{h}$,

$$[\alpha_X, \alpha_Y] - \alpha_{[X,Y]_{\mathfrak{h}}} = \mathrm{ad}_{\omega(X,Y)} \tag{B1}$$

and

$$\sum_{\text{cyclic}} \left(\alpha_X \omega(Y, Z) - \omega([X, Y]_{\mathfrak{h}}, Z)\right) = 0. \tag{B2}$$

Then, one may verify that, for $X_1 + V_1, X_2 + V_2 \in \mathfrak{h} \oplus \mathfrak{v}$,

$$[X_1 + V_1, X_2 + V_2]_{\mathfrak{g}} := [X_1, X_2]_{\mathfrak{h}} + \omega(X_1, X_2) + \alpha_{X_1} V_2 - \alpha_{X_2} V_1 + [V_1, V_2]_{\mathfrak{v}}$$

defines a Lie bracket on $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{v}$, and we say $\mathfrak{g}$ is an extension of $\mathfrak{h}$ over $\mathfrak{v}$. That is, $\mathfrak{g}$ is the Lie algebra with ideal $\mathfrak{v}$ and quotient algebra $\mathfrak{g}/\mathfrak{v} = \mathfrak{h}$. The associated exact sequence is

$$0 \to \mathfrak{v} \xrightarrow{\iota_1} \mathfrak{g} \xrightarrow{\pi_2} \mathfrak{h} \to 0,$$

where ι_1 is inclusion and π_2 is projection. In fact, the following theorem (see, for example, [2]) states that these are the only extensions of $\mathfrak{h}$ over $\mathfrak{v}$.

Theorem 2.3. *Isomorphism classes of extensions of $\mathfrak{h}$ over $\mathfrak{v}$ (that is, short exact sequences of Lie algebras $0 \to \mathfrak{v} \to \mathfrak{g} \to \mathfrak{h} \to 0$) modulo the equivalence described by the commutative diagram of Lie algebra homomorphisms*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{v} & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{h} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathfrak{v} & \longrightarrow & \mathfrak{g}' & \longrightarrow & \mathfrak{h} & \longrightarrow & 0, \end{array}$$

correspond bijectively to equivalence classes of pairs of linear maps $\alpha : \mathfrak{h} \to \mathrm{Der}(\mathfrak{v})$ and skew-symmetric bilinear maps $\omega : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{v}$ satisfying (B1) and (B2), where $(\alpha, \omega) \equiv (\alpha', \omega')$ if there exists a linear $b : \mathfrak{h} \to \mathfrak{v}$ such that

$$\alpha'_X = \alpha_X + \mathrm{ad}_{b(X)},$$

and

$$\omega'(X, Y) = \omega(X, Y) + \alpha_X b(Y) - \alpha_Y b(X) - b([X, Y]) + [b(X), b(Y)]_{\mathfrak{v}}.$$

The corresponding isomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is given by $\varphi(X + V) = X - b(X) + V$.

When $\mathfrak{v} = V$ is an abelian Lie algebra, these pairs consist of a Lie algebra homomorphism $\alpha : \mathfrak{h} \to gl(V)$ and $\omega \in H^2(\mathfrak{h}, V)$, a Chevalley cohomology class with coefficients in the $\mathfrak{h}$-module V (see [16], Chapter 1, Sections 3.1 and 4.5). For definitions and details on extensions of Lie algebras, see Section XIV.5 of [7]. Reference [2] also gives a nice (although unpublished) summary. Reference [28] gives some conditions under which the extension of $\mathfrak{h}$ over $\mathfrak{v}$ is nilpotent (when $\mathfrak{h}$

and $\mathfrak{v}$ are nilpotent); [24] gives a characterization of extensions of a Lie algebra over a Heisenberg Lie algebra.

3. Semi-infinite Lie algebras and groups

Throughout the rest of this paper (W, H, μ) will denote a real abstract Wiener space, and $\mathfrak{v}$ will denote a nilpotent Lie algebra with $\dim(\mathfrak{v}) = N < \infty$, equipped with an inner product $\langle \cdot, \cdot \rangle_{\mathfrak{v}}$ and a Lie bracket $[\cdot, \cdot]_{\mathfrak{v}}$. Since $\mathfrak{v}$ is finite dimensional, its bracket is necessarily continuous and there exists a constant $c_0 < \infty$ such that

$$\|[X, Y]\|_{\mathfrak{v}} \leq c_0 \|X\|_{\mathfrak{v}} \|Y\|_{\mathfrak{v}},$$

for all $X, Y \in \mathfrak{v}$. For simplicity, we will assume that $c_0 \equiv 1$. Also, $\mathrm{Der}(\mathfrak{v})$ will denote the derivations of $\mathfrak{v}$, equipped with the norm defined in (2.4).

We will consider the vector spaces $\mathfrak{g} := W \oplus \mathfrak{v}$ and $\mathfrak{g}_{CM} := H \oplus \mathfrak{v}$. Note that $\mathfrak{g}$ is a Banach space in the norm

$$\|(w, v)\|_{\mathfrak{g}} := \|w\|_W + \|v\|_{\mathfrak{v}},$$

and $\mathfrak{g}_{CM}$ is a Hilbert space with respect to the inner product

$$\langle (A, a), (B, b) \rangle_{\mathfrak{g}_{CM}} := \langle A, B \rangle_H + \langle a, b \rangle_{\mathfrak{v}}.$$

The associated Hilbertian norm on $\mathfrak{g}_{CM}$ is given by

$$\|(A, a)\|_{\mathfrak{g}_{CM}} := \sqrt{\|A\|_H^2 + \|a\|_{\mathfrak{v}}^2}.$$

Motivated by the discussion in Section 2.2, we may consider W as an abelian Lie algebra and construct extensions of W over $\mathfrak{v}$. So suppose there is a skew-symmetric continuous bilinear mapping

$$\omega : W \times W \to \mathfrak{v}$$

and a continuous linear mapping

$$\alpha : W \to \mathrm{Der}(\mathfrak{v})$$

such that α and ω satisfy (B1) and (B2), which in this setting become

$$[\alpha_X, \alpha_Y] = \mathrm{ad}_{\omega(X,Y)} \tag{C1}$$

and

$$\alpha_X \omega(Y, Z) + \alpha_Y \omega(Z, X) + \alpha_Z \omega(X, Y) = 0, \tag{C2}$$

for all $X, Y, Z \in W$. Then we may define a Lie algebra structure on $\mathfrak{g} = W \oplus \mathfrak{v}$ via the Lie bracket

$$[(X_1, V_1), (X_2, V_2)]_{\mathfrak{g}} := (0, \omega(X_1, X_2) + \alpha_{X_1} V_2 - \alpha_{X_2} V_1 + [V_1, V_2]_{\mathfrak{v}}).$$

Theorem 2.3 indicates that these are the only extensions of W over $\mathfrak{v}$. Since $\mathfrak{v}$ is nilpotent, we may choose ω and α so that $\mathfrak{g}$ is a nilpotent Lie algebra (see Section 3.1 for some examples). Thus, we make the following definition.

Definition 3.1. Let (W, H, μ) be an abstract Wiener space and $\mathfrak{v}$ a finite dimensional nilpotent Lie algebra. Then $\mathfrak{g} = W \oplus \mathfrak{v}$ endowed with a Lie bracket satisfying

(1) $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{v}$,
(2) $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is continuous, and
(3) there exists $r \in \mathbb{N}$ such that $\mathrm{ad}_x^m = 0$, for all $m \geq r$ and $x \in \mathfrak{g}$

will be called a *semi-infinite Lie algebra*.

Each extension $\mathfrak{g}$ will depend on (the equivalence class of) a given ω and α.

Notation 3.2. Let

$$\|\omega\|_0 := \sup\{\|\omega(w_1, w_2)\|_{\mathfrak{v}} : \|w_1\|_W = \|w_2\|_W = 1\}$$

and

$$\|\alpha\|_0 := \sup\{\|\alpha_w v\|_{\mathfrak{v}} : \|w\|_W = \|v\|_{\mathfrak{v}} = 1\}$$

be the uniform norms of ω and α, which are finite by their assumed continuity.

It will be useful to note that

$$\|[\cdot,\cdot]\|_0 := \sup\{\|[g_1, g_2]\|_{\mathfrak{v}} : \|g_1\|_{\mathfrak{g}} = \|g_2\|_{\mathfrak{g}} = 1\} \leq \|\omega\|_0 + 2\|\alpha\|_0 + 1 < \infty, \tag{3.1}$$

and similarly

$$C := C(\omega, \alpha) := \sup\{\|[h, k]\|_{\mathfrak{v}} : \|h\|_{\mathfrak{g}_{CM}} = \|k\|_{\mathfrak{g}_{CM}} = 1\} \leq \|[\cdot,\cdot]\|_0 < \infty. \tag{3.2}$$

Thus, for all $h, k \in \mathfrak{g}_{CM}$,

$$\|\mathrm{ad}_h^\ell k\|_{\mathfrak{v}} \leq C^\ell \|h\|_{\mathfrak{g}_{CM}}^\ell \|k\|_{\mathfrak{g}_{CM}},$$

for $\ell = 1, \cdots, r-1$.

The Baker-Campbell-Hausdorff-Dynkin formula implies that

$$\log(e^A e^B) = A + B + \sum_{k=1}^{r-1} \sum_{(n,m)\in\mathcal{I}_k} a_{n,m}^k \mathrm{ad}_A^{n_1} \mathrm{ad}_B^{m_1} \cdots \mathrm{ad}_A^{n_k} \mathrm{ad}_B^{m_k} A,$$

for all $A, B \in \mathfrak{g}$, where

$$a_{n,m}^k := \frac{(-1)^k}{(k+1)m!n!(|n|+1)}, \tag{3.3}$$

$\mathcal{I}_k := \{(n,m) \in \mathbb{Z}_+^k \times \mathbb{Z}_+^k : n_i + m_i > 0 \text{ for all } 1 \leq i \leq k\}$, and for each multi-index $n \in \mathbb{Z}_+^k$,

$$n! = n_1! \cdots n_k! \quad \text{and} \quad |n| = n_1 + \cdots + n_k;$$

see, for example, [15]. Since $\mathfrak{g}$ is nilpotent of step r,

$$\mathrm{ad}_A^{n_1} \mathrm{ad}_B^{m_1} \cdots \mathrm{ad}_A^{n_k} \mathrm{ad}_B^{m_k} A = 0 \quad \text{if } |n| + |m| \geq r.$$

for $A, B \in \mathfrak{g}$. Since $\mathfrak{g}$ is simply connected and nilpotent, the exponential map is a global diffeomorphism (see, for example, Theorems 3.6.2 of [27] or 1.2.1 of [9]). In particular, we may view $\mathfrak{g}$ as both a Lie algebra and Lie group, and one may verify that

$$g \cdot h = g + h + \sum_{k=1}^{r-1} \sum_{(n,m)\in\mathcal{I}_k} a_{n,m}^k \mathrm{ad}_g^{n_1} \mathrm{ad}_h^{m_1} \cdots \mathrm{ad}_g^{n_k} \mathrm{ad}_h^{m_k} g \tag{3.4}$$

defines a group structure on $\mathfrak{g}$. Note that $g^{-1} = -g$ and the identity $\mathbf{e} = (0,0)$.

Definition 3.3. When we wish to emphasize the group structure on $\mathfrak{g}$, we will denote $\mathfrak{g}$ by G. Similarly, when we wish to view $\mathfrak{g}_{CM}$ as a subgroup of G, it will be denoted by G_{CM} and will be called the *Cameron-Martin subgroup*.

Remark 3.4. Note that, for the purpose of making Definition 3.1, it is not really necessary to assume the continuity of the bracket or that $\mathfrak{g}$ be nilpotent. That is, Definition 3.1 is reasonable if $\mathfrak{v}$ is a (not necessarily nilpotent) finite dimensional Lie algebra and we only require that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{v}$. However, the group operation given here, and, in fact, all subsequent results included in this paper, rely on the nilpotence, and many results require the continuity of the bracket. Thus, we include these assumptions in the definition above.

Lemma 3.5. *The Banach space topologies on $\mathfrak{g}$ and $\mathfrak{g}_{CM}$ make G and G_{CM} into topological groups.*

Proof. Since $\mathfrak{g}$ and $\mathfrak{g}_{CM}$ are topological vector spaces, $g \mapsto g^{-1} = -g$ and $(g_1, g_2) \mapsto g_1 + g_2$ are continuous by definition. The map $(g_1, g_2) \mapsto [g_1, g_2]$ is continuous in both the $\mathfrak{g}$ and $\mathfrak{g}_{CM}$ topologies by the estimates in equations (3.1) and (3.2). It then follows from (3.4) that $(g_1, g_2) \mapsto g_1 \cdot g_2$ is continuous as well. ■

3.1. **Examples.** In this section, we give a few simple examples of semi-infinite Lie algebras.

Example 3.6. If $\mathfrak{v}$ is a finite dimensional inner product space, we may consider $\mathfrak{v}$ as an abelian Lie algebra, and taking $\alpha \equiv 0$ yields the infinite dimensional (step 2, stratified) Heisenberg like Lie algebras described in [10].

Example 3.7. Suppose $\mathfrak{v}$ is an N-dimensional nilpotent Lie algebra. One standard way to construct Lie algebra extensions is as follows. Let $\beta : W \to \mathfrak{v}$ be a continuous linear map, and define $\alpha : W \to \mathrm{Der}(\mathfrak{v})$ as the inner derivation $\alpha_X := \mathrm{ad}_{\beta(X)}$. In this case, (C1) and (C2) are both satisfied if $\omega : W \times W \to \mathfrak{v}$ is given by $\omega(X, Y) := [\beta(X), \beta(Y)]_{\mathfrak{v}}$. Thus, $\mathfrak{g}$ has Lie bracket

$$[(X, V), (Y, U)]_{\mathfrak{g}} = (0, [\beta(X), \beta(Y)]_{\mathfrak{v}} + [\beta(X), U]_{\mathfrak{v}} - [\beta(Y), V]_{\mathfrak{v}} + [V, U]_{\mathfrak{v}}),$$

and, if $\mathfrak{v}$ is nilpotent Lie algebra of step r, then $\mathfrak{g}$ is nilpotent of step r.

One should note for this construction that, since β is linear, we have the decomposition $W = \mathrm{Nul}(\beta) \oplus \mathrm{Nul}(\beta)^\perp$, where $\dim(\mathrm{Nul}(\beta)^\perp) \leq \dim(\mathfrak{v}) = N$. Thus, for $X = X_1 + X_2, Y = Y_1 + Y_2 \in W$,

$$\omega(X_1 + X_2, Y_1 + Y_2) = [\beta(X_1 + X_2), \beta(Y_1 + Y_2)] = [\beta(X_2), \beta(Y_2)],$$

and ω is a map on $\mathrm{Nul}(\beta)^\perp \times \mathrm{Nul}(\beta)^\perp$. Thus, $[\mathrm{Nul}(\beta), \mathrm{Nul}(\beta)] = \{0\}$ and similarly $[\mathrm{Nul}(\beta), \mathfrak{v}] = \{0\}$. So

$$\mathfrak{g} = W \oplus \mathfrak{v} = \mathrm{Nul}(\beta) \oplus \mathrm{Nul}(\beta)^\perp \oplus \mathfrak{v}$$

is in a sense just an extension of the finite dimensional subspace $\mathrm{Nul}(\beta)^\perp$ by $\mathfrak{v}$.

Example 3.8. One can generalize the previous example by taking a linear map $\beta : W \to \mathfrak{h}$, where $\mathfrak{h}$ is nilpotent Lie algebra, and constructing an extension of $\mathfrak{h}$ by a nilpotent Lie algebra. For the sake of a concrete example, consider the following.

Let

$$W = W(\mathbb{R}^3) = \{\sigma : [0, 1] \to \mathbb{R}^3 : \sigma \text{ is continuous and } \sigma(0) = 0\}$$

and

$$H = \left\{ \sigma \in W : \sigma \text{ is absolutely continuous and } \int_0^1 \|\dot{\sigma}(s)\|^2 \, ds < \infty \right\},$$

so that (W, H) is classical Wiener space. Let $\mathfrak{v} = \mathbb{R}^3$ be an abelian Lie algebra. Let $\bar{\sigma} = \int_0^1 \sigma(s) \, ds = (\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3)$, and define $\omega : W \times W \to \mathbb{R}^3$ by

$$\omega(\sigma, \tau) = (\bar{\sigma}_1 \bar{\tau}_2 - \bar{\tau}_1 \bar{\sigma}_2, \bar{\sigma}_2 \bar{\tau}_3 - \bar{\tau}_2 \bar{\sigma}_3, 0)$$

and $\alpha_\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\alpha_\sigma(x, y, z) = (0, 0, \bar{\sigma}_1 y - \bar{\sigma}_3 x).$$

Then $\alpha_\sigma \alpha_\tau = 0$ and (C1) is trivially satisfied. Using that

$$\alpha_\kappa \omega(\sigma, \tau) = (0, 0, \bar{\kappa}_1 (\bar{\sigma}_2 \bar{\tau}_3 - \bar{\tau}_2 \bar{\sigma}_3) - \bar{\kappa}_3 (\bar{\sigma}_1 \bar{\tau}_2 - \bar{\tau}_1 \bar{\sigma}_2))$$

one may verify that (C2) is satisfied. Thus, the Lie bracket for this extension $\mathfrak{g} = W \oplus \mathbb{R}^3$ is given by

$$[(\sigma, v), (\tau, u)] = (0, \bar{\sigma}_1\bar{\tau}_2 - \bar{\tau}_1\bar{\sigma}_2, \bar{\sigma}_2\bar{\tau}_3 - \bar{\tau}_2\bar{\sigma}_3, \bar{\sigma}_1 u_2 - \bar{\sigma}_3 u_1 - \bar{\tau}_1 v_2 + \bar{\tau}_3 v_1),$$
$$[(\kappa, w), [(\sigma, v), (\tau, u)]] = (0, 0, 0, \bar{\kappa}_1(\bar{\sigma}_2\bar{\tau}_3 - \bar{\tau}_2\bar{\sigma}_3) - \bar{\kappa}_3(\bar{\sigma}_1\bar{\tau}_2 - \bar{\tau}_1\bar{\sigma}_2)),$$

and all higher order brackets are 0.

As an aside, note that the bracket in this extension is essentially defined as the bracket of a linear Lie algebra, and the extension itself is analogous to a (standard) construction of $T_4 = \{4 \times 4$ strictly upper triangular matrices$\}$ as an extension of $\mathbb{R}^3$ by $\mathbb{R}^3$. To see this, for $A = (a, b, c) \in \mathbb{R}^3$, define the isomorphisms

$$f(A) = \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix} = \tilde{A} \quad \text{and} \quad g(A) = \begin{pmatrix} 0 & 0 & a & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \bar{A}.$$

Let $U = \mathbb{R}^3$ and $V = \mathfrak{v} = \mathbb{R}^3$, and define $\omega' : U \times U \to g(V)$ by

$$\omega'(A, A') = f(A)f(A') - f(A')f(A) = \tilde{A}\tilde{A}' - \tilde{A}'\tilde{A}$$

and $\alpha' : U \times V \to g(V)$ by

$$\alpha'_A A' = f(A)g(A') - g(A')f(A) = \tilde{A}\bar{A}' - \bar{A}'\tilde{A}.$$

Thus, $T_4 \cong \mathbb{R}^6 = U \oplus V$ with bracket determined by the pair $(g^{-1} \circ \omega', g^{-1} \circ \alpha')$.

In particular, for the extension $\mathfrak{g} = W \oplus \mathbb{R}^3$ as given in this example, we have that $\omega = g^{-1} \circ \omega' \circ \beta$ and $\alpha = g^{-1} \circ \alpha' \circ \beta$ where $\beta : W \to U$ is given by $\beta(\sigma) = (\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3)$.

Example 3.9. Consider $\mathfrak{v} = \mathbb{R}^n \oplus \mathbb{R}$ as an abelian Lie algebra. For $\omega : W \times W \to \mathbb{R}^n$, we may write $\omega = (\omega_1, \cdots, \omega_n)$, where $\omega_i : W \times W \to \mathbb{R}$ are bilinear, anti-symmetric, continuous maps. Similarly, for $\alpha : W \times \mathbb{R}^n \to \mathbb{R}$, we have $\alpha_i(\cdot) = \alpha.e_i$, where $\{e_i\}_{i=1}^n$ is the standard basis for $\mathbb{R}^n$. Thus,

$$\alpha_w(a_1, \ldots, a_n) = \sum_{i=1}^n a_i \alpha_i(w).$$

Then α and ω satisfy (C2) as long as

$$\alpha_1 \wedge \omega_1 + \cdots + \alpha_n \wedge \omega_n = 0.$$

In the case $n = 1$, this is not very interesting, since $\alpha \wedge \omega = 0$ implies that $\omega = \alpha \wedge \beta$ for some $\beta \in W^*$.

For $n = 2$, we have $\mathfrak{v} = \mathbb{R}^2 \oplus \mathbb{R}$. Let $\Omega : W \times W \to \mathbb{R}$ be bilinear, antisymmetric, and continuous, and $\gamma : W \to \mathbb{R}$ be linear and continuous. Then define $\omega : W \times W \to \mathbb{R}^2$ by $\omega = (\Omega, \Omega)$ and $\alpha : W \times \mathbb{R}^2 \to \mathbb{R}$ by $\alpha_1 = \gamma$ and $\alpha_2 = -\gamma$, so that, for any $u, w \in W$ and $v = (v_1, v_2) \in \mathbb{R}^2$,

$$\omega(w, u) = (\Omega(w, u), \Omega(w, u)) \quad \text{and} \quad \alpha_w v = \gamma(w)(v_1 - v_2).$$

Note that, for any $w, u, h \in W$, ω and α satisfy

$$\alpha_h \omega(w, u) = \alpha_h(\Omega(w, u), \Omega(w, u)) = \gamma(h)(\Omega(w, u) - \Omega(w, u)) = 0.$$

Thus, for any $(w, v, x), (w', v', x'), (w'', v'', x'') \in W \oplus \mathfrak{v}$,

$$\begin{aligned}[(w, v, x), (w', v', x')] &= (0, \omega(w, w'), \alpha_w v' - \alpha_{w'} v) \\ &= (0, (\Omega(w, w'), \Omega(w, w')), \gamma(w)(v'_1 - v'_2) - \gamma(w')(v_1 - v_2)),\end{aligned}$$

$$[(w'',v'',x''),[(w,v,x),(w',v',x')]] = (0,0,\alpha_{w''}\omega(w,w')) = 0,$$

and $\mathfrak{g}$ is a step 2 Lie algebra. The group operation is given by

$$(w,v,x)\cdot(w',v',x') = (w+w', v+v'+\frac{1}{2}(\Omega(w,w'),\Omega(w,w')),$$
$$x+x'+\frac{1}{2}(\gamma(w)(v_1'-v_2')-\gamma(w')(v_1-v_2)).$$

As an example of a particular appropriate Ω and γ, again let $W=W(\mathbb{R}^3)$ and H be as in Example 3.8. Suppose φ is an anti-symmetric bilinear form on $\mathbb{R}^3$, $\rho:\mathbb{R}^3\to\mathbb{R}$ is a linear map, and η is a finite measure on $[0,1]$. Then we may define

$$\Omega(\sigma,\tau) = \int_0^1 \varphi(\sigma(s),\tau(s))\,d\eta(s)$$

and

$$\gamma(\sigma) = \int_0^1 \rho(\sigma(s))\,d\eta(s).$$

Example 3.10. Here we make a slight modification on the previous example to construct a stratified step 3 Lie algebra. Let $\mathfrak{v}=\mathbb{R}^6=\mathbb{R}^3\oplus\mathbb{R}^2\oplus\mathbb{R}$ be an abelian Lie algebra. Let Ω and γ be as in the previous example. Define $\omega: W\times W\to\mathbb{R}^3$ by

$$\omega(w,u) = (\Omega(w,u),\Omega(w,u),\Omega(w,u))$$

and $\alpha: W\times\mathfrak{v}\to\mathfrak{v}$ by

$$\alpha_w((v_1,v_2,v_3),(x_1,x_2),y) = (0,(\gamma(w)(v_1-v_2),\gamma(w)(v_2-v_3)),\gamma(w)(x_1-x_2))$$

(so α_w is a particular element of the 6×6 strictly lower triangular matrices). Then $\alpha_w\alpha_u=\alpha_u\alpha_w$ and so α satisfies (C1), and also

$$\alpha_v\omega(w,u) = (0,(\gamma(v)(\Omega(w,u)-\Omega(w,u)),\gamma(v)(\Omega(w,u)-\Omega(w,u))),0)=0,$$

so α and ω satisfy (C2) trivially. The Lie bracket is given by

$$[(w,v,x,y),(w',v',x',y')] = (0,\omega(w,w'),\alpha_w v'-\alpha_{w'}v,\alpha_w x'-\alpha_{w'}x),$$

or, more explicitly, this may be written componentwise as

$$[(w,v,x,y),(w',v',x',y')]_2 = (\Omega(w,w'),\Omega(w,w'),\Omega(w,w'))\in\mathbb{R}^3,$$

$$[(w,v,x,y),(w',v',x',y')]_3$$
$$= (\gamma(w)(v_1'-v_2')-\gamma(w')(v_1-v_2),\gamma(w)(v_2'-v_3')-\gamma(w')(v_2-v_3))\in\mathbb{R}^2,$$

and

$$[(w,v,x,y),(w',v',x',y')]_4 = \gamma(w)(x_1'-x_2')-\gamma(w')(x_1-x_2)\in\mathbb{R}.$$

Thus,

$$\begin{aligned}[(w'',v'',x'',y''),&[(w,v,x,y),(w',v',x',y')]]\\ &= (0,0,\alpha_{w''}\omega(w,w'),\alpha_{w''}(\alpha_w v'-\alpha_{w'}v))\\ &= (0,0,0,\alpha_{w''}\alpha_w v'-\alpha_{w''}\alpha_{w'}v)\\ &= (0,0,0,\gamma(w'')\gamma(w)(v_1'-v_3')-\gamma(w'')\gamma(w')(v_1-v_3)),\end{aligned}$$

and all higher order brackets are 0. So for $g=(w,v,x,y)$ and $g'=(w',v',x',y')$, the group operation is given by

$$\begin{aligned}
(g\cdot g')_1 &= w+w' \\
(g\cdot g')_2 &= v+v'+\frac{1}{2}\omega(w,w') \\
(g\cdot g')_3 &= x+x'+\frac{1}{2}(\alpha_w v'-\alpha_{w'}v) \\
(g\cdot g')_4 &= y+y'+\frac{1}{2}(\alpha_w x'-\alpha_{w'}x)+\frac{1}{12}(\alpha_w^2 v'+\alpha_{w'}^2 v-\alpha_w\alpha_{w'}(v-v')).
\end{aligned}$$

Clearly, this example may be further modified to make nilpotent Lie algebras of arbitrary step.

3.2. **Hilbert-Schmidt norms.** In this section, we will show that the assumed continuity of ω and α makes the Lie bracket into a Hilbert-Schmidt operator on $\mathfrak{g}_{CM}$. This result will be needed later in guaranteeing that our stochastic integrals are well-defined.

Notation 3.11. Let $H_1,\dots,H_n$ and V be Hilbert spaces, and let $\{h_j^i\}_{j=1}^{\dim(H_i)}$ denote an orthonormal basis for each H_i. If $\rho: H_1\times\cdots\times H_n\to V$ is a multilinear map, then the Hilbert-Schmidt norm of ρ is defined by

$$\|\rho\|_2^2 := \|\rho\|^2_{H_1^*\otimes\cdots\otimes H_n^*\otimes V} = \sum_{j_1,\dots,j_n} \|\rho(h^1_{j_1},\dots,h^n_{j_n})\|_V^2.$$

In particular, for H an infinite dimensional Hilbert space with orthonormal basis $\{h_i\}_{i=1}^\infty$, $\rho: H^{\otimes n}\to V$ is Hilbert-Schmidt if

$$\|\rho\|_2^2 = \|\rho\|^2_{(H^*)^{\otimes n}\otimes V} = \sum_{j_1,\dots,j_n=1}^\infty \|\rho(h_{j_1},\dots,h_{j_n})\|_V^2 < \infty.$$

One may verify directly that these norms are independent of the chosen bases.

Proposition 3.12. *For all $w\in W$ and $x\in\mathfrak{v}$,*

$$\|\alpha_w\cdot\|^2_{\mathfrak{v}^*\otimes\mathfrak{v}} \le N\|\alpha\|_0^2\|w\|_W^2 \quad \text{and} \quad \|\alpha.x\|^2_{H^*\otimes\mathfrak{v}} \le C_2\|\alpha\|_0^2\|x\|^2_{\mathfrak{v}},$$

where $N=\dim(\mathfrak{v})$, C_2 is as in equation (2.1), and $\|\cdot\|_0$ is as defined in Notation 3.2. Also,

$$\|\omega(w,\cdot)\|^2_{H^*\otimes\mathfrak{v}} \le C_2\|\omega\|_0^2\|w\|_W^2.$$

Furthermore,

$$\|\alpha\|_2^2 \le NC_2\|\alpha\|_0^2<\infty \quad \text{and} \quad \|\omega\|_2^2\le C_2^2\|\omega\|_0^2<\infty.$$

Proof. Let $\{e_i\}_{i=1}^N$ be an orthonormal basis of $\mathfrak{v}$. Then, for any $w\in W$,

$$\|\alpha_w\cdot\|^2_{\mathfrak{v}^*\otimes\mathfrak{v}} = \sum_{i=1}^N\|\alpha_we_i\|^2_{\mathfrak{v}} \le \sum_{i=1}^N \|\alpha\|_0^2\|w\|_W^2\|e_i\|^2_{\mathfrak{v}} = N\|\alpha\|_0^2\|w\|_W^2.$$

For fixed $x\in\mathfrak{v}$, $\alpha.x: W\to\mathfrak{v}$ is a continuous linear map. Thus, equation (2.2) gives

$$\begin{aligned}
\|\alpha.x\|^2_{H^*\otimes\mathfrak{v}} &= \int_W \|\alpha_wx\|^2_{\mathfrak{v}}\,d\mu(w) \\
&\le \int_W \|\alpha\|_0^2\|w\|_W^2\|x\|^2_{\mathfrak{v}}\,d\mu(w) = C_2\|\alpha\|_0^2\|x\|^2_{\mathfrak{v}}.
\end{aligned}$$

Similarly, for fixed $w \in W$ and $\omega(w, \cdot) : W \to \mathfrak{v}$,

$$\begin{aligned}\|\omega(w,\cdot)\|^2_{H^*\otimes\mathfrak{v}} &= \int_W \|\omega(w,w')\|^2_{\mathfrak{v}}\, d\mu(w') \\ &\le \int_W \|\omega\|_0^2 \|w\|_W^2 \|w'\|_W^2\, d\mu(w') = C_2 \|\omega\|_0^2 \|w\|_W^2.\end{aligned}$$

Since $w \mapsto \alpha_w$ is a continuous linear map from W to $\mathfrak{v}^* \otimes \mathfrak{v}$, it follows from equations (2.2) and (2.3) that

$$\|\alpha\|_2^2 = \int_W \|\alpha_w \cdot\|^2_{\mathfrak{v}^*\otimes\mathfrak{v}}\, d\mu(w) \le \int_W N\|\alpha\|_0^2 \|w\|_W^2\, d\mu(w) = NC_2\|\alpha\|_0^2,$$

and since $w \mapsto \omega(w, \cdot)$ is a continuous linear map from W to $H^* \otimes \mathfrak{v}$,

$$\begin{aligned}\|\omega\|_2^2 = \|h \mapsto \omega(h,\cdot)\|^2_{H^*\otimes(H^*\otimes\mathfrak{v})} &= \int_W \|\omega(w,\cdot)\|^2_{H^*\otimes\mathfrak{v}}\, d\mu(w) \\ \le \int_W C_2 \|\omega\|_0^2 \|w\|_W^2\, d\mu(w) &= C_2^2 \|\omega\|_0^2.\end{aligned}$$

■

This proposition easily gives the following result.

Corollary 3.13. *For all $m \ge 2$, $[[[\cdot,\cdot],\ldots],\cdot] : \mathfrak{g}_{CM}^{\otimes m} \to \mathfrak{v}$ is Hilbert-Schmidt.*

Proof. For $m = 2$, this follows from the previous proposition and the continuity of the Lie bracket on $\mathfrak{v}$, since taking $\{h_i\}_{i=1}^\infty = \{k_i\}_{i=1}^\infty \cup \{e_j\}_{j=1}^N$, where $\{k_i\}_{i=1}^\infty$ and $\{e_j\}_{j=1}^N$ are orthonormal bases of H and $\mathfrak{v}$, respectively, gives

$$\begin{aligned}\|[\cdot,\cdot]\|_2^2 = \|[\cdot,\cdot]\|^2_{\mathfrak{g}^*_{CM}\otimes\mathfrak{g}^*_{CM}\otimes\mathfrak{v}} &= \sum_{i_1,i_2=1}^\infty \|[h_{i_1}, h_{i_2}]\|^2_{\mathfrak{v}} \\ &= \sum_{i_1,i_2=1}^\infty \|\omega(k_{i_1}, k_{i_2})\|^2_{\mathfrak{v}} + \sum_{i_1=1}^\infty \sum_{j_2=1}^N \|\alpha_{k_{i_1}} e_{j_2}\|^2_{\mathfrak{v}} \\ &\quad + \sum_{i_2=1}^\infty \sum_{j_1=1}^N \|\alpha_{k_{i_2}} e_{j_1}\|^2_{\mathfrak{v}} + \sum_{j_1,j_2=1}^N \|[e_{j_1}, e_{j_2}]\|^2_{\mathfrak{v}} \\ &= \|\omega\|_2^2 + 2\|\alpha\|_2^2 + N < \infty.\end{aligned}$$

Now assume the statement is true for all $m = 2, \ldots, \ell$. Consider $m = \ell + 1$. Writing $[[h_{i_1}, h_{i_2}], \cdots, h_{i_\ell}] \in \mathfrak{v}$ in terms of the orthonormal basis $\{e_j\}_{j=1}^N$ and using multiple applications of the Cauchy-Schwarz inequality gives

$$\begin{aligned}
\|[[[\cdot,\cdot],\ldots],\cdot]\|_2^2 &= \|[[[\cdot,\cdot],\ldots],\cdot]\|_{(\mathfrak{g}_{CM}^*)^{\otimes \ell+1}\otimes\mathfrak{v}} \\
&= \sum_{i_1,\ldots,i_{\ell+1}=1}^{\infty} \|[[[h_{i_1},h_{i_2}],\cdots,h_{i_\ell}],h_{i_{\ell+1}}]\|_{\mathfrak{v}}^2 \\
&= \sum_{i_1,\ldots,i_{\ell+1}=1}^{\infty} \left\| \sum_{j=1}^{N} [e_j,h_{i_{\ell+1}}]\langle e_j,[[h_{i_1},h_{i_2}],\cdots,h_{i_\ell}]\rangle \right\|_{\mathfrak{v}}^2 \\
&\leq N \sum_{i_1,\ldots,i_{\ell+1}=1}^{\infty} \sum_{j=1}^{N} \|[e_j,h_{i_{\ell+1}}]\|_{\mathfrak{v}}^2 |\langle e_j,[[h_{i_1},h_{i_2}],\cdots,h_{i_\ell}]\rangle|^2 \\
&\leq N \left(\sum_{i_{\ell+1}=1}^{\infty} \sum_{j=1}^{N} \|[e_j,h_{i_{\ell+1}}]\|_{\mathfrak{v}}^2 \right) \left(\sum_{i_1,\ldots,i_\ell=1}^{\infty} \sum_{j=1}^{N} |\langle e_j,[[h_{i_1},h_{i_2}],\cdots,h_{i_\ell}]\rangle|^2 \right) \\
&\leq N \|[\cdot,\cdot]\|_{\mathfrak{g}_{CM}^{\otimes 2}\otimes\mathfrak{v}}^2 \cdot \|[[[\cdot,\cdot],\ldots],\cdot]\|_{\mathfrak{g}_{CM}^{\otimes \ell}\otimes\mathfrak{v}}^2,
\end{aligned}$$

where in the penultimate inequality we have used that all terms in the sums are positive. The last line is finite by the induction hypothesis. ∎

3.3. **Length and distance.** In this section, we define the Riemannian distance on G_{CM} and show that the topology induced by this metric is equivalent to the Hilbert topology induced by $\|\cdot\|_{\mathfrak{g}_{CM}}$.

For $g \in G$, let $L_g : G \to G$ and $R_g : G \to G$ denote left and right multiplication by g, respectively. As G is a vector space, to each $g \in G$ we can associate the tangent space T_gG to G at g, which is naturally isomorphic to G.

Notation 3.14. For $f : G \to \mathbb{R}$ a Frechét smooth function and $v, x \in G$ and $h \in \mathfrak{g}$, let

$$f'(x)h := \partial_h f(x) = \frac{d}{dt}\bigg|_0 f(x+th),$$

and let $v_x \in T_xG$ denote the tangent vector satisfying $v_x f = f'(x)v$. If $\sigma(t)$ is any smooth curve in G such that $\sigma(0) = x$ and $\dot\sigma(0) = v$ (for example, $\sigma(t) = x + tv$), then

$$L_{g*}v_x = \frac{d}{dt}\bigg|_0 g \cdot \sigma(t).$$

Notation 3.15. Let $T > 0$ and $C^1([0,T], G_{CM})$ denote the collection of C^1-paths $g : [0,T] \to G_{CM}$. The length of g is defined as

$$\ell_{CM}(g) := \int_0^T \|L_{g^{-1}(s)*}g'(s)\|_{\mathfrak{g}_{CM}}\, ds.$$

The Riemannian distance between $x, y \in G_{CM}$ then takes the usual form

$$d_{CM}(x,y) := \inf\{\ell_{CM}(g) : g \in C^1([0,T], G_{CM}) \text{ such that } g(0) = x \text{ and } g(T) = y\}.$$

Note that the value of T in the definition of d_{CM} is irrelevant since the length functional is invariant under reparameterization.

Proposition 3.16. *For $g, x \in G$ and $v_x \in T_xG$,*

(3.5)

$$L_{g*}v_x = v + \sum_{k=1}^{r-1} \sum_{(n,m)\in\mathcal{I}_k} a^k_{n,m} \times \sum_{\substack{j \in \{1,\dots,k\} \\ m_j > 0}} \sum_{\ell=0}^{m_j-1} \mathrm{ad}_g^{n_1}\mathrm{ad}_x^{m_1}\cdots\mathrm{ad}_g^{n_j}\mathrm{ad}_x^{\ell}\mathrm{ad}_v\mathrm{ad}_x^{m_j-\ell-1}\mathrm{ad}_g^{n_{j-1}}\cdots\mathrm{ad}_g^{n_k}\mathrm{ad}_x^{m_k}g,$$

where $a^k_{n,m}$ are the coefficients in the group multiplication given in equation (3.3).

Proof. The proof is a simple computation. Let $x(t) = x + tv$, and first note that

$$\begin{aligned}\frac{d}{dt}\Big|_0 &\mathrm{ad}_g^{n_1}\mathrm{ad}_{x(t)}^{m_1}\cdots\mathrm{ad}_g^{n_k}\mathrm{ad}_{x(t)}^{m_k}g \\ &= \sum_{\substack{j \in \{1,\dots,k\} \\ m_j > 0}} \sum_{\ell=0}^{m_j-1} \mathrm{ad}_g^{n_1}\mathrm{ad}_x^{m_1}\cdots\mathrm{ad}_g^{n_j}\mathrm{ad}_x^{\ell}\mathrm{ad}_v\mathrm{ad}_x^{m_j-\ell-1}\mathrm{ad}_g^{n_{j-1}}\cdots\mathrm{ad}_g^{n_k}\mathrm{ad}_x^{m_k}g.\end{aligned}$$

Then using (3.4) and plugging this into

$$\begin{aligned}L_{g*}v_x &= \frac{d}{dt}\Big|_0 g\cdot x(t) \\ &= \frac{d}{dt}\Big|_0 \left(g + x(t) + \sum_{k=1}^{r-1}\sum_{(n,m)\in\mathcal{I}_k} a^k_{n,m}\mathrm{ad}_g^{n_1}\mathrm{ad}_{x(t)}^{m_1}\cdots\mathrm{ad}_g^{n_k}\mathrm{ad}_{x(t)}^{m_k}g\right)\end{aligned}$$

yields the desired result. ■

Example 3.17 (The step 3 case). When $r = 3$, the group operation is

$$g\cdot h = g + h + \frac{1}{2}[g,h] + \frac{1}{12}([g,[g,h]] + [h,[h,g]]).$$

Thus,

$$\begin{aligned}L_{g*}v_x &= \frac{d}{dt}\Big|_0 g\cdot x(t) \\ &= \frac{d}{dt}\Big|_0 \left(g + x(t) + \frac{1}{2}[g,x(t)] + \frac{1}{12}([g,[g,x(t)]] + [x(t),[x(t),g]])\right) \\ &= v + \frac{1}{2}[g,v] + \frac{1}{12}([g,[g,v]] + [v,[x,g]] + [x,[v,g]]).\end{aligned}$$

Lemma 3.18. *There exists a continuous decreasing function $\varepsilon > 0$ such that, for all $g \in G_{CM}$ and $v \in \mathfrak{g}_{CM}$,*

$$\|L_{g^{-1}*}v_g\|_{\mathfrak{g}_{CM}} \geq \varepsilon(\|g\|_{\mathfrak{g}_{CM}})\|v\|_{\mathfrak{g}_{CM}}.$$

Proof. Let $g \in G_{CM}$ and $v \in \mathfrak{g}_{CM}$. By equation (3.5),

$$L_{g^{-1}*}v_g = v + \sum_{k=1}^{r-1} \sum_{(n,m)\in\mathcal{I}_k} a^k_{n,m} \sum_{m_j>0} \sum_{\ell=0}^{m_j-1} \mathrm{ad}^{n_1}_{g^{-1}} \mathrm{ad}^{m_1}_g \cdots \mathrm{ad}^{n_j}_{g^{-1}} \mathrm{ad}^{\ell}_g \mathrm{ad}_v \mathrm{ad}^{m_j-\ell-1}_g \cdots \mathrm{ad}^{n_k}_{g^{-1}} \mathrm{ad}^{m_k}_g g^{-1}$$

$$(3.6) \qquad = v + \sum_{k=1}^{r-1} \sum_{(n,m)\in\mathcal{I}_k} (-1)^{|n|} 1_{\{m_k>0\}} a^k_{n,m} \mathrm{ad}^{|m|+|n|}_g v = (I - \Lambda(g))v,$$

where

$$\Lambda(g) := -\sum_{\ell=1}^{r-1} d_\ell \mathrm{ad}^\ell_g$$

with

$$d_\ell := \sum_{k=1}^{\ell} \sum_{\substack{(n,m)\in\mathcal{I}_k \\ |m|+|n|=\ell}} (-1)^{|n|} 1_{\{m_k>0\}} a^k_{n,m}.$$

Since $\mathfrak{g}$ is nilpotent, the operator $\Lambda(g)$ is nilpotent. Thus, there exists $M \in \mathbb{N}$ so that

$$(I - \Lambda(g))^{-1} = \sum_{k=0}^{M} \Lambda(g)^k,$$

and $v = (I-\Lambda(g))^{-1} L_{g^{-1}*}v_g$. Note that the operator norm satisfies

$$\|\Lambda(g)\| \le \sum_{\ell=1}^{r-1} |d_\ell| C^\ell \|g\|^\ell_{\mathfrak{g}_{CM}} < \infty,$$

where $C = C(\omega, \alpha)$ is as defined in (3.2). Therefore, taking

$$\varepsilon(\|g\|_{\mathfrak{g}_{CM}}) := \left[\sum_{k=0}^{M} \left(\sum_{\ell=1}^{r-1} |d_\ell| C^\ell \|g\|^\ell_{\mathfrak{g}_{CM}} \right)^k \right]^{-1}$$

satisfies the desired estimate. ∎

Proposition 3.19. *For all $x \in G_{CM}$ and $R > 0$, there exists $\delta = \delta(x, R)$ such that $d_{CM}(x, y) < \delta$ implies that $\|y - x\|_{\mathfrak{g}_{CM}} < R$.*

Proof. Fix $x \in G_{CM}$ and $R > 0$. We will determine $\delta = \delta(x, R)$ so that $\|y - x\|_{\mathfrak{g}_{CM}} \ge R$ implies $d_{CM}(x, y) \ge \delta(x, R)$. Let $B(x, R) := \{z \in G_{CM} : \|z - x\|_{\mathfrak{g}_{CM}} < R\}$, and consider $y \in G_{CM}$ such that $y \notin B(x, R)$. Then, for any C^1-path $g : [0, 1] \to G_{CM}$ such that $g(0) = x$ and $g(1) = y$, there is some first time t_0 such that g exits $B(x, R)$, and

$$\begin{aligned} \ell_{CM}(g) &= \int_0^1 \|L_{g^{-1}(s)*}g'(s)\|_{\mathfrak{g}_{CM}}\,ds \ge \int_0^{t_0} \|L_{g^{-1}(s)*}g'(s)\|_{\mathfrak{g}_{CM}}\,ds \\ &\ge \varepsilon(\|x\|_{\mathfrak{g}_{CM}} + R) \int_0^{t_0} \|g'(s)\|_{\mathfrak{g}_{CM}}\,ds \\ &\ge \varepsilon(\|x\|_{\mathfrak{g}_{CM}} + R)\|g(t_0) - x\|_{\mathfrak{g}_{CM}} = \varepsilon(\|x\|_{\mathfrak{g}_{CM}} + R)R =: \delta(x, R), \end{aligned}$$

for ε as given in Lemma 3.18. Since this estimate is true for any C^1-path g from x to $y \notin B(x, R)$, optimizing over all such paths gives the desired result. ∎

Proposition 3.20. *There exists a continuous increasing function $K < \infty$ such that $K(0) = 0$ and, for all $x, y \in G_{CM}$,*

$$d_{CM}(x,y) \leq (1 + K(\|x\|_{\mathfrak{g}_{CM}} \wedge \|y\|_{\mathfrak{g}_{CM}}))\|y-x\|_{\mathfrak{g}_{CM}} + o\left(\|y-x\|_{\mathfrak{g}_{CM}}\right),$$

as $\|y-x\|_{\mathfrak{g}_{CM}} \to 0$, where the implied constants in $o\left(\|y-x\|_{\mathfrak{g}_{CM}}\right)$ also depend on $\|x\|_{\mathfrak{g}_{CM}} \wedge \|y\|_{\mathfrak{g}_{CM}}$.

Proof. For notational simplicity, let $T = 1$. Let $g(s)$ be a path in $C^1([0,1], G_{CM})$. By equation (3.6), taking $g = g(s)$ and $v_{g(s)} = g'(s)$,

$$\ell_{CM}(g) = \int_0^1 \left\| g'(s) + \sum_{\ell=1}^{r-1} d_\ell \mathrm{ad}^\ell_{g(s)} g'(s) \right\|_{\mathfrak{g}_{CM}} ds. \tag{3.7}$$

Now suppose that $x, y \in G_{CM}$, and take $g(s) = x + s(y-x)$ for $0 \leq s \leq 1$. Then (3.7) gives

$$\begin{aligned} d_{CM}(x,y) &\leq \ell_{CM}(g) \\ &= \int_0^1 \left\| (y-x) + \sum_{\ell=1}^{r-1} d_\ell \mathrm{ad}^\ell_{x+s(y-x)}(y-x) \right\|_{\mathfrak{g}_{CM}} ds \\ &= \int_0^1 \left\| (y-x) + \sum_{\ell=1}^{r-1} d_\ell \sum_{\substack{(n,m)\in\mathcal{I}_\ell \\ |m|+|n|=\ell}} s^{|n|} \mathrm{ad}^{m_1}_x \mathrm{ad}^{n_1}_{y-x} \cdots \mathrm{ad}^{m_\ell}_x \mathrm{ad}^{n_\ell}_{y-x}(y-x) \right\|_{\mathfrak{g}_{CM}} ds. \end{aligned}$$

Splitting off all terms in the sum of order two or higher and evaluating the integral,

$$\begin{aligned} d_{CM}(x,y) \leq & \left\| (y-x) + \sum_{\ell=1}^{r-1} d_\ell \mathrm{ad}^\ell_x (y-x) \right\| \\ & + \left\| \sum_{\ell=1}^{r-1} d_\ell \sum_{\substack{(n,m)\in\mathcal{I}_\ell \\ |m|+|n|=\ell}} \frac{1_{\{|n|>0\}}}{|n|+1} \mathrm{ad}^{m_1}_x \mathrm{ad}^{n_1}_{y-x} \cdots \mathrm{ad}^{m_\ell}_x \mathrm{ad}^{n_\ell}_{y-x}(y-x) \right\|_{\mathfrak{g}_{CM}} \\ \leq & \left(1 + \sum_{\ell=1}^{r-1} \sum_{\substack{(n,m)\in\mathcal{I}_\ell \\ |m|+|n|=\ell}} |d_\ell| C^\ell \|x\|^\ell_{\mathfrak{g}_{CM}} \right) \|y-x\|_{\mathfrak{g}_{CM}} + o\left(\|y-x\|_{\mathfrak{g}_{CM}}\right), \end{aligned}$$

where $C = C(\omega, \alpha)$ is as defined in (3.2). Interchanging the roles of x and y in $g(s)$, and thus in this inequality, completes the proof. ■

Propositions 3.19 and 3.20 yield the following corollary.

Corollary 3.21. *The topology on G_{CM} induced by d_{CM} is equivalent to the Hilbert topology induced by $\|\cdot\|_{\mathfrak{g}_{CM}}$.*

3.4. **Ricci curvature.** In this section, we compute the Ricci curvature of certain finite dimensional approximations of G and show that it is bounded below uniformly. This result will be used in Section 5.1 to give L^p-bounds on Radon Nikodym derivatives of ν_t. It will also be applied in Section 5.2 to prove a logarithmic Sobolev inequality for ν_t. First we must define the appropriate approximations.

Let $i : H \to W$ be the inclusion map, and $i^* : W^* \to H^*$ be its transpose. That is, $i^*\ell := \ell \circ i$ for all $\ell \in W^*$. Also, let

$$H_* := \{h \in H : \langle \cdot, h \rangle_H \in \mathrm{Range}(i^*) \subset H^*\}.$$

That is, for $h \in H$, $h \in H_*$ if and only if $\langle \cdot, h\rangle_H \in H^*$ extends to a continuous linear functional on W, which we will continue to denote by $\langle \cdot, h\rangle_H$. Because H is a dense subspace of W, i^* is injective and thus has a dense range. Since $h \mapsto \langle \cdot, h\rangle_H$ as a map from H to H^* is a linear isometric isomorphism, it follows that $H_* \ni h \mapsto \langle \cdot, h\rangle_H \in W^*$ is a linear isomorphism also, and so H_* is a dense subspace of H.

Now suppose that $P : H \to H$ is a finite rank orthogonal projection such that $PH \subset H_*$. Let $\{k_j\}_{j=1}^m$ be an orthonormal basis for PH. Then we may extend P to a (unique) continuous operator from $W \to H$ (still denoted by P) by letting

$$Pw := \sum_{j=1}^m \langle w, k_j\rangle_H k_j \tag{3.8}$$

for all $w \in W$. For later purposes, we will also define

$$\pi_P(w, x) := (Pw, x).$$

Notation 3.22. Let $\mathrm{Proj}(W)$ denote the collection of finite rank projections on W such that $PW \subset H_*$ and $P|_H : H \to H$ is an orthogonal projection, that is, P has the form given in equation (3.8). Further, let $G_P := PW \oplus \mathfrak{v}$ (a subgroup of G_{CM}), and we equip G_P with the left invariant Riemannian metric induced from the restriction of the inner product on $\mathfrak{g}_{CM} = H \oplus \mathfrak{v}$ to $\mathrm{Lie}(G_P) = PH \oplus \mathfrak{v} =: \mathfrak{g}_{CM}^P$. Let Ric^P denote the associated Ricci tensor at the identity in G_P.

Proposition 3.23. *For $X = (A, a) \in \mathfrak{g}_{CM}^P$,*

$$\langle \mathrm{Ric}^P X, X\rangle_{\mathfrak{g}_{CM}^P} = \frac{1}{4}\|\langle a, [\cdot,\cdot]\rangle\|^2_{(\mathfrak{g}_{CM}^P)^*\otimes(\mathfrak{g}_{CM}^P)^*} - \frac{1}{2}\|[\cdot, X]\|^2_{(\mathfrak{g}_{CM}^P)^*\otimes\mathfrak{v}},$$

where $(\mathfrak{g}_{CM}^P)^* = (PH)^* \otimes \mathfrak{v}^*$.

Proof. For $\mathfrak{g}$ any nilpotent Lie algebra with orthonormal basis Γ,

$$\langle \mathrm{Ric}\, X, X\rangle = \frac{1}{4}\sum_{Y\in\Gamma} \|\mathrm{ad}_Y^* X\|^2 - \frac{1}{2}\sum_{Y\in\Gamma}\|\mathrm{ad}_Y X\|^2, \tag{3.9}$$

for all $X \in \mathfrak{g}$; see for example Theorem 7.30 and Corollary 7.33 of [5].

So let $\Gamma_m := \{h_i\}_{i=1}^{m+N} = \{(k_i, 0)\}_{i=1}^m \cup \{(0, e_j)\}_{j=1}^N$ be an orthonormal basis of $\mathfrak{g}_{CM}^P = PH \oplus \mathfrak{v}$, where $\{k_i\}_{i=1}^m$ and $\{e_j\}_{j=1}^N$ are orthonormal bases of PH and $\mathfrak{v}$, respectively. Then, for $Y \in \mathfrak{g}_{CM}^P$,

$$\mathrm{ad}_Y^* X = \sum_{h_i\in\Gamma_m} \langle \mathrm{ad}_Y^* X, h_i\rangle_{\mathfrak{g}_{CM}} h_i = \sum_{h_i\in\Gamma_m} \langle X, \mathrm{ad}_Y h_i\rangle_{\mathfrak{g}_{CM}} h_i.$$

Thus,

$$\sum_{h_i\in\Gamma_m} \|\mathrm{ad}_{h_i}^* X\|^2_{\mathfrak{g}_{CM}} = \sum_{h_i\in\Gamma_m}\sum_{h_j\in\Gamma_m} \langle X, \mathrm{ad}_{h_i} h_j\rangle^2_{\mathfrak{g}_{CM}} = \sum_{h_i,h_j\in\Gamma_m} \langle X, [h_i, h_j]\rangle^2_{\mathfrak{g}_{CM}}.$$

Plugging this into (3.9) gives

$$\begin{aligned}\langle \mathrm{Ric}^P X, X\rangle_{\mathfrak{g}_{CM}^P} &= \frac{1}{4}\sum_{h_i,h_j\in\Gamma_m} \langle X, [h_i, h_j]\rangle^2_{\mathfrak{g}_{CM}} - \frac{1}{2}\sum_{h_i\in\Gamma_m} \|[h_i, X]\|^2_{\mathfrak{g}_{CM}} \\ &= \frac{1}{4}\sum_{h_i,h_j\in\Gamma_m} \langle a, [h_i, h_j]\rangle^2_{\mathfrak{v}} - \frac{1}{2}\sum_{h_i\in\Gamma_m} \|[h_i, X]\|^2_{\mathfrak{v}}.\end{aligned}$$

■

Corollary 3.24. *Let*

$$K := -\frac{1}{2}\sup\left\{\|[\cdot, X]\|^2_{\mathfrak{g}^*_{CM}\otimes\mathfrak{v}} : \|X\|_{\mathfrak{g}_{CM}} = 1\right\}.$$

Then $K > -\infty$ and K is the largest constant such that

$$\langle \mathrm{Ric}^P X, X\rangle_{\mathfrak{g}^P_{CM}} \geq K\|X\|^2_{\mathfrak{g}^P_{CM}}, \quad \text{for all } X \in \mathfrak{g}^P_{CM},$$

holds uniformly for all $P \in \mathrm{Proj}(W)$.

Proof. The first assertion is simple, since

$$K \geq -\frac{1}{2}\|[\cdot,\cdot]\|_2^2 > -\infty,$$

by Corollary 3.13. Now, for $P \in \mathrm{Proj}(W)$ as in Notation 3.22, Proposition 3.23 implies that

$$\langle \mathrm{Ric}^P X, X\rangle_{\mathfrak{g}^P_{CM}} \geq -\frac{1}{2}\|[\cdot, X]\|^2_{(\mathfrak{g}^P_{CM})^*\otimes\mathfrak{v}}.$$

Thus,

$$\begin{aligned}\frac{\langle \mathrm{Ric}^P X, X\rangle_{\mathfrak{g}^P_{CM}}}{\|X\|^2_{\mathfrak{g}^P_{CM}}} &\geq -\frac{1}{2}\frac{\|[\cdot, X]\|^2_{(\mathfrak{g}^P_{CM})^*\otimes\mathfrak{v}}}{\|X\|^2_{\mathfrak{g}^P_{CM}}}\\ &\geq -\frac{1}{2}\sup\left\{\|[\cdot, X]\|^2_{(\mathfrak{g}^P_{CM})^*\otimes\mathfrak{v}} : \quad \|X\|_{\mathfrak{g}^P_{CM}} = 1\right\} =: K_P. \end{aligned} \tag{3.10}$$

Noting that the infimum of K_P over all $P \in \mathrm{Proj}(W)$ is K completes the proof. ■

Remark 3.25. Of course, one can compute the Ricci curvature for G_{CM} just as in Proposition 3.23. Choose an orthonormal basis $\Gamma = \{h_i\}_{i=1}^\infty = \{(k_i, 0)\}_{i=1}^\infty \cup \{(0, e_j)\}_{j=1}^N$ of $\mathfrak{g}_{CM} = H \oplus \mathfrak{v}$, where $\{k_i\}_{i=1}^\infty$ is an orthonormal basis of H, and $\{e_j\}_{j=1}^N$ is an orthonormal basis of $\mathfrak{v}$. Then, for all $X = (A, a) \in \mathfrak{g}_{CM}$,

$$\begin{aligned}\langle \mathrm{Ric}\, X, X\rangle_{\mathfrak{g}_{CM}} &= \frac{1}{4}\sum_{i,j=1}^\infty \langle a, [h_i, h_j]\rangle^2_{\mathfrak{v}} - \frac{1}{2}\sum_{i=1}^\infty \|[h_i, X]\|^2_{\mathfrak{v}}\\ &= \frac{1}{4}\|\langle a, [\cdot,\cdot]\rangle\|^2_{\mathfrak{g}^*_{CM}\otimes\mathfrak{g}^*_{CM}} - \frac{1}{2}\|[\cdot, X]\|^2_{\mathfrak{g}^*_{CM}\otimes\mathfrak{v}} \geq K\|X\|^2_{\mathfrak{g}_{CM}}.\end{aligned}$$

4. Brownian motion

Suppose that B_t is a smooth curve in $\mathfrak{g}_{CM}$ with $B_0 = 0$, and consider the differential equation

$$\dot{g}_t = g_t\dot{B}_t := L_{g_t*}\dot{B}_t, \quad \text{with } g_0 = \mathbf{e}.$$

The solution g_t may be written as follows (see [26]): For $t > 0$, let $\Delta_n(t)$ denote the simplex in $\mathbb{R}^n$ given by

$$\{s = (s_1, \cdots, s_n) \in \mathbb{R}^n : 0 < s_1 < s_2 < \cdots < s_n < t\}.$$

Let $\mathcal{S}_n$ denote the permutation group on $(1, \cdots, n)$, and, for each $\sigma \in \mathcal{S}_n$, let $e(\sigma)$ denote the number of "errors" in the ordering $(\sigma(1), \sigma(2), \cdots, \sigma(n))$, that is,

$e(\sigma) = \#\{j < n : \sigma(j) > \sigma(j+1)\}$. Then

$$g_t = \sum_{n=1}^{r} \sum_{\sigma \in \mathcal{S}_n} \left((-1)^{e(\sigma)} \Big/ n^2 \begin{bmatrix} n-1 \\ e(\sigma) \end{bmatrix} \right) \times \int_{\Delta_n(t)} [\cdots[\dot{B}_{s_{\sigma(1)}}, \dot{B}_{s_{\sigma(2)}}], \ldots,]\dot{B}_{s_{\sigma(n)}}]\, ds. \tag{4.1}$$

For $n \in \{1, \cdots, r\}$ and $\sigma \in \mathcal{S}_n$, let $F_n^\sigma : \mathfrak{g}_{CM}^{\otimes n} \to \mathfrak{v}$ be the linear map given by

$$F_n^\sigma(k_1 \otimes \cdots \otimes k_n) := [[\cdots[k_{\sigma(1)}, k_{\sigma(2)}], \cdots], k_{\sigma(n)}]. \tag{4.2}$$

Recall that F_n^σ is Hilbert-Schmidt by Corollary 3.13. Then we may write

$$g_t = \sum_{n=1}^{r-1} \sum_{\sigma \in \mathcal{S}_n} c_n^\sigma F_n^\sigma \left(\int_{\Delta_n(t)} \dot{B}_{s_1} \otimes \cdots \otimes \dot{B}_{s_n}\, ds \right), \tag{4.3}$$

where the coefficients c_n^σ are determined by (4.1). Using this as our motivation, we first explore stochastic integral analogues of equation (4.3) where the smooth curve B is replaced by Brownian motion on $\mathfrak{g}$.

4.1. **Multiple Itô integrals.** Let $\langle \cdot, \cdot \rangle_{\mathfrak{g}_{CM}^{\otimes n}}$ denote the inner product on $\mathfrak{g}_{CM}^{\otimes n}$ arising from the inner product on $\mathfrak{g}_{CM}$. Also, let $\{k_i\}_{i=1}^\infty \subset H_*$ be an orthonormal basis of H, and define $P_m \in \mathrm{Proj}(W)$ by

$$P_m(w) = \sum_{i=1}^{m} \langle w, k_i \rangle_H k_i, \qquad \text{for all } w \in W, \tag{4.4}$$

as in equation (3.8), and define

$$\pi_m(w, x) := \pi_{P_m}(w, x) = (P_m(w), x) \in G_{P_m}.$$

Of course, $\dim(G_{P_m}) = m + N$, but in a mild abuse of notation, in this section we will use $\{h_i\}_{i=1}^m$ to denote an orthonormal basis of G_{P_m}, rather than the more cumbersome $\{h_i\}_{i=1}^{m+N} = \{(k_i, 0)\}_{i=1}^m \cup \{(0, e_i)\}_{i=1}^N$, where $\{e_i\}_{i=1}^N$ is an orthonormal basis of $\mathfrak{v}$.

Let $\{B_t\}_{t\geq 0} = \{(\beta_t, \beta_t^{\mathfrak{v}})\}_{t\geq 0}$ be a Brownian motion on $\mathfrak{g} = W \oplus \mathfrak{v}$ with variance determined by

$$\mathbb{E}\left[\langle B_s, h \rangle_{\mathfrak{g}_{CM}} \langle B_t, k \rangle_{\mathfrak{g}_{CM}}\right] = \langle h, k \rangle_{\mathfrak{g}_{CM}} \min(s, t),$$

for all $s, t \geq 0$ and $h = (A, a)$ and $k = (C, c)$, such that $A, C \in H_*$ and $a, c \in \mathfrak{v}$. Then $\pi_m B = (P_m \beta, \beta^{\mathfrak{v}})$ is a Brownian motion on $\mathfrak{g}^{P_m} = P_m W \oplus \mathfrak{v} \subset \mathfrak{g}_{CM}$.

Proposition 4.1. *For $\xi \in L^2(\Delta_n(t), \mathfrak{g}_{CM}^{\otimes n})$ a continuous mapping, let*

$$\begin{aligned} J_n^m(\xi)_t &:= \int_{\Delta_n(t)} \langle \pi_m^{\otimes n} \xi(s), dB_{s_1} \otimes \cdots \otimes dB_{s_n} \rangle_{\mathfrak{g}_{CM}^{\otimes n}} \\ &= \int_{\Delta_n(t)} \langle \xi(s), d\pi_m B_{s_1} \otimes \cdots \otimes d\pi_m B_{s_n} \rangle_{\mathfrak{g}_{CM}^{\otimes n}}. \end{aligned}$$

Then $\{J_n^m(\xi)_t\}_{t\geq 0}$ is a continuous L^2-martingale such that, for all m,

$$\mathbb{E}|J_n^m(\xi)_t|^2 \leq \|\xi\|^2_{L^2(\Delta_n(t), \mathfrak{g}_{CM}^{\otimes n})},$$

and there exists a continuous L^2-martingale $\{J_n(\xi)_t\}_{t\geq 0}$ such that

$$\lim_{m\to\infty} \mathbb{E}\left[\sup_{\tau\leq t} |J_n^m(\xi)_\tau - J_n(\xi)_\tau|^2\right] = 0, \tag{4.5}$$

for all $t<\infty$. $J_n(\xi)_t$ is well-defined independent of the choice of orthonormal basis $\{h_i\}_{i=1}^\infty$ in (4.4), and so will be denoted by

$$J_n(\xi)_t = \int_{\Delta_n(t)} \langle \xi(s), dB_{s_1}\otimes\cdots\otimes dB_{s_n}\rangle_{\mathfrak{g}_{CM}^{\otimes n}}.$$

Proof. Note first that,

$$J_n^m(\xi)_t = \sum_{i_1,\ldots,i_n=1}^m \int_{\Delta_n(t)} \langle \xi(s), h_{i_1}\otimes\cdots\otimes h_{i_n}\rangle_{\mathfrak{g}_{CM}^{\otimes n}} dB_{s_1}^{i_1}\cdots dB_{s_n}^{i_n}$$

where $\{B^i\}_{i=1}^m$ are independent real valued Brownian motions. Let $\xi_{i_1,\ldots,i_n} := \langle \xi, h_{i_1}\otimes\cdots\otimes h_{i_n}\rangle$. Then

$$|\xi_{i_1,\ldots,i_n}(s)|^2 \leq \|\xi(s)\|^2_{\mathfrak{g}_{CM}^{\otimes n}}$$

and $\xi_{i_1,\ldots,i_n}\in L^2(\Delta_n(t))$. Thus, $J_n^m(\xi)_t$ is defined as a (finite dimensional) vector-valued multiple Wiener-Itô integral, see for example [20, 25].

Now note that

$$\begin{aligned} dJ_n^m(\xi)_t &= \int_{\Delta_{n-1}(t)} \langle \xi(s_1,\ldots,s_{n-1},t), d\pi_m B_{s_1}\otimes\cdots\otimes d\pi_m B_{s_{n-1}}\otimes d\pi_m B_t\rangle_{\mathfrak{g}_{CM}^{\otimes n}} \\ &= \sum_{i=1}^m \int_{\Delta_{n-1}(t)} \langle \xi(s_1,\ldots,s_{n-1},t), d\pi_m B_{s_1}\otimes\cdots\otimes d\pi_m B_{s_{n-1}}\otimes h_i\rangle_{\mathfrak{g}_{CM}^{\otimes n}} dB_t^i. \end{aligned}$$

Thus, the quadratic variation $\langle J_n^m(\xi)\rangle_t$ is given by

$$\sum_{i=1}^m \int_0^t \left| \int_{\Delta_{n-1}(\tau)} \langle \xi(s_1,\ldots,s_{n-1},\tau), d\pi_m B_{s_1}\otimes\cdots\otimes d\pi_m B_{s_{n-1}}\otimes h_i\rangle_{\mathfrak{g}_{CM}^{\otimes n}}\right|^2 d\tau,$$

and

$$\begin{aligned} \mathbb{E}|J_n^m(\xi)_t|^2 &= \mathbb{E}\langle J_n^m(\xi)\rangle_t \\ &= \sum_{i_1=1}^m \int_0^t \mathbb{E}\bigg[\sum_{i_2=1}^m \int_0^{\tau_1} \bigg| \int_{\Delta_{n-2}(\tau_2)} \langle \xi(s_1,\ldots,s_{n-2},\tau_2,\tau_1), d\pi_m B_{s_1}\otimes\cdots \\ &\qquad\qquad \cdots\otimes d\pi_m B_{s_{n-2}}\otimes h_{i_2}\otimes h_{i_1}\rangle_{\mathfrak{g}_{CM}^{\otimes n}}\bigg|^2 d\tau_2\bigg] d\tau_1. \end{aligned}$$

Iterating this procedure n times gives

$$\begin{aligned} \mathbb{E}|J_n^m(\xi)_t|^2 &= \sum_{i_1,\ldots,i_n=1}^m \int_{\Delta_n(t)} \left|\langle \xi(\tau_1,\cdots,\tau_n), h_{i_1}\otimes\cdots\otimes h_{i_n}\rangle_{\mathfrak{g}_{CM}^{\otimes n}}\right|^2 d\tau_1\cdots d\tau_n \\ &= \int_{\Delta_n(t)} \|\pi_m^{\otimes n}\xi(s)\|^2_{\mathfrak{g}_{CM}^{\otimes n}} \leq \|\xi\|^2_{L^2(\Delta_n(t),\mathfrak{g}_{CM}^{\otimes n})}, \end{aligned}$$

and thus, for each n, $J_n^m(\xi)_t$ is bounded uniformly in L^2 independent of m.

Now, for $P \in \mathrm{Proj}(W)$, let $J_n^P(\xi)_t := \int_{\Delta_n(t)} \langle \pi_P^{\otimes n}\xi(s), dB_{s_1} \otimes \cdots \otimes dB_{s_n}\rangle_{\mathfrak{g}_{CM}^{\otimes n}}$. For $P, Q \in \mathrm{Proj}(W)$, a similar argument to the above implies that

$$\mathbb{E}|J_n^P(\xi)_t - J_n^Q(\xi)_t|^2 = \int_{\Delta_n(t)} \|\pi_P^{\otimes n}\xi(s) - \pi_Q^{\otimes n}\xi(s)\|^2_{\mathfrak{g}_{CM}^{\otimes n}}\, ds. \tag{4.6}$$

In particular, take $P = P_m$ and $Q = P_\ell$ with $\ell \leq m$, and note that

$$\begin{aligned}
&\|\pi_m^{\otimes n}\xi(s) - \pi_\ell^{\otimes n}\xi(s)\|^2_{\mathfrak{g}_{CM}^{\otimes n}} \\
&\quad = \sum_{i_1,\dots,i_n=1}^\infty \left| \sum_{j=1}^n \langle \pi_m^{\otimes j-1} \otimes (\pi_m - \pi_\ell) \otimes \pi_\ell^{n-j-1}\xi(s), h_{i_1} \otimes \cdots \otimes h_{i_n}\rangle_{\mathfrak{g}_{CM}^{\otimes n}} \right|^2 \\
&\quad \leq n \sum_{j=1}^n \sum_{i_1,\dots,i_n=1}^\infty \left| \langle \pi_m^{\otimes j-1} \otimes (\pi_m - \pi_\ell) \otimes \pi_\ell^{n-j-1}\xi(s), h_{i_1} \otimes \cdots \otimes h_{i_n}\rangle_{\mathfrak{g}_{CM}^{\otimes n}} \right|^2 \\
&\quad = n \sum_{j=1}^n \sum_{i_1,\dots,i_{j-1}=1}^m \sum_{i_j=\ell+1}^m \sum_{i_{j+1},\dots,i_n=1}^\ell \left| \langle \xi(s), h_{i_1} \otimes \cdots \otimes h_{i_n}\rangle_{\mathfrak{g}_{CM}^{\otimes n}} \right|^2 \to 0,
\end{aligned} \tag{4.7}$$

as $\ell, m \to \infty$, for all $s \in \Delta_n(t)$, since $\|\xi\|^2_{L^2(\Delta_n(t), \mathfrak{g}_{CM}^{\otimes n})} < \infty$. Thus, equation (4.6) implies that

$$\lim_{\ell,m\to\infty} \mathbb{E}\left|J_n^m(\xi)_t - J_n^\ell(\xi)_t\right|^2 = 0,$$

by dominated convergence, and $\{J_n^m(\xi)_t\}_{m=1}^\infty$ is Cauchy in L^2. Since the space of continuous L^2-martingales is complete in the norm $M \mapsto \mathbb{E}|M_t|^2$, there exists a continuous martingale $\{J_n(\xi)_t\}_{t\geq 0}$ such that

$$\lim_{m\to\infty} \mathbb{E}|J_n^m(\xi)_t - J_n(\xi)_t|^2 = 0.$$

Combining this with Doob's maximal inequality proves equation (4.5).

To see that $J_n(\xi)_t$ is independent of the choice of basis, suppose now that $\{h'_j\}_{j=1}^\infty \subset H_*$ is another orthonormal basis for H, and let $P'_m : W \to H_*$ and $\pi'_m : G \to G_{P'_m}$ be the corresponding orthogonal projections. Consider the inequality (4.7) with π_ℓ replaced by π'_m. Writing $\pi_m - \pi'_m = (\pi_m - I) + (I - \pi'_m)$, and considering terms for each fixed j, we have

$$\begin{aligned}
&\sum_{i_1,\dots,i_n=1}^\infty \left| \langle \pi_m^{\otimes j-1} \otimes (\pi_m - I) \otimes (\pi'_m)^{n-j-1}\xi(s), h_{i_1} \otimes \cdots \otimes h_{i_n}\rangle_{\mathfrak{g}_{CM}^{\otimes n}} \right|^2 \\
&= \sum_{i_1,\dots,i_{j-1}=1}^m \sum_{i_j=m+1}^\infty \sum_{i_{j+1},\dots,i_n=1}^\infty \left| \langle \xi(s), h_{i_1} \otimes \cdots \otimes h_{i_j} \otimes \pi'_m h_{i_{j+1}} \otimes \cdots \otimes \pi'_m h_{i_n}\rangle_{\mathfrak{g}_{CM}^{\otimes n}} \right|^2 \\
&\leq \sum_{i_1,\dots,i_{j-1}=1}^m \sum_{i_j=m+1}^\infty \sum_{i_{j+1},\dots,i_n=1}^\infty \left| \langle \xi(s), h_{i_1} \otimes \cdots \otimes h_{i_n}\rangle_{\mathfrak{g}_{CM}^{\otimes n}} \right|^2 \to 0,
\end{aligned}$$

as $m \to \infty$. Similarly,

$$\sum_{i_1,\dots,i_n=1}^{\infty} \left| \langle \pi_m^{\otimes j-1} \otimes (I-\pi'_m) \otimes (\pi'_m)^{n-j-1} \xi(s), h_{i_1} \otimes \cdots \otimes h_{i_n} \rangle_{\mathfrak{g}_{CM}^{\otimes n}} \right|^2$$
$$= \sum_{i_1,\dots,i_n=1}^{\infty} \left| \langle \pi_m^{\otimes j-1} \otimes (I-\pi'_m) \otimes (\pi'_m)^{n-j-1} \xi(s), h'_{i_1} \otimes \cdots \otimes h'_{i_n} \rangle_{\mathfrak{g}_{CM}^{\otimes n}} \right|^2 \to 0,$$

as $m \to \infty$. Thus,

$$\lim_{m\to\infty} \|\pi_m^{\otimes n}\xi(s) - (\pi'_m)^{\otimes n}\xi(s)\|^2_{\mathfrak{g}_{CM}^{\otimes n}} = 0,$$

for each $s \in \Delta_n(t)$. Thus, for $J_n^{m'}(\xi)_t := J_n^{P'_m}(\xi)_t$, using equation (4.6) with $P = P_m$ and $Q = P'_m$ shows that

$$\lim_{m\to\infty} \mathbb{E}|J_n^m(\xi)_t - J_n^{m'}(\xi)_t|^2 = 0,$$

again by dominated convergence. ■

A simple linearity argument extends the map J_n to functions taking values in $(\mathfrak{g}_{CM}^*)^{\otimes n} \otimes \mathfrak{v}$.

Corollary 4.2. *Let $F \in L^2(\Delta_n(t), (\mathfrak{g}_{CM}^*)^{\otimes n} \otimes \mathfrak{v})$ be a continuous map. That is, $F : \Delta_n(t) \times \mathfrak{g}_{CM}^{\otimes n} \to \mathfrak{v}$ is a map continuous in s and linear on $\mathfrak{g}_{CM}^{\otimes n}$ such that*

$$\int_{\Delta_n(t)} \|F(s)\|_2^2\, ds = \int_{\Delta_n(t)} \sum_{j_1,\dots,j_n=1}^{\infty} \|F(s)(h_{j_1} \otimes \cdots \otimes h_{j_n})\|^2_{\mathfrak{v}}\, ds < \infty.$$

Then

$$J_n^m(F)_t := \int_{\Delta_n(t)} F(s)(d\pi_m B_{s_1} \otimes \cdots \otimes d\pi_m B_{s_n})$$

is a continuous L^2-martingale, and there exists a continuous $\mathfrak{v}$-valued L^2-martingale $\{J_n(F)_t\}_{t\geq 0}$ such that

$$\lim_{m\to\infty} \mathbb{E}\left[\sup_{\tau\leq t} \|J_n^m(\xi)_\tau - J_n(\xi)_\tau\|^2_{\mathfrak{v}}\right] = 0,$$

for all $t < \infty$. The martingale $J_n(\xi)_t$ is well-defined independent of the choice of orthonormal basis $\{h_i\}_{i=1}^\infty$ in (4.4), and thus will be denoted by

$$J_n(F)_t = \int_{\Delta_n(t)} F(s)(dB_{s_1} \otimes \cdots \otimes dB_{s_n}).$$

Proof. Let $\{e_j\}_{j=1}^N$ be an orthonormal basis of $\mathfrak{v}$. Then for any $k_1, \dots, k_n \in \mathfrak{g}_{CM}$,

$$F(s)(k_1 \otimes \cdots \otimes k_n) = \sum_{j=1}^N \langle F(s)(k_1 \otimes \cdots \otimes k_n), e_j\rangle e_j.$$

Since $\langle F(s)(\cdot), e_j\rangle$ is linear on $\mathfrak{g}_{CM}^{\otimes n}$, for each s there exists $\xi_j(s) \in \mathfrak{g}_{CM}^{\otimes n}$ such that

$$\langle \xi_j(s), k_1 \otimes \cdots \otimes k_n\rangle = \langle F(s)(k_1 \otimes \cdots \otimes k_n), e_j\rangle. \tag{4.8}$$

If $\xi_j : \Delta_n(t) \to \mathfrak{g}_{CM}^{\otimes n}$ is defined by equation (4.8), then

$$\|\xi_j\|_{L^2(\Delta_n(t), \mathfrak{g}_{CM}^{\otimes n})} \leq \int_{\Delta_n(t)} \|F(s)\|_2^2\, ds < \infty.$$

Thus,

$$J_n(F)_t = \sum_{j=1}^N \int_{\Delta_n(t)} \langle \xi_j(s), dB_{s_1} \otimes \cdots \otimes dB_{s_n} \rangle e_j = \sum_{j=1}^N J_n(\xi_j)_t e_j,$$

is well-defined, and, for each j, $J_n(\xi_j)$ is a martingale as defined in Proposition 4.1. ■

4.2. **Brownian motion and finite dimensional approximations.** Again let B_t denote Brownian motion on $\mathfrak{g}$. By equation (4.1), the solution to the Stratonovich stochastic differential equation

$$\delta g_t = L_{g_t *} \delta B_t, \qquad \text{with } g_0 = \mathbf{e},$$

should be given by

$$g_t = \sum_{n=1}^{r-1} \sum_{\sigma \in \mathcal{S}_n} c_n^\sigma \int_{\Delta_n(t)} [[\cdots [\delta B_{s_{\sigma(1)}}, \delta B_{s_{\sigma(2)}}], \cdots], \delta B_{s_{\sigma(n)}}], \tag{4.9}$$

for coefficients c_n^σ determined by equation (4.1).

To understand the integrals in (4.9), consider the following heuristic computation. Let $\{M_n(t)\}_{t\geq 0}$ denote the process in $\mathfrak{g}^{\otimes n}$ defined by

$$M_n(t) := \int_{\Delta_n(t)} \delta B_{s_1} \otimes \cdots \otimes \delta B_{s_n}.$$

By repeatedly applying the definition of the Stratonovich integral, the iterated Stratonovich integral $M_n(t)$ may be realized as a linear combination of iterated Itô integrals:

$$M_n(t) = \sum_{m=\lceil n/2 \rceil}^n \frac{1}{2^{n-m}} \sum_{\alpha \in \mathcal{J}_n^m} I_t^n(\alpha),$$

where

$$\mathcal{J}_n^m := \left\{ (\alpha_1, \ldots, \alpha_m) \in \{1,2\}^m : \sum_{i=1}^m \alpha_i = n \right\},$$

and, for $\alpha \in \mathcal{J}_n^m$, $I_t^n(\alpha)$ is the iterated Itô integral

$$I_t^n(\alpha) = \int_{\Delta_m(t)} dX_{s_1}^1 \otimes \cdots \otimes dX_{s_m}^m$$

with

$$dX_s^i = \begin{cases} dB_s & \text{if } \alpha_i = 1 \\ \sum_{j=1}^\infty h_j \otimes h_j \, ds & \text{if } \alpha_i = 2 \end{cases};$$

compare with Proposition 1 of [4]. This change from multiple Stratonovich integrals to multiple Itô integrals may also be recognized as a specific case of the Hu-Meyer formulas [17, 18], but we will compute more explicitly to verify that our integrals are well-defined.

As in equation (4.2), letting

$$F_n^\sigma(k_1 \otimes \cdots \otimes k_n) := [[\cdots [k_{\sigma(1)}, k_{\sigma(2)}], \cdots], k_{\sigma(n)}],$$

we may write

$$g_t = \sum_{n=1}^{r-1} \sum_{\sigma \in \mathcal{S}_n} c_n^\sigma F_n^\sigma(M_n(t)) = \sum_{n=1}^{r-1} \sum_{\sigma \in \mathcal{S}_n} \sum_{m=\lceil n/2 \rceil}^n \frac{c_n^\sigma}{2^{n-m}} \sum_{\alpha \in \mathcal{J}_n^m} F_n^\sigma(I_t^n(\alpha)),$$

presuming we can make sense of the integrals $F_n^\sigma(I_t^n(\alpha))$.

For each α, let $p_\alpha = \#\{i : \alpha_i = 1\}$ and $q_\alpha = \#\{i : \alpha_i = 2\}$ (so that $p_\alpha + q_\alpha = m$ when $\alpha \in \mathcal{J}_n^m$), and let

$$\mathcal{J}_n := \bigcup_{m=\lceil n/2 \rceil}^{n} \mathcal{J}_n^m.$$

Then, for each $\sigma \in \mathcal{S}_n$ and $\alpha \in \mathcal{J}_n$,

$$F_n^\sigma(I_t^n(\alpha)) = \int_{\Delta_{p_\alpha}(t)} f_\alpha(s,t) \hat{F}_n^{\sigma,\alpha}(dB_{s_1} \otimes \cdots \otimes dB_{s_{p_\alpha}}),$$

where $\hat{F}_n^{\sigma,\alpha}$ and f_α are as follows.

The map $\hat{F}_n^{\sigma,\alpha} : \mathfrak{g}^{\otimes p_\alpha} \to \mathfrak{g}$ is defined by

$$\begin{aligned} \hat{F}_n^{\sigma,\alpha}&(k_1 \otimes \cdots \otimes k_{p_\alpha}) \\ &:= \sum_{j_1,\ldots,j_{q_\alpha}=1}^{\infty} F_n^{\sigma'}(k_1 \otimes \cdots \otimes k_{p_\alpha} \otimes h_{j_1} \otimes h_{j_1} \otimes \cdots \otimes h_{j_{q_\alpha}} \otimes h_{j_{q_\alpha}}), \end{aligned} \tag{4.10}$$

for $\{h_j\}_{j=1}^\infty$ an orthonormal basis of $\mathfrak{g}_{CM}$ and $\sigma' = \sigma'(\alpha) \in \mathcal{S}_n$ given by $\sigma' = \sigma \circ \tau^{-1}$, for any $\tau \in \mathcal{S}_n$ such that

$$\begin{aligned} \tau&(dX_{s_1}^1 \otimes \cdots \otimes dX_{s_m}^m) \\ &= \sum_{j_1,\cdots,j_{q_\alpha}=1}^{\infty} dB_{s_1} \otimes \cdots \otimes dB_{s_{p_\alpha}} \otimes h_{j_1} \otimes h_{j_1} \otimes \cdots \otimes h_{j_{q_\alpha}} \otimes h_{j_{q_\alpha}} ds_1 \cdots ds_{q_\alpha}. \end{aligned}$$

To define f_α, first consider the polynomial of order q_α, in the variables s_i with i such that $\alpha_i = 1$ and in the variable t, given by evaluating the integral

$$f'_\alpha((s_i : \alpha_i = 1), t) = \int_{\Delta'_{q_\alpha}(t)} \prod_{i:\alpha_i=2} ds_i, \tag{4.11}$$

where $\Delta'_{q_\alpha}(t) = \{s_{i-1} < s_i < s_{i+1} : \alpha_i = 2\}$ with $s_0 = 0$ and $s_{m+1} = t$. Then f_α is f'_α with the variables reindexed by the bijection $\{i : \alpha_i = 1\} \to \{1, \ldots, p_\alpha\}$ that maintains the natural ordering of these sets. (For example, for $\alpha = (1,2,1,2) \in \mathcal{J}_6^4$,

$$f'_\alpha(s_1, s_3, t) = \int_{\{s_1<s_2<s_3, s_3<s_4<t\}} ds_2\, ds_4 = (t - s_3)(s_3 - s_1),$$

so that $f_\alpha(s_1, s_2, t) = (t - s_2)(s_2 - s_1)$.)

This explicit realization of f_α is not critical to the sequel. It is really only necessary to know that f_α is a polynomial of order q_α in $s = (s_1, \ldots, s_{p_\alpha})$ and t, and thus may be written as

$$f_\alpha(s,t) = \sum_{a=0}^{q_\alpha} b_\alpha^a t^a \tilde{f}_{\alpha,a}(s),$$

for some coefficients $b_\alpha^a \in \mathbb{R}$ and polynomials $\tilde{f}_{\alpha,a}$ of degree $q_\alpha - a$ in s. Now, if $\hat{F}_n^{\sigma,\alpha}$ is Hilbert-Schmidt on $\mathfrak{g}_{CM}^{\otimes p_\alpha}$, then

$$\int_{\Delta_{p_\alpha}(t)} \left\| \tilde{f}_{\alpha,a}(s) \hat{F}_n^{\sigma,\alpha} \right\|_2^2 ds = \left\| \tilde{f}_{\alpha,a} \right\|_{L^2(\Delta_{p_\alpha}(t))} \left\| \hat{F}_n^{\sigma,\alpha} \right\|_2^2 < \infty,$$

and

$$F_n^\sigma(I_t^n(\alpha)) = \sum_{a=0}^{q_\alpha} b_\alpha^a t^a J_n(\tilde{f}_{\alpha,a}\hat{F}_n^{\sigma,\alpha})_t \tag{4.12}$$

may be understood in the sense of the limit integrals in Corollary 4.2. (In particular, if $\alpha_m = 1$, then $f_\alpha = f_\alpha(s)$ does not depend on t, and Corollary 4.2 implies that $F_n^\sigma(I_t^n(\alpha))$ is a $\mathfrak{v}$-valued L^2-martingale.)

The above computations show that, if for all n $\hat{F}_n^{\sigma,\alpha}$ is Hilbert-Schmidt for all $\sigma \in \mathcal{S}_n$ and $\alpha \in \mathcal{J}_n$, then we may rewrite (4.9) as

$$g_t = \sum_{n=1}^{r-1} \sum_{\sigma\in\mathcal{S}_n} \sum_{m=\lceil n/2\rceil}^{n} \frac{c_n^\sigma}{2^{n-m}} \sum_{\alpha\in\mathcal{J}_n^m} \sum_{a=0}^{q_\alpha} b_\alpha^a t^a J_n(\tilde{f}_{\alpha,a}\hat{F}_n^{\sigma,\alpha})_t,$$

where J_n is as defined in Corollary 4.2. The next two results show that $\hat{F}_n^{\sigma,\alpha}$ is Hilbert-Schmidt as desired, and thus g_t in (4.9) is well-defined.

Lemma 4.3. *Let $n \in \{2,\dots,r\}$, $\sigma \in \mathcal{S}_n$, and $\alpha \in \mathcal{J}_n$. For any $v \in \mathfrak{v}$, $\langle \hat{F}_n^{\sigma,\alpha}, v\rangle$ is a Hilbert-Schmidt operator on $\mathfrak{g}_{CM}^{\otimes p_\alpha}$.*

Proof. First consider the case $n = 2$. In this case, $p_\alpha = 0$ or $p_\alpha = 2$. If $p_\alpha = 0$, then $\hat{F}_2^{\sigma,\alpha} = \sum_{i=1}^\infty F_2^\sigma(h_i \otimes h_i) = 0$. If $p_\alpha = 2$, then $\hat{F}_2^{\sigma,\alpha}(k_1\otimes k_2) = F_2^\sigma(k_1 \otimes k_2) = [k_{\sigma(1)}, k_{\sigma(2)})]$ is Hilbert-Schmidt by Corollary 3.13, and thus $\langle \hat{F}_2^{\sigma,\alpha}, v\rangle$ is Hilbert-Schmidt. For $n = 3$, $p_\alpha = 1$ or $p_\alpha = 3$. If $p_\alpha = 3$, then $\alpha = (1,1,1)$ and

$$\hat{F}_3^{\sigma,\alpha}(k_1\otimes k_2\otimes k_3) = F_3^\sigma(k_1\otimes k_2\otimes k_3) = [[k_{\sigma(1)}, k_{\sigma(2)}], k_{\sigma(3)}]$$

is Hilbert-Schmidt, again by Corollary 3.13. If $p_\alpha = 1$, then $\alpha = (1,2)$ or $\alpha = (2,1)$ and

$$\hat{F}_3^{\sigma,\alpha}(k) = \sum_{i=1}^\infty F_3^{\sigma'}(k\otimes h_i \otimes h_i),$$

and we need only consider the case that

$$F_3^{\sigma'}(k\otimes h\otimes h) = [[h,k],h].$$

So let $\{k_i\}_{i=1}^\infty$ be an orthonormal basis of $\mathfrak{g}_{CM}$ and $\{e_\ell\}_{\ell=1}^N$ be an orthonormal basis of $\mathfrak{v}$. As in the proof of Corollary 3.13, expanding terms in an orthonormal basis of $\mathfrak{v}$ and applying the Cauchy-Schwarz inequality gives

$$\begin{aligned}
\|\langle \hat{F}_3^{\sigma,\alpha}, v\rangle\|_2^2 &= \sum_{i=1}^\infty \left|\sum_{j=1}^\infty \langle [[h_j,k_i],h_j],v\rangle\right|^2 = \sum_{i=1}^\infty \left|\sum_{j=1}^\infty\sum_{\ell=1}^N \langle [e_\ell,h_j],v\rangle\langle e_\ell,[h_j,k_i]\rangle\right|^2 \\
&\le N\sum_{i=1}^\infty\sum_{\ell=1}^N \left|\sum_{j=1}^\infty \langle [e_\ell,h_j],v\rangle\langle e_\ell,[h_j,k_i]\rangle\right|^2 \\
&\le N\sum_{i=1}^\infty\sum_{\ell=1}^N \left(\sum_{j=1}^\infty |\langle [e_\ell,h_j],v\rangle|^2\right)\left(\sum_{j=1}^\infty |\langle e_\ell,[h_j,k_i]\rangle|^2\right) \\
&\le N\left(\sum_{j=1}^\infty\sum_{\ell=1}^N |\langle [e_\ell,h_j],v\rangle|^2\right)\left(\sum_{i,j=1}^\infty\sum_{\ell=1}^N |\langle e_\ell,[h_j,k_i]\rangle|^2\right) \\
&\le N^2\|v\|^2\|[\cdot,\cdot]\|_2^2\cdot\|[\cdot,\cdot]\|_2^2.
\end{aligned}$$

Now assume $\langle \hat{F}_{n-1}^{\sigma,\alpha}, v\rangle$ is Hilbert-Schmidt for all $\sigma \in \mathcal{S}_{n-1}$ and $\alpha \in \mathcal{J}_{n-1}$, and consider $\langle \hat{F}_n^{\sigma,\alpha}, v\rangle$ for some $\sigma \in \mathcal{S}_n$ and $\alpha \in \mathcal{J}_n^m$. Let $a = p_\alpha$ and $b = q_\alpha$, and note that either $a \geq 1$ and

$$\begin{aligned}
&\hat{F}_n^{\sigma,\alpha}(k_1 \otimes \cdots \otimes k_a) \\
&\qquad = \sum_{j_1,\dots,j_b=1}^\infty F_n^{\sigma'}(k_1 \otimes \cdots \otimes k_a \otimes h_{j_1} \otimes h_{j_1} \otimes \cdots \otimes h_{j_b} \otimes h_{j_b}) \\
&\qquad = \sum_{j_1,\dots,j_b=1}^\infty [F_{n-1}^{\sigma''}(k_1 \otimes \cdots \otimes k_{d-1} \otimes k_{d+1} \otimes \cdots \otimes k_a \otimes h_{j_1} \otimes \cdots \otimes h_{j_b}), k_d] \\
&\qquad = [\hat{F}_{n-1}^{\tau,\beta}(k_1 \otimes \cdots \otimes k_{d-1} \otimes k_{d+1} \otimes \cdots \otimes k_a), k_d], \qquad (4.13)
\end{aligned}$$

for some $d \in \{1, \dots, a\}$, $\sigma'', \tau \in \mathcal{S}_{n-1}$, and $\beta \in \mathcal{J}_{n-1}^{m-1}$ such that $p_\beta = p_\alpha - 1$ and $q_\beta = q_\alpha$, or $b \geq 1$ and

$$\begin{aligned}
&\hat{F}_n^{\sigma,\alpha}(k_1 \otimes \cdots \otimes k_a) \\
&= \sum_{j_1,\dots,j_b=1}^\infty [F_{n-1}^{\sigma''}(k_1 \otimes \cdots \otimes k_a \otimes h_{j_1} \otimes \cdots \otimes h_{j_{d-1}} \otimes h_{j_d} \otimes h_{j_{d+1}} \otimes \cdots \otimes h_{j_b}), h_{j_d}] \\
&= \sum_{j_d=1}^\infty [\hat{F}_{n-1}^{\tau,\beta}(k_1 \otimes \cdots \otimes k_a \otimes h_{j_d}), h_{j_d}], \qquad (4.14)
\end{aligned}$$

for some $d \in \{1, \dots, b\}$, $\sigma'', \tau \in \mathcal{S}_{n-1}$ and $\beta \in \mathcal{J}_{n-1}^m$ such that $p_\beta = p_\alpha + 1$ and $q_\beta = q_\alpha - 1$. In the first case, working as above for $n = 3$,

$$\begin{aligned}
\left\| \langle \hat{F}_n^{\sigma,\alpha}, v\rangle \right\|_2^2 &= \sum_{i_1,\dots,i_a=1}^\infty \left| \sum_{j_1,\dots,j_b=1}^\infty \langle F_n^{\sigma'}(k_{i_1} \otimes \cdots \otimes k_{i_a} \otimes h_{j_1} \otimes \cdots \otimes h_{j_b}), v\rangle \right|^2 \\
&= \sum_{i_1,\dots,i_a=1}^\infty \left| \sum_{j_1,\dots,j_b=1}^\infty \langle [F_{n-1}^{\sigma''}(k_{i_1} \otimes \cdots \otimes h_{j_b}), k_{i_d}], v\rangle \right|^2 \\
&\leq N \sum_{i_1,\dots,i_a=1}^\infty \sum_{\ell=1}^N \left| \sum_{j_1,\dots,j_b=1}^\infty \langle F_{n-1}^{\sigma''}(k_{i_1} \otimes \cdots \otimes h_{j_b}), e_\ell\rangle \langle [e_\ell, k_{i_d}], v\rangle \right|^2 \\
&= N \sum_{i_1,\dots,i_a=1}^\infty \sum_{\ell=1}^N |\langle [e_\ell, k_{i_d}], v\rangle|^2 \left| \sum_{j_1,\dots,j_b=1}^\infty \langle F_{n-1}^{\sigma''}(k_{i_1} \otimes \cdots \otimes h_{j_b}), e_\ell\rangle \right|^2 \\
&\leq N\|v\|^2 \|[\cdot,\cdot]\|_2^2 \sum_{\ell=1}^N \left\| \langle \hat{F}_{n-1}^{\tau,\beta}, e_\ell\rangle \right\|_2^2,
\end{aligned}$$

which is finite by the induction hypothesis. Similarly, in the second case

$$\left\|\langle \hat{F}_n^{\sigma,\alpha}, v\rangle\right\|_2^2 = \sum_{i_1,\ldots,i_a=1}^{\infty} \left| \sum_{j_1,\ldots,j_b=1}^{\infty} \langle [F_{n-1}^{\sigma''}(k_{i_1}\otimes\cdots\otimes h_{j_b}), h_{j_d}], v\rangle \right|^2$$

$$\leq N \sum_{i_1,\ldots,i_a=1}^{\infty} \sum_{\ell=1}^{N} \left| \sum_{j_1,\ldots,j_b=1}^{\infty} \langle F_{n-1}^{\sigma''}(k_{i_1}\otimes\cdots\otimes h_{j_b}), e_\ell\rangle \langle [e_\ell, h_{j_d}], v\rangle \right|^2$$

$$\leq N \left(\sum_{i_1,\ldots,i_a=1}^{\infty} \sum_{\ell=1}^{N} \sum_{j_d=1}^{\infty} \left| \sum_{j_1,\ldots,j_{d-1},j_{d+1},\ldots,j_b=1}^{\infty} \langle F_{n-1}^{\sigma''}(k_{i_1}\otimes\cdots\otimes h_{j_b}), e_\ell\rangle \right|^2 \right)$$

$$\times \left(\sum_{\ell=1}^{N} \sum_{j_d=1}^{\infty} |\langle [e_\ell, h_{j_d}], v\rangle|^2 \right)$$

$$\leq N \sum_{\ell=1}^{N} \left\| \langle \hat{F}_{n-1}^{\tau,\beta}, e_\ell\rangle \right\|_2^2 \cdot \|v\|^2 \|[\cdot,\cdot]\|_2^2.$$

■

Proposition 4.4. *Let $n \in \{2,\ldots,r\}$, $\sigma \in \mathcal{S}_n$, and $\alpha \in \mathcal{J}_n$. Then $\hat{F}_n^{\sigma,\alpha} : \mathfrak{g}_{CM}^{\otimes p_\alpha} \to \mathfrak{v}$ is Hilbert-Schmidt.*

Proof. This proof is analogous to that of Lemma 4.3. For $\hat{F}_n^{\sigma,\alpha}$ as in equation (4.14), we have

$$\|\hat{F}_n^{\sigma,\alpha}\|_2^2 = \sum_{i_1,\ldots,i_a=1}^{\infty} \left\| \sum_{j_1,\ldots,j_b=1}^{\infty} [F_{n-1}^{\sigma''}(k_{i_1}\otimes\cdots\otimes h_{j_b}), h_{j_d}] \right\|^2$$

$$\leq N \sum_{i_1,\ldots,i_a=1}^{\infty} \sum_{\ell=1}^{N} \left\| \sum_{j_1,\ldots,j_b=1}^{\infty} \langle F_{n-1}^{\sigma''}(k_{i_1}\otimes\cdots\otimes h_{j_b}), e_\ell\rangle [e_\ell, h_{j_d}] \right\|^2$$

$$\leq N \left(\sum_{\ell=1}^{\infty} \sum_{j_\ell=1}^{\infty} \|[e_\ell, h_{j_d}]\|^2 \right)$$

$$\times \left(\sum_{i_1,\ldots,i_a=1}^{\infty} \sum_{\ell=1}^{N} \sum_{j_d=1}^{\infty} \left| \sum_{j_1,\ldots,j_{d-1},j_{d+1},\ldots,j_b=1}^{\infty} \langle F_{n-1}^{\sigma''}(k_{i_1}\otimes\cdots\otimes h_{j_b}), e_\ell\rangle \right|^2 \right)$$

$$\leq N \|[\cdot,\cdot]\|_2^2 \sum_{d=1}^{N} \left\| \langle \hat{F}_{n-1}^{\tau,\beta}, e_\ell\rangle \right\|_2^2,$$

which is finite by Corollary 3.13 and Lemma 4.3. In a similar way, one may show in the second case that the same estimate holds for $\hat{F}_n^{\sigma,\alpha}$ as in equation (4.13). ■

Remark 4.5. The proofs of the previous propositions rely strongly on $\mathfrak{v}$ being finite dimensional. Thus, if we wished to extend the results of this paper to $\mathfrak{v}$ an infinite dimensional Lie algebra, another proof would be required here, or more likely, some trace class requirements on the Lie bracket of $\mathfrak{g}$.

Proposition 4.4 allows us to make the following definition.

Definition 4.6. A *Brownian motion* on G is the continuous G-valued process defined by

$$g_t = \sum_{n=1}^{r} \sum_{\sigma\in\mathcal{S}_n} \sum_{m=\lceil n/2\rceil}^{n} \frac{c_n^\sigma}{2^{n-m}} \sum_{\alpha\in\mathcal{J}_n^m} \int_{\Delta_{p_\alpha}(t)} f_\alpha(s,t)\hat{F}_n^{\sigma,\alpha}(dB_{s_1}\otimes\cdots\otimes dB_{s_{p_\alpha}}),$$

where

$$c_n^\sigma = (-1)^{e(\sigma)} \Big/ n^2 \begin{bmatrix} n-1 \\ e(\sigma) \end{bmatrix},$$

$\hat{F}_n^{\sigma,\alpha}$ is as defined in equation (4.10), and f_α is as defined below equation (4.11). For $t>0$, let $\nu_t = \mathrm{Law}(g_t)$ be the *heat kernel measure at time* t, a probability measure on G.

Example 4.7 (The step 3 case). Suppose that $\mathfrak{g}$ is nilpotent of step 3. Then

$$\begin{aligned} g_t &= \sum_{n=1}^{3} \sum_{\sigma\in\mathcal{S}_n} c_n^\sigma F_n^\sigma(M_n(t)) \\ &= \sum_{n=1}^{3} \sum_{\sigma\in\mathcal{S}_n} \sum_{m=\lceil n/2\rceil}^{n} \frac{c_n^\sigma}{2^{n-m}} \sum_{\alpha\in\mathcal{J}_n^m} F_n^\sigma(I_t^n(\alpha)) \\ &= \sum_{n=1}^{3} \sum_{\sigma\in\mathcal{S}_n} \sum_{m=\lceil n/2\rceil}^{n} \frac{c_n^\sigma}{2^{n-m}} \sum_{\alpha\in\mathcal{J}_n^m} \int_{\Delta_{p_\alpha}(t)} f_\alpha(s,t)\hat{F}_n^{\sigma,\alpha}(dB_{s_1}\otimes\cdots\otimes dB_{s_{p_\alpha}}). \end{aligned}$$

For $n=1$, there is the single term given by

$$M_1(t) = \int_0^t \delta B_s = B_t.$$

For $n=2$, $\mathcal{J}_2 = \{(1,1),(2)\}$, and so

$$\begin{aligned} M_2(t) &= I_t^2((1,1)) + \frac{1}{2} I_t^2((2)) \\ &= \int_{\Delta_2(t)} dB_{s_1}\otimes dB_{s_2} + \frac{1}{2}\int_0^t h_i\otimes h_i\, ds_2 \\ &= \int_{\Delta_2(t)} dB_{s_1}\otimes dB_{s_2} + \frac{1}{2} t \sum_{i=1}^{\infty} h_i\otimes h_i. \end{aligned}$$

(Again, we use slightly heuristic computations to determine the correct form for the Brownian motion, but the integrals in the end are well-defined.) There are of course just two permutations: $\sigma=(12)$ with $e(\sigma)=0$ and $c_2^\sigma = \frac{1}{4}$, and $\tau=(21)$ with $e(\tau)=1$ and $c_2^\tau = -\frac{1}{4}$, and, by the antisymmetry of the Lie bracket,

$$\sum_{\sigma\in\mathcal{S}_2} c_2^\sigma F_2^\sigma(M_2(t)) = \frac{1}{4}[dB_{s_1}, dB_{s_2}] - \frac{1}{4}[dB_{s_2}, dB_{s_1}] = \frac{1}{2}[dB_{s_1}, dB_{s_2}].$$

For $n = 3$, the permutations are (123) with $e = 0$, (213), (132), (312), (231) with $e = 1$, and (321) with $e = 2$. Thus,

$$\sum_{\sigma\in\mathcal{S}_3} c_3^\sigma F_3^\sigma(k_1\otimes k_2\otimes k_3) = \frac{1}{9}[[k_1,k_2],k_3] - \frac{1}{18}[[k_2,k_1,],k_3] - \frac{1}{18}[[k_1,k_3],k_2]$$
$$- \frac{1}{18}[[k_3,k_1],k_2] - \frac{1}{18}[[k_2,k_3],k_1] + \frac{1}{9}[[k_3,k_2,],k_1]$$
$$= \frac{1}{6}[[k_1,k_2],k_3] + \frac{1}{6}[[k_3,k_2,],k_1]. \tag{4.15}$$

Also, $\mathcal{J}_3 = \{(1,1,1),(1,2),(2,1)\}$, and so

$$\begin{aligned} M_3(t) &= I_t^3((1,1,1)) + \frac{1}{2}I_t^3((1,2)) + \frac{1}{2}I_t^3((2,1)) \\ &= \int_{\Delta_3(t)} dB_{s_1}\otimes dB_{s_2}\otimes dB_{s_3} + \frac{1}{2}\int_{\Delta_2(t)}\sum_{i=1}^\infty dB_{s_1}\otimes h_i\otimes h_i ds_3 \\ &\quad + \frac{1}{2}\int_0^t \sum_{i=1}^\infty s_3 h_i\otimes h_i\otimes dB_{s_3} \\ &= \int_{\Delta_3(t)} dB_{s_1}\otimes dB_{s_2}\otimes dB_{s_3} + \frac{1}{2}\int_0^t\sum_{i=1}^\infty (t-s_1) dB_{s_1}\otimes h_i\otimes h_i \\ &\quad + \frac{1}{2}\int_0^t \sum_{i=1}^\infty s_3 h_i\otimes h_i\otimes dB_{s_3}. \end{aligned}$$

Note that $f_{(1,2)}(s,t) = t - s_1$ and $f_{(2,1)}(s,t) = s_3$. Plugging this into equation (4.15) gives, for the $\alpha = (1,1,1)\in\mathcal{J}_3^3$ term,

$$\begin{aligned} \sum_{\sigma\in\mathcal{S}_3} c_3^\sigma F_3^\sigma(I_t^3((1,1,1))) &= \sum_{\sigma\in\mathcal{S}_3} c_3^\sigma \int_{\Delta_3(t)} F_3^\sigma(dB_{s_1}\otimes dB_{s_2}\otimes dB_{s_3}) \\ &= \frac{1}{6}\int_{\Delta_3(t)} ([[dB_{s_1},dB_{s_2}],dB_{s_3}] + [[dB_{s_3},dB_{s_2}],dB_{s_1}]). \end{aligned}$$

For $\alpha = (1,2)\in\mathcal{J}_3^2$,

$$\sum_{\sigma\in\mathcal{S}_3} c_3^\sigma F_3^\sigma(I_t(1,2)) = \frac{1}{6}\int_0^t \sum_{i=1}^\infty (t-s_1)[[dB_{s_1},h_i],h_i],$$

and

$$\hat{F}_3^{\sigma,(1,2)}(k) = \sum_{i=1}^\infty F_3^\sigma(k\otimes h_i\otimes h_i)$$

with $\sigma' = \sigma$. For $\alpha = (2,1)\in\mathcal{J}_3^2$,

$$\sum_{\sigma\in\mathcal{S}_3} c_3^\sigma F_3^\sigma(I_t((2,1))) = \frac{1}{6}\int_0^t \sum_{i=1}^\infty s_3[[dB_{s_3},h_i],h_i],$$

and note that, in this case,

$$\hat{F}_3^{\sigma,(2,1)}(k) = \sum_{i=1}^\infty F_3^{\sigma'}(k\otimes h_i\otimes h_i) = \sum_{i=1}^\infty F_3^\sigma(h_i\otimes h_i\otimes k),$$

and so $\sigma' = \sigma\circ(231)$ (or $\sigma' = \sigma\circ(321)$). Combining the above, Brownian motion on G may be written as

$$\begin{aligned} g_t &= B_t + \frac{1}{2}\int_{\Delta_2(t)} [dB_{s_1}, dB_{s_2}] \\ &\quad + \frac{1}{12}\int_{\Delta_3(t)} ([[dB_{s_1}, dB_{s_2}], dB_{s_3}] + [[dB_{s_3}, dB_{s_2}], dB_{s_1}]) \\ &\quad + \frac{1}{24}\sum_{i=1}^\infty \int_0^t ((t-s)[[dB_s, h_i], h_i] + s[[dB_s, h_i], h_i]) \\ &= B_t + \frac{1}{2}\int_0^t [B_s, dB_s] + \frac{1}{12}\int_{\Delta_2(t)} ([[B_{s_1}, dB_{s_1}], dB_{s_2}] + [[dB_{s_2}, dB_{s_1}], B_{s_1}]) \\ &\quad + \frac{1}{24}\sum_{i=1}^\infty t[[B_t, h_i], h_i]. \end{aligned}$$

Remark 4.8. In principle, the Brownian motion on G has generator

$$\Delta = \sum_{i=1}^\infty \tilde{h}_i^2,$$

where $\{h_i\}_{i=1}^\infty$ is an orthonormal basis of $\mathfrak{g}_{CM} = H\oplus\mathfrak{v}$ and $\tilde{h}$ is the unique left invariant vector field on G such that $\tilde{h}(\mathbf{e}) = h$, and Δ is well-defined independent of the choice of orthonormal basis. Then the heat kernel measure $\{\nu_t\}_{t>0}$ has the standard characterization as the unique family of probability measures such that $\nu_t(f) := \int_G f\,d\nu_t$ is continuously differentiable in t for all $f\in C_b^2(G)$ and satisfies

$$\frac{d}{dt}\nu_t(f) = \frac{1}{2}\nu_t(\Delta f) \quad \text{with } \lim_{t\downarrow 0}\nu_t(f) = f(e).$$

However, this realization of ν_t is not necessary for our results.

Proposition 4.9 (Finite dimensional approximations)**.** *For $P\in\mathrm{Proj}(W)$, let g_t^P be the continuous process on G_P defined by*

$$g_t^P = \sum_{n=1}^r \sum_{\sigma\in\mathcal{S}_n} \sum_{m=\lceil n/2\rceil}^n \frac{c_n^\sigma}{2^{n-m}} \sum_{\alpha\in\mathcal{J}_n^m} \int_{\Delta_{p_\alpha}(t)} f_\alpha(s,t)\hat{F}_n^{\sigma,\alpha}(d\pi B_{s_1}\otimes\cdots\otimes d\pi B_{s_{p_\alpha}}),$$

for $\pi(w,x) = (Pw,x)$. Then g_t^P is Brownian motion on G_P. In particular, let $g_t^\ell = g_t^{P_\ell}$, for projections $\{P_\ell\}_{\ell=1}^\infty\subset\mathrm{Proj}(W)$ as in equation (4.4). Then, for all $p\in[1,\infty)$ and $t<\infty$,

$$\lim_{\ell\to\infty}\mathbb{E}\left[\sup_{\tau\le t}\left\|g_\tau^\ell - g_\tau\right\|_{\mathfrak{g}}^p\right] = 0. \tag{4.16}$$

Proof. First note that g_t^P solves the Stratonovich equation $\delta g_t^P = L_{g_t^P *}\delta PB_t$ with $g_0^P = \mathbf{e}$, see [4, 8, 3]. Thus, g_t^P is a G_P-valued Brownian motion.

Now, if β_t a Brownian motion on W, then, for all $p\in[1,\infty)$,

$$\lim_{\ell\to\infty}\mathbb{E}\left[\sup_{\tau\le t}\|P_\ell\beta_\tau - \beta_\tau\|_W^p\right] = 0;$$

see, for example, Proposition 4.6 of [10]. Thus,

$$\lim_{\ell\to\infty}\mathbb{E}\left[\sup_{\tau\le t}\|\pi_\ell B_\tau - B_\tau\|_{\mathfrak{g}}^p\right]=0.$$

By equation (4.12) and its preceding discussion,

$$g_t^\ell=\sum_{n=1}^r\sum_{\sigma\in\mathcal{S}_n}\sum_{m=\lceil n/2\rceil}^n\frac{c_n^\sigma}{2^{n-m}}\sum_{\alpha\in\mathcal{J}_n^m}\sum_{a=0}^{q_\alpha}b_\alpha^a t^a J_n^\ell(\tilde{f}_\alpha\hat{F}_n^{\sigma,\alpha})_t,$$

and thus, to verify (4.16), it suffices to show that, for all $p\in[1,\infty)$,

$$\lim_{\ell\to\infty}\mathbb{E}\left[\sup_{\tau\le t}\left\|J_n^\ell(\tilde{f}_\alpha\hat{F}_n^{\sigma,\alpha})_\tau - J_n(\tilde{f}_\alpha\hat{F}_n^{\sigma,\alpha})_\tau\right\|_{\mathfrak{v}}^p\right]=0,$$

for all $n\in\{2,\dots,r\}$, $\sigma\in\mathcal{S}_n$ and $\alpha\in\mathcal{J}_n$. By Proposition 4.4, $\hat{F}_n^{\sigma,\alpha}$ is Hilbert-Schmidt, and recall that $\tilde{f}_\alpha$ is a deterministic polynomial function in s. Thus $J_n^\ell(\tilde{f}_\alpha\hat{F}_n^{\sigma,\alpha})$ and $J_n(\tilde{f}_\alpha\hat{F}_n^{\sigma,\alpha})$ are $\mathfrak{v}$-valued martingales as defined in Corollary 4.2. So, by Doob's maximal inequality, it suffices to show that

$$\lim_{\ell\to\infty}\mathbb{E}\left\|J_n^\ell(\tilde{f}_\alpha\hat{F}_n^{\sigma,\alpha})_t - J_n(\tilde{f}_\alpha\hat{F}_n^{\sigma,\alpha})_t\right\|_{\mathfrak{v}}^p=0.$$

Corollary 4.2 gives the limit for $p=2$. For $p>2$, since each $J_n^\ell(\tilde{f}_\alpha\hat{F}_n^{\sigma,\alpha})$ and $J_n(\tilde{f}_\alpha\hat{F}_n^{\sigma,\alpha})$ has chaos expansion terminating at degree n, a theorem of Nelson (see Lemma 2 of [23] and pp. 216-217 of [22]) implies that, for each $j\in\mathbb{N}$, there exists $c_j<\infty$ such that

$$\mathbb{E}\left\|J_n^\ell(\tilde{f}_\alpha\hat{F}_n^{\sigma,\alpha})_t - J_n(\tilde{f}_\alpha\hat{F}_n^{\sigma,\alpha})_t\right\|_{\mathfrak{v}}^{2j}\le c_j\left(\mathbb{E}\left\|J_n^\ell(\tilde{f}_\alpha\hat{F}_n^{\sigma,\alpha})_t - J_n(\tilde{f}_\alpha\hat{F}_n^{\sigma,\alpha})_t\right\|_{\mathfrak{v}}^2\right)^j.$$

■

5. Heat kernel measure

We collect here some properties of the heat kernel measure on G. The following two propositions are completely analogous to Corollary 4.9 of [10] and Proposition 4.6 in [12]. The proofs are included here for the convenience of the reader.

Proposition 5.1. *For any $t>0$, the heat kernel measure ν_t is invariant under the inversion map $g\mapsto g^{-1}$ for any $g\in G$.*

Proof. The heat kernel measures $\nu_t^{P_n}=\mathrm{Law}(g_t^n)$ on the finite dimensional groups G_{P_n} are invariant under inversion (see, for example, [13]). Suppose that $f: G\to\mathbb{R}$ is a bounded continuous function. By passing to a subsequence if necessary, we may assume that the sequence of G_{P_n}-valued random variables $\{g_t^n\}_{n=1}^\infty$ in Proposition 4.9 converges almost surely to g_t. Thus, by dominated convergence,

$$\mathbb{E}\left[f\left(g_t^{-1}\right)\right]=\lim_{n\to\infty}\mathbb{E}\left[f\left((g_t^n)^{-1}\right)\right]=\lim_{n\to\infty}\mathbb{E}\left[f\left(g_t^n\right)\right]=\mathbb{E}\left[f\left(g_t\right)\right].$$

Since ν_t is the law of g_t, this completes the proof. ■

Proposition 5.2. *For all $t>0$, $\nu_t(G_{CM})=0$.*

Proof. Let μ_t denote Wiener measure on W with variance t. Then for a bounded measurable function f on $G=W\oplus\mathfrak{v}$ such that $f(w,x)=f(w)$,

$$\int_G f(w)\,d\nu_t(w,x)=\mathbb{E}[f(\beta_t)]=\int_W f(w)\,d\mu_t(w).$$

Let $\pi : W \times \mathfrak{v} \to W$ be the projection $\pi(w,x) = w$. Then $\pi_*\nu_t = \mu_t$, and thus

$$\nu_t(G_{CM}) = \nu_t\left(\pi^{-1}(H)\right) = \pi_*\nu_t(H) = \mu_t(H) = 0.$$

■

This proposition gives some justification to our calling G_{CM} the Cameron-Martin subgroup of G. In the next section, we further justify this by showing that a Cameron-Martin type quasi-invariance theorem holds for ν_t.

5.1. **Quasi-invariance and Radon-Nikodym derivative estimates.** The following theorem states that the heat kernel measure $\nu_t = \mathrm{Law}(g_t)$ is quasi-invariant under left and right translation by elements of G_{CM} and gives estimates for the Radon-Nikodym derivatives of the translated measures.

Theorem 5.3. *For all $h \in G_{CM}$ and $t > 0$, $\nu_t \circ L_h^{-1}$ and $\nu_t \circ R_h^{-1}$ are absolutely continuous with respect to ν_t. Let*

$$Z_h^l := \frac{d(\nu_t \circ L_h^{-1})}{d\nu_t} \qquad \text{and} \qquad Z_h^r := \frac{d(\nu_t \circ R_h^{-1})}{d\nu_t}$$

be the Radon-Nikodym derivatives, K be lower bound on the Ricci curvature of G as in Corollary 3.24, and

$$c(t) := \frac{t}{e^t - 1}, \qquad \text{for all } t \in \mathbb{R},$$

with the convention that $c(0) = 1$. Then, $Z_h^l, Z_h^r \in L^p(\nu_t)$ for all $p \in [1,\infty)$, and both satisfy the estimate

$$\|Z_h^*\|_{L^p(\nu_t)} \le \exp\left(\frac{c(Kt)(p-1)}{2t} d_{CM}^2(\mathbf{e},h)\right),$$

where $* = l$ or $* = r$.

Proof. As in [10], the proof of this theorem is an application of Theorem 7.3 and Corollary 7.4 in [11] on the quasi-invariance of heat kernel measures for inductive limits of finite dimensional Lie groups. In applying these results, the reader should take $G_0 = G_{CM}$, $A = \mathrm{Proj}(W)$, $s_P = \pi_P$, $\nu_P = \mathrm{Law}(g_t^P)$, and $\nu = \nu_t = \mathrm{Law}(g_t)$. We now verify that the hypotheses of Theorem 7.3 of [11] are satisfied.

By Corollary 3.21, the inductive limit group $\cup_{P\in\mathrm{Proj}(W)} G_P$ is a dense subgroup of G_{CM}. By Proposition 4.9, for any $\{P_n\}_{n=1}^\infty \subset \mathrm{Proj}(W)$ with $P_n|_H \uparrow I_H$ and $f : G \to \mathbb{R}$ a bounded continuous function,

$$\int_G f\, d\nu_t = \lim_{n\to\infty} \int_{G_{P_n}} (f \circ i_{P_n})\, d\nu_t^{P_n}, \tag{5.1}$$

where i_{P_n} is the inclusion map, and thus the heat kernel measure is consistent on the finite dimensional projections. Corollary 3.24 says that $K > -\infty$ and $\mathrm{Ric}^P \ge K g^P$, for all $P \in \mathrm{Proj}(W)$, and thus the Ricci curvature is uniformly bounded on these projections. Lastly, the length of a path in the inductive limit group can be approximated by the lengths of paths in the finite dimensional projections. That is, for any $P_0 \in \mathrm{Proj}(W)$ and $\varphi \in C^1([0,1], G_{CM})$ with $\varphi(0) = \mathbf{e}$, there exists an increasing sequence $\{P_n\}_{n=1}^\infty \subset \mathrm{Proj}(W)$ such that $P_0 \subset P_n$, $P_n|_H \uparrow I_H$, and

$$\ell_{CM}(\varphi) = \lim_{n\to\infty} \ell_{G_{P_n}}(\pi_n \circ \varphi).$$

To see this, let $\varphi(t) = (A(t), a(t))$ be a path in G_{CM}, and recall that, by equation (3.7),

$$\ell_{G_{P_n}}(\pi_n \circ \varphi) = \int_0^1 \left\| \pi_n \varphi'(s) + \sum_{\ell=1}^{r-1} d_\ell \mathrm{ad}^\ell_{\pi_n\varphi(s)} \pi_n \varphi'(s) \right\|_{\mathfrak{g}_{CM}} ds$$
$$= \int_0^1 \sqrt{\|P_n A'(s)\|_H^2 + \left\| a'(s) + \sum_{\ell=1}^{r-1} d_\ell \mathrm{ad}^\ell_{\pi_n\varphi(s)} \pi_n \varphi'(s) \right\|_{\mathfrak{v}}^2} ds$$

Applying dominated convergence to this equation shows that (5.1) holds for any such choice of $P_n|_H \uparrow I_H$ such that $P_0 \subset P_n$. ■

We also get the usual strong converse to quasi-invariance of ν_t under translations by elements in G_{CM}.

Proposition 5.4. *For $h \in G \setminus G_{CM}$ and $t > 0$, $(\nu_t \circ L_h^{-1})$ and ν_t are singular and $(\nu_t \circ R_h^{-1})$ and ν_t are singular.*

Proof. Again, let μ_t denote Wiener measure on W with variance t. Let $h = (A, a) \in G \setminus G_{CM}$ with $A \in W \setminus H$ and $a \in \mathfrak{v}$. Given a measurable subset $U \subset W$,

$$\nu_t(U \times \mathfrak{v}) = P(\beta_t \in U) = \mu_t(U).$$

If $A \in W \setminus H$, $\mu_t(\cdot - A)$ and μ_t are singular; for example, see Corollary 2.5.3 of [6]. Thus, there are disjoint subsets W_0 and W_1 of W such that $\mu_t(W_0) = 1 = \mu_t(W_1 - A)$. Note that

$$L_k^{-1}(U \times \mathfrak{v}) = R_k^{-1}(U \times \mathfrak{v}) = (U - A) \times \mathfrak{v}.$$

Thus, for $G_i := W_i \times \mathfrak{v}$ for $i = 0, 1$, G is the disjoint union of G_0 and G_1, and $\nu_t(G_0) = \mu_t(W_0) = 1$ while

$$\nu_t\left(R_k^{-1}(G_1)\right) = \nu_t\left(L_k^{-1}(G_1)\right) = \nu_t((W_1 - A) \times \mathfrak{v}) = \mu_t(W_1 - A) = 1.$$

■

Proposition 5.5. *For all $h \in G_{CM}$ and $t > 0$, $Z_h^r(g) = Z_{h^{-1}}^l(g^{-1})$.*

Proof. By Proposition 5.1, ν_t is invariant under inversions. Thus

$$\int_G f(g \cdot h)\, d\nu_t(g) = \int_G f\left(g^{-1} \cdot h\right)\, d\nu_t(g) = \int_G f\left(\left(h^{-1} \cdot g\right)^{-1}\right)\, d\nu_t(g)$$
$$= \int_G f\left(g^{-1}\right) Z_{h^{-1}}^l(g)\, d\nu_t(g) = \int_G f(g) Z_{h^{-1}}^l\left(g^{-1}\right)\, d\nu_t(g).$$

■

5.2. **Logarithmic Sobolev inequality.**

Definition 5.6. A function $f : G \to \mathbb{R}$ is said to be a *(smooth) cylinder function* if $f = F \circ \pi_P$ for some $P \in \mathrm{Proj}(W)$ and some (smooth) function $F : G_P \to \mathbb{R}$. Also, f is a *cylinder polynomial* if $f = F \circ \pi_P$ for F a polynomial function on G_P.

Theorem 5.7. *Given a cylinder polynomial f on G, let $\nabla f : G \to \mathfrak{g}_{CM}$ be the gradient of f, the unique element of $\mathfrak{g}_{CM}$ such that*

$$\langle \nabla f(g), h \rangle_{\mathfrak{g}_{CM}} = \tilde{h} f(g) := f'(g)(L_{g*} h_{\mathbf{e}}),$$

for all $h \in \mathfrak{g}_{CM}$. Then for K as in Corollary 3.24,

$$\int_G (f^2 \ln f^2)\, d\nu_t - \left(\int_G f^2\, d\nu_t\right) \cdot \ln\left(\int_G f^2\, d\nu_t\right) \leq 2\frac{1-e^{-Kt}}{K} \int_G \|\nabla f\|^2_{\mathfrak{g}_{CM}}\, d\nu_t.$$

Proof. Following the method of Bakry and Ledoux applied to G_P (see Theorem 2.9 of [14] for the case needed here) shows that

$$\mathbb{E}\left[\left(f^2 \ln f^2\right)\left(g_t^P\right)\right] - \mathbb{E}\left[f^2\left(g_t^P\right)\right] \ln \mathbb{E}\left[f^2\left(g_t^P\right)\right] \leq 2\frac{1-e^{-K_P t}}{K_P} \mathbb{E}\left\|\left(\nabla^P f\right)\left(g_t^P\right)\right\|^2_{\mathfrak{g}^P_{CM}},$$

for K_P as in equation (3.10). Since the function $x \mapsto (1-e^{-x})/x$ is decreasing and $K \leq K_P$ for all $P \in \mathrm{Proj}(W)$, this estimate also holds with K_P replaced with K. Now applying Proposition 4.9 to pass to the limit as $P \uparrow I$ gives the desired result. ∎

Remark 5.8. It is desirable to state Theorem 5.7 for a larger class of functions in $L^2(\nu_t)$. To do this, one must prove that the gradient operator $\nabla : L^2(\nu_t) \to L^2(\nu_t) \otimes \mathfrak{g}_{CM}$ is closable. Unfortunately, Theorem 5.3 doesn't give good information on the dependence of the Radon-Nikodym derivatives Z_h^l and Z_h^r on h, and so at this point we can't prove the necessary integration by parts formulae to show closability.

References

[1] Hélène Airault and Paul Malliavin, *Quasi-invariance of Brownian measures on the group of circle homeomorphisms and infinite-dimensional Riemannian geometry*, J. Funct. Anal. **241** (2006), no. 1, 99–142. MR MR2264248 (2008b:60119)

[2] Dmitri Alekseevsky, Peter W. Michor, and Wolfgang A.F. Ruppert, *Extensions of Lie algebras*, 2000.

[3] Fabrice Baudoin, *An introduction to the geometry of stochastic flows*, Imperial College Press, London, 2004. MR MR2154760 (2006f:60003)

[4] Gérard Ben Arous, *Flots et séries de Taylor stochastiques*, Probab. Theory Related Fields **81** (1989), no. 1, 29–77. MR MR981567 (90a:60106)

[5] Arthur L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10, Springer-Verlag, Berlin, 1987. MR MR867684 (88f:53087)

[6] Vladimir I. Bogachev, *Gaussian measures*, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998. MR MR1642391 (2000a:60004)

[7] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999, With an appendix by David A. Buchsbaum, Reprint of the 1956 original. MR MR1731415 (2000h:18022)

[8] Fabienne Castell, *Asymptotic expansion of stochastic flows*, Probab. Theory Related Fields **96** (1993), no. 2, 225–239. MR MR1227033 (94g:60110)

[9] Lawrence J. Corwin and Frederick P. Greenleaf, *Representations of nilpotent Lie groups and their applications. Part I*, Cambridge Studies in Advanced Mathematics, vol. 18, Cambridge University Press, Cambridge, 1990, Basic theory and examples. MR MR1070979 (92b:22007)

[10] B. Driver and M. Gordina, *Heat kernel analysis on infinite-dimensional Heisenberg groups*, J. Funct. Anal. **255** (2008), no. 2, 2395–2461.

[11] ______, *Integrated Harnack inequalities on Lie groups*, Submitted, arXiv:0711.4392 (2008).

[12] ______, *Square integrable holomorphic functions on infinite-dimensional Heisenberg type groups*, to appear in Probab. Theory Related Fields (2008).

[13] Bruce K. Driver, *Integration by parts and quasi-invariance for heat kernel measures on loop groups*, J. Funct. Anal. **149** (1997), no. 2, 470–547. MR MR1472366 (99a:60054a)

[14] Bruce K. Driver and Terry Lohrenz, *Logarithmic Sobolev inequalities for pinned loop groups*, J. Funct. Anal. **140** (1996), no. 2, 381–448. MR MR1409043 (97h:58176)

[15] J. J. Duistermaat and J. A. C. Kolk, *Lie groups*, Universitext, Springer-Verlag, Berlin, 2000. MR MR1738431 (2001j:22008)

[16] D. B. Fuks, *Cohomology of infinite-dimensional Lie algebras*, Contemporary Soviet Mathematics, Consultants Bureau, New York, 1986, Translated from the Russian by A. B. Sosinskiĭ. MR MR874337 (88b:17001)

[17] Y. Z. Hu and P.-A. Meyer, *Chaos de Wiener et intégrale de Feynman*, Séminaire de Probabilités, XXII, Lecture Notes in Math., vol. 1321, Springer, Berlin, 1988, pp. 51–71. MR MR960508 (89m:60193)

[18] ———, *Sur les intégrales multiples de Stratonovitch*, Séminaire de Probabilités, XXII, Lecture Notes in Math., vol. 1321, Springer, Berlin, 1988, pp. 72–81. MR MR960509 (89k:60070)

[19] Yuzuru Inahama, *Logarithmic Sobolev inequality for H_0^s-metric on pinned loop groups*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **7** (2004), no. 1, 1–26. MR MR2021645 (2005d:58061)

[20] Kiyosi Itô, *Multiple Wiener integral*, J. Math. Soc. Japan **3** (1951), 157–169. MR MR0044064 (13,364a)

[21] Hui Hsiung Kuo, *Gaussian measures in Banach spaces*, Lecture Notes in Mathematics, Vol. 463, Springer-Verlag, Berlin, 1975. MR MR0461643 (57 #1628)

[22] Edward Nelson, *The free Markoff field*, J. Functional Analysis **12** (1973), 211–227. MR MR0343816 (49 #8556)

[23] ———, *Quantum fields and Markoff fields*, Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971), Amer. Math. Soc., Providence, R.I., 1973, pp. 413–420. MR MR0337206 (49 #1978)

[24] S. T. Sadètov, *On extensions of Lie algebras by the Heisenberg algebra*, Mat. Zametki **78** (2005), no. 5, 745–747. MR MR2252954 (2007e:17016)

[25] Ichiro Shigekawa, *Stochastic analysis*, Translations of Mathematical Monographs, vol. 224, American Mathematical Society, Providence, RI, 2004, Translated from the 1998 Japanese original by the author, Iwanami Series in Modern Mathematics. MR MR2060917 (2005k:60002)

[26] Robert S. Strichartz, *The Campbell-Baker-Hausdorff-Dynkin formula and solutions of differential equations*, J. Funct. Anal. **72** (1987), no. 2, 320–345. MR MR886816 (89b:22011)

[27] V. S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Graduate Texts in Mathematics, vol. 102, Springer-Verlag, New York, 1984, Reprint of the 1974 edition. MR 85e:22001

[28] Bill Yankosky, *On nilpotent extensions of Lie algebras*, Houston J. Math. **27** (2001), no. 4, 719–724. MR MR1874666 (2002k:17026)

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