The Integer Arithmetic of Legislative Dynamics*

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Abstract

Every legislature may be defined by a finite integer partition of legislative seats between parties and a winning vote quota defining the minimum coalition size required to pass decisions. In this paper we explore the finite set of integer partitions of legislatures, categorizing all legislatures as one of five basic types. As legislatures approach the partition thresholds between these categories, they approach the thresholds for changing the set of winning coalitions. The criteria defining such partitions of legislatures thus define important thresholds for shifting the legislature from one bargaining environment to another. These different bargaining environments will generate different bargaining expectations for at least some of the set of parties, which in turn may create incentives for defections of legislators from one party to another, in order to shift the legislature from one bargaining environment to another and thereby create a gain in bargaining expectations for the switching legislator and the party to which she defects. The paper proceeds in five parts. First, we develop a theoretical categorization of legislatures into those likely *a priori* to have different bargaining environments. Second, we analyze the role of a particular type of party that is especially privileged under particular integer partitions of a legislature, which we call a *k*-dominant party. Third, we examine the category frequencies and partition characteristics in a "metadata" set that contains to universe of all logically possible integer partitions of seats between up to ten parties in a 100-seat legislature. Fourth, we examine similar patterns in real legislatures, linking the categories to outcomes such as control of the prime minister and single-party minority government. Finally, we explore the extent to which all of this helps us to explain party switching by individual legislators.

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1. INTRODUCTION

The argument in this paper is driven by a very simple intuition. This is that every legislature is in one sense defined by an integer partition of a finite number of legislative sets between parties and a winning quota; winning any legislative vote requires the tacit or explicit assent a coalition of legislators whose number must exceed this quota. This remains true whether the legislative proposal voted on concerns the setting of public policy, the allocation of public offices, or indeed the distribution of pork. If some coalition of legislators is one vote short of the winning quota, then this is the same as being many votes far short of the quota, and very different indeed from having one extra vote and reaching the winning quota. This means that the set of integer partitions that describes the universe of possible legislatures is itself partitioned into categories. There are pairs of such partitions such that reallocating one vote from one party to another – which we call a "minimal integer repartition" – causes the legislature to move from one category to another, and by implication from one bargaining environment to another, since the set of winning coalitions is changed. The criteria defining such partitions of legislatures thus define important thresholds, whereby a minimal integer repartition shifts the legislature from one bargaining environment to another. These different bargaining environments will, almost by definition, generate different bargaining expectations for at least some of the set of parties (in a sense we make more precise below). This in turn may create incentives for defections of legislators from one party to another, to bring about the minimal integer repartition that shifts the legislature from one bargaining environment to another and thereby creates a gain in bargaining expectations for the party to which the legislator defects, which gain can be shared between the receiving party and the defector. In this sense, legislature close to category-defining thresholds should be more prone to defection and party-switching than legislators that are far away from them, for which no individual defection changes the parties' bargaining expectations. The overall aim is to identify legislatures that will be prone, in this sense, to party switching by legislators, and which parties are likely to attract such defections. We focus in this paper on minimal integer repartitions – defection by individual legislators – leaving for later work an analysis of the collective action problems involved in the defection *en masse* of coalitions, of factions, of legislators.

The argument proceeds in five parts. First, we develop a theoretical categorization of legislatures into those likely *a priori* to have different bargaining environments. Second, we analyze the role of a particular type of party that is especially privileged under particular integer partitions of a legislature – which we call a *k*-dominant party. Such parties are much more likely to have super-proportional constant-sum bargaining weight, as well as being much more likely to be median on any arbitrary policy dimension. Third, we look at the incidence of the various legislative categories, and of *k*-dominant parties, in a "metadata" set that contains to universe of all logically possible integer partitions of seats between up to ten parties in both 100-seat and 101-seat legislature. The number of these is large but not infinite and we have generated them all. Fourth, we look at the incidence of the various legislative categories, and of dominant parties in real legislatures, showing that our classification of legislatures very much helps us to explain circumstances in which the largest party controls the position of prime minister, and in which the largest party is able to government alone as a one-party minority government. Finally, we explore the extent to which all of this helps us to explain party switching by individual legislators.

2. *A PRIORI* CLASSIFICATION OF LEGISLATURES USING SIMPLE INTEGER ARTIHMETIC

2.1 Notation and terminology

Any state of any legislature can be described as the partition of a constant *M* seats between *N* well-disciplined parties or factions. In this context a well-disciplined party or party faction can be defined as a set of legislators who are affiliated to the same party or faction and who can be expected – for reasons to do with the modeling of intra-party politics that we do not investigate here – to vote in the same way on each of the set of matters to be determined by the legislature. For simplicity in what follows, a "well-disciplined party or faction" is called a "party". Let the set of *N* parties be $\{P_1, P_2, \ldots P_n\}$ and without loss of generality order the parties by the number of seats each party controls. Write a coalition between P_x and P_y as P_xP_y. Let the number of seats controlled by Party *i* be S_i. Thus $S_1 \geq S_2 \geq \ldots \geq S_n$. Let the winning quota for making legislative decisions, that is the number of legislative votes needed to pass a decision, be *W*. All quantities $\{M, W, S_1, S_2, \ldots S_n\}$ are integers. Require the winning quota to be decisive in the sense that, if some coalition, *C*, of legislators is winning then its complement, *C'*, is losing.[1] The winning quota must thus be a

[1] If the winning threshold is not decisive in this sense, as it always is in actual legislatures, then two utterly contradictory legislative motions could be passed at the same time.

least a simple majority of legislators ($W > M/2$), though most of what follows does not constrain W to be a simple majority.

2.2 Legislatures with a single winning party

Different partitions of M seats between N parties in a legislature with a winning threshold W create different decision-making environments, which can be described in terms of the sizes of the largest parties in the system relative to W. The best known of these environments arises when $S_1 \geq W$. Since W is decisive, $S_2 < W$. There is a single winning party, which must be the largest, and this party can control all legislative decisions.[2]

Since this legislative environment is rather well understood, we do not consider it further in itself. It is nonetheless worth noting that, in a dynamic environment where legislators may switch parties or form new parties – in other words when the party affiliation of legislators is endogenous – a legislature controlled by a single winning party may not be in steady state. For the most part in what follows, however, we explore decision-making in legislatures for which $S_1 < W$. In such legislatures, a *coalition* of two or more parties is required to assemble the votes needed to achieve the winning quota.

2.3 Legislatures with a strongly dominant party

Even in a legislature for which $S_1 < W$, legislative arithmetic may put the largest party in a privileged position. One concept from the literature on co-operative game theory that has already attracted some attention and characterizes this privileged position is that of a "dominant party". A dominant party is defined as a party P_d such that there is at least one pair of mutually exclusive losing coalitions excluding P_d, each of which P_d can join to make winning, but which cannot combine with each other to form a winning coalition. The intuition is that a dominant party can play off the two losing coalitions against each other while these coalitions, not being winning even if they join forces, cannot combine to put pressure on the dominant party. It has been shown that the dominant party must be the

[2] Even if there are only two parties it may not be the case that $S_1 \geq W$ if W is greater than a simple majority; if $S_1 < W$ in this context there will be no winning party.

largest party. (For this result, plus general definitions and discussion of the dominant party, see Peleg, 1981; Einy, 1985; van Deemen, 1989; van Roozendaal, 1992).

Laver and Benoit (2003) show that dominant parties tend strongly to have higher (super-proportional) bargaining power, in the sense of being essential members of more winning coalitions and measured using the Shapley value, than non-dominant parties of the same size. They also showed analytically that, for any dominant P_1, it must be true that $S_1 \geq W/2$.[3] This is the first of a number of results that focus our attention on $W/2$ in addition to W.

The definition of a dominant party refers to mutually exclusive losing *coalitions* made winning by adding the largest party, P_1, but the intuition is much more striking if we think in terms of individual losing *parties*. Define a "*strongly* dominant" party, P_s, as one for which there exist two other parties P_i and P_j for which $S_s + S_i \geq W$ and $S_s + S_j \geq W$ and $S_i + S_j < W$. Thus a strongly dominant party is one made dominant by joining with losing *parties* to form winning coalitions, as opposed to joining with losing *coalitions* of parties. Define a "*weakly* dominant" party as a party that is dominant but not strongly dominant. In other words a weakly dominant party is a party made dominant only by being able to join mutually exclusive losing coalitions of parties to make these winning.

The simple integer arithmetic of legislative decision-making allows us to infer quite a lot about a legislature with a strongly dominant party. First, note that since any party can be described as a coalition of one party, any strongly dominant party also satisfies the conditions for being a dominant party and has the characteristics of a dominant party. Hence a strongly dominant party must be the largest party, P_1. Furthermore, for any strongly dominant P_1, it must be that $S_1 \geq W/2$**.**

Second, note that if two non-winning parties, P_i and P_j, ranked by size, render P_1 strongly dominant because $S_1 + S_i \geq W$ and $S_1 + S_j \geq W$ and $S_i + S_j < W$, then P_2 and P_3 also render P_1 strongly dominant. Since $S_2 \geq S_3 \geq S_i \geq S_j$, if the first two conditions strong dominance hold for S_i and S_j, they hold *a fortiori* for S_2 and S_3. To see that the third condition also holds, note that if P_1P_j is winning then its complement (P_1P_j)' is losing. For any $j > 3$, P_2P_3 is a subset of (P_1Pj)' and thus $S_2 + S_3 < W$. Thus, if the defining inequalities of strong dominance are fulfilled for any P_1, P_i and P_j, they are fulfilled for P_1,

[3] Consider a pair of mutually exclusive losing coalitions, $\{C\ C^*\}$, each of which excludes P_1 but can be made winning by adding P_1. P_1 is dominant by definition *iff* $S_c + S_{c^*} <$ W and $S_1 + S_c \geq W$ and $S_1 + S_{c^*} \geq W$. Imagine $S_1 < W/2$. This implies $S_c > W/2$ and $S_{c^*} > W/2$. This implies $S_c + S_{c'} > W$. Contradiction. It must be that $S_1 \geq W/2$ if P_1 is dominant.

P_2 and P_3. In other words, the inequalities $S_1 + S_2 \geq W$ and $S_1 + S_3 \geq W$ and $S_2 + S_3 < W$ are necessary and sufficient conditions for P_1 to be strongly dominant. ***The sizes of the three largest parties determine whether any party is strongly dominant.***

Third, note that P_1 strongly dominant implies a constraint on the size of S_3. Since $S_2 + S_3 < W$ and $S_2 \geq S_3$, ***if P_1 is strongly dominant, then $S_3 < W/2$***. Indeed this is a necessary condition for P_1 to be strongly dominant. This is the second result that focuses our attention on $W/2$ in addition to W.

Fourth, note that ***P_1 strongly dominant implies that both P_2 and P_3 must be members of any winning coalition excluding P_1***. Since the coalition P_1P_2 is winning by definition of strong dominance, its complement $(P_1P_2)'$ is losing. Thus $(P_1P_2)'$ must add either P_1 or P_2 to become winning. If it excludes P_1 it must add P_2. Thus if P_1 is strongly dominant, any winning coalition excluding P_1 must include P_2. An identical argument applies to P_3. This highlights the special position of a strongly dominant P_1, since there are severe constraints on any coalition that excludes it.

Fifth, note that that ***P_1 strongly dominant implies that P_1 and only P_1 is a member of every two-party winning coalition***. Since the largest possible two-party coalition excluding P_1, which is P_2P_3, is losing, then every possible two-party coalition excluding P_1 is losing. This is another aspect of the privileged position of a strongly dominant P_1.

To summarize the argument so far, some simple arithmetical constraints give us strong intuitions about the distinguished position of a strongly dominant party. Any time a strongly dominant P_1 is excluded from a winning coalition, both P_2 and P_3 must be members of that winning coalition. But P_1 can form a winning coalition with either P_2 or P_3. Any two-party winning coalition must include a strongly dominant P_1. According to almost any conceivable model of legislative decision-making, P_1 can thus make offers to both P_2 and P_3, to induce them to break any of the potentially many winning coalitions from which P_1 is excluded, and these offers can be implemented by the winning coalitions P_1P_2 and P_1P_3 without recourse to any other party. The constrained legislative arithmetic means that no party other than P_1 can be in this privileged position. We shall see below that these results are empirically significant because legislatures with strongly dominant parties are actually rather common in the real world.

2.4 Legislatures with a *k*-dominant party

We can go beyond the conclusions in the previous section, however. There are increasing levels of strong dominance for the largest party. Having defined strong dominance in terms of the sizes of the top three parties, consider how far down the rank order of parties we can go, combining the largest party with party P_k such that $S_1 + S_k \geq W$, while $S_2 + S_3 < W$. The limiting case arises when the largest party can form a majority coalition with any of the other parties, while $S_2 + S_3 < W$.[4] Call such a party "system-dominant". In such a case, using a similar argument to that which showed that any winning coalition excluding a strongly dominant P_1 must include both P_2 and P_3, we can see that ***any winning coalition excluding a system-dominant party must include all other parties.*** Indeed, this is a necessary and sufficient condition for system dominance.

More generally, we can think of a largest party as being *k*-dominant if the smallest party with which it can form a majority coalition, subject to $S_2 + S_3 < W$, is P_k.[5] A strongly dominant P_1 must be at least *3*-dominant, being able to form winning coalitions with at least P_2 and P_3. A system dominant P_1 in an *n*-party system must be *n*-dominant, being able to form winning coalitions with each of the other parties. A straightforward extension of the proof that any winning coalition excluding a strongly dominant P_1 must include both P_2 and P_3 implies that ***any winning coalition excluding a k-dominant party must include all parties from P_2 to P_k.***[6]

2.5. Partitioning the universe of all possible legislatures

The definition of strong dominance and the distinctive decision-making environment generated when there is a strongly dominant P_1, depend on the sizes of the three largest parties relative to W. More generally, we can exclusively and exhaustively partition the universe of all possible legislatures using a set of inequalities describing the relative sizes of the three largest parties and W, and each cell in this partition seems likely to generate a different decision making environment. This partition is described using a dendrogam in Figure 1. Reading this from the top, the universe of possible legislatures can first be partitioned according to whether $S_1 \geq W$ or $S_1 < W$. In the former case, as we have already noted, there is a single winning P_1, in the latter there is not.

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[4] For example, in a 100-seat legislature with a simple majority rule, this would arise if the partition of seats between 6 parties was (40 , 12, 12, 12, 12, 12)

[5] That is, P_1 is *k*-dominant if $S_2 + S_3 < W$ and $S_1 + S_k \geq W$ and $S_1 + S_{k+1} < W$

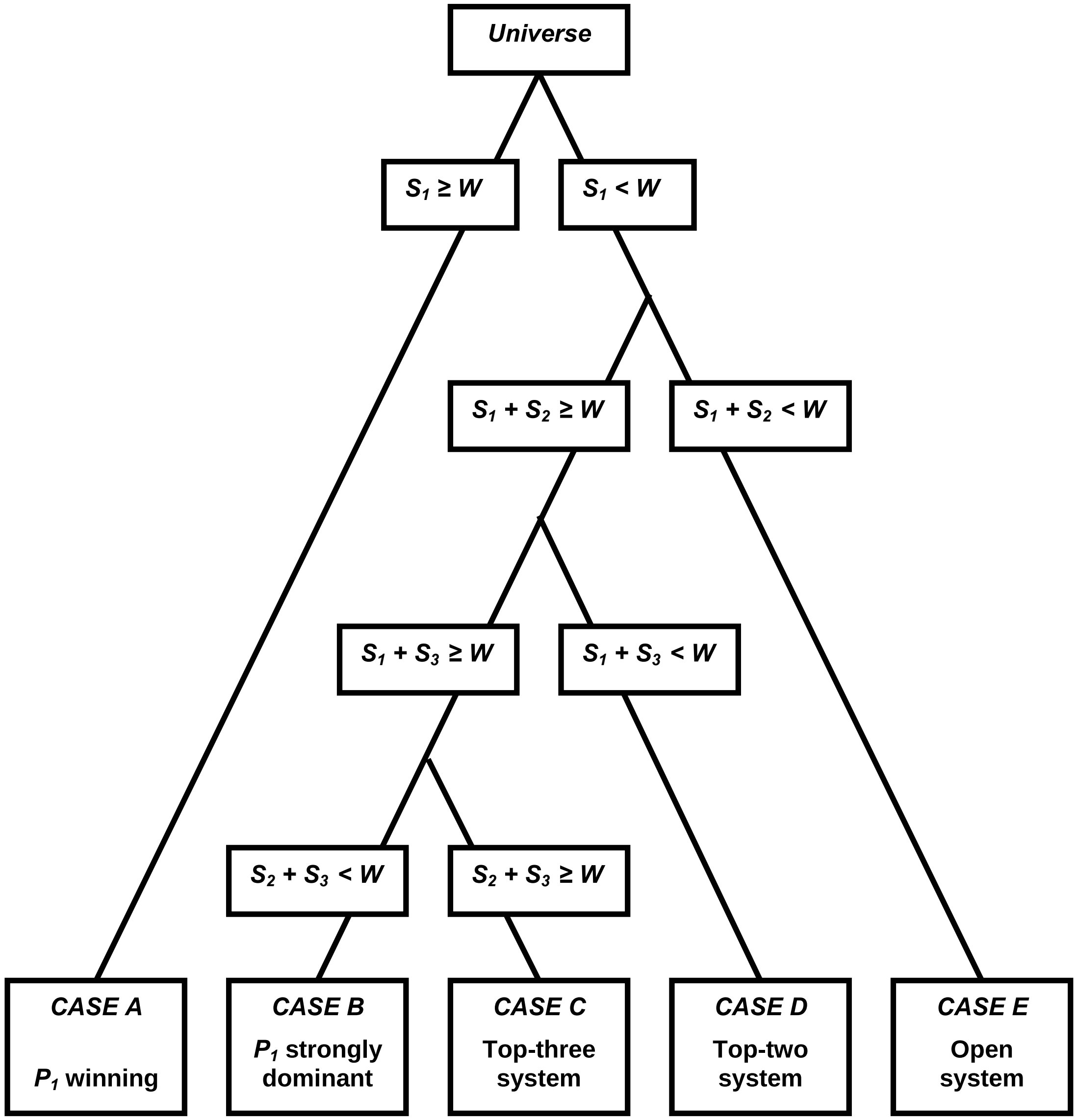
[6] Since P_1P_k is winning, $(P_1P_k)'$ is losing, thus $(P_1P_k)'$ must add either P_1 or P_k to be winning.

- The set of legislatures for which $S_1 < W$ can then be partitioned according to whether $S_1 + S_2 \geq W$ or $S_1 + S_2 < W$.

- The set of legislatures for which $S_1 < W$ and $S_1 + S_2 \geq W$ can then be further partitioned according to whether $S_1 + S_3 \geq W$ or $S_1 + S_3 < W$.

- The set of legislatures for which $S_1 < W$ and $S_1 + S_2 \geq W$ and $S_1 + S_3 \geq W$ can then be further partitioned according to whether $S_2 + S_3 \geq W$ or $S_2 + S_3 < W$.

Figure 1: A partition of the universe of all possible legislatures



The net result, shown at the bottom of Figure 1, is a set of five exclusive and exhaustive partitions of the universe of possible legislatures, described as Case A to Case E. Every possible legislature fits one of these five cases. Since the partitioning in Figure 1 always makes the leftmost of the two inequalities the one that is more favorable to P_1, the cases can be read from A to E in terms of the distinguished position of the largest party. As we have seen, Case A legislatures arise when there is a single winning P_1, while the inequalities defining Case B legislatures are those defining a strongly dominant P_1. As we will now see, each of the other three cases also defines a distinctive decision making environment.

2.6 Case C: "Top-three" systems

Case C legislatures arise when no one, but any two, of the three largest parties can form a winning coalition. We can thus think of these as "top-three" systems. Defining a "pivotal" party as a party that can turn at least one losing coalition into a winning coalition by joining it, a very striking feature of any top-three system is that no party outside the largest three is ever pivotal. In other words ***no party outside the top three in a Case C legislature ever makes the difference between winning and losing a legislative decision.***

This is because, by definition of a Case C legislature, ***any coalition excluding two of the top three parties is losing***. Thus if P_2P_3 is winning this implies that its complement, $(P_2P_3)'$, the coalition between P_1 and all parties outside the top three, is losing. Similarly, P_1P_3 winning implies $(P_1P_3)'$ losing, and P_1P_2 winning implies $(P_1P_2)'$ losing. Thus no party outside the top three can render winning a coalition *excluding* two of the top three parties, since every such coalition must be losing. Yet, by definition of the case, every coalition *including* two of the top three parties is winning regardless of the addition or subtraction of another party outside the top three. In a nutshell, in a top-three system, only the three largest parties have any impact at all on legislative decision-making since no other party is ever pivotal.

Note that $S_2 + S_3 \geq W$, the key inequality distinguishing Case C from Case B, implies that $S_2 \geq W/2$ since $S_2 \geq S_3$. ***Indeed $S_2 \geq W/2$ is a necessary condition for a top-three legislature.*** This is the third result focusing our attention on $W/2$.

Since, $S_1 + S_3 \geq W$, this in turn implies that $S_1 + S_2 + S_3 \geq 3W/2$. The top three parties must control one and a half times the winning threshold in a Case C legislature. Thus ***a top-three legislature can never arise when the winning quota is higher than two-thirds of total seats***.

Finally, note that there is no natural extension of the notion of a "top-three" legislature to that of a "top-four" legislature and beyond. Four parties cannot all find themselves in the same position in this sense at the top of the system. $S_2 + S_3 \geq W$ implies $S_1 + S_4 < W$. Thus no symmetrical set of inequalities can be constructed to define a "top four" system based on sets of two-party winning coalitions.[7]

2.7 Case D: "Top-two" systems

Case D legislatures arise when the P_1 and P_2 can form a majority coalition but P_1 and P_3 (hence P_2 and P_3) cannot. In this case, ***P_1P_2 is the only two-party winning coalition,*** since P_1P_3, the next-largest two-party coalition, is losing. This implies that in a Case D legislature ***one or other of the two largest parties is a member of every winning coalition***. Thus we can think of Case D as a "top-two" legislature. Since the one of the crucial inequalities defining a top-two legislature is that $S_1 + S_3 < W$, we know that it must be true that $S_3 < W/2$; indeed this is a necessary condition for a top-two legislature. This is the fourth result focusing our attention on $W/2$.

The salient feature of a top-two legislature is that one or both of the two largest parties is essential to any winning coalition, and that this is only true for the two largest parties. This does not however mean that the top two parties must have an equal role in parliamentary decision-making, since it is quite possible for $S_1 + S_3 + S_4 \geq W$ but $S_2 + S_3 + S_4 < W$, giving P_1 more options that P_2.[8] Nonetheless P_1 and P_2 are at the "top" of a Case D legislature in the sense that one or the other of them must be a part of every legislative majority, while they and only they can form a winning coalition between themselves that excludes all other parties.

[7] Note that the case in which there are four parties, all of equal weight, falls into Case E. Analogous symmetrical sets of inequalities can be constructed, for legislatures with more than five parties, involving sets of three-party winning coalitions comprising any three of the top five parties, etc.. These generate subsets of Case E.

[8] For example, in a 100-seat legislature with a simple majority rule, this would arise if the partition of seats between 6 parties was (35, 20, 13, 12, 10, 10). Thus there are subsets of Case D that somewhat privilege the largest party.

When the winning threshold W is a simple majority of the legislature, and thus $W-1 \leq M/2$, we note that ***there must be at least five parties in a top-two legislature***. Imagine there were three parties. If $S_1 < W$ then $S_2 < W$ and under the simple majority threshold $S_1 + S_3 \geq W$, contradicting the Case D inequalities. Imagine there were four parties. If $S_2 + S_3 < W$, then $S_1 + S_4 \geq W$ under a simple majority threshold. But by the definition of the case $S_1 + S_3 < W$. Hence $S_1 + S_4 < W$. Contradiction. It is easy to construct examples of five-party Case D legislatures for which W is a simple majority.[9] This result demonstrates that that any model of legislative decision-making using a simple majority decision rule that cannot accommodate a legislature with at least five parties does not cover all possible legislative cases.

2.8 Case E: "Open" systems

The defining characteristic of a Case E legislature is that a coalition between the two largest parties is losing and thus ***there is no winning two-party coalition***. Since the one of the crucial inequalities defining a Case E legislature is that $S_1 + S_2 < W$, we know that it must be true that $S_2 < W/2$; indeed this is a necessary condition for a Case E legislature and is the fifth and final result directing our attention to $W/2$. In such a legislature it is never possible for some party, P_x, that is excluded from some winning coalition to tempt a single pivotal member, P_p, of that coalition with an offer that can be implemented exclusively by P_x and P_p. This is because $P_x P_p$, as a two-party coalition, must be losing. In this case what is striking and distinctive is that even the largest party must deal with coalitions of other parties – ***and the potential collective action problems within such coalitions*** – in order to put together a winning coalition.[10] To go further in such a situation we would need a more explicit model of bargaining between parties and collective action within coalitions of parties. For this reason, we call a Case E legislature an "open" system. ***In every case other than an open system, if the largest party does not win on its own, it can win by forming a coalition with no more than one other party.***

2.9 *W/2* as a threshold between legislative types

[9] For example, in a 100-seat legislature with a simple majority rule, this would arise if the partition of seats between 5 parties was (35, 30, 13, 12, 10).
[10] Thus note that the largest party may be *weakly* dominant in an open system.

Putting together a number of the conclusions we have reached above we note, intriguingly, that *W/2* is an important threshold for individual party sizes when partitioning legislatures between cases. First, consider what must be true when $S_1 < W/2$. By definition, the legislature cannot be in Case A. We have also seen that it cannot be in Case B. In fact in cannot be in Cases B, C or D, all of which require $S_1 + S_2 \geq W$, impossible if $S_1 < W/2$. Thus ***if*** $S_1 < W/2$ ***then the legislature must be in Case E, an open legislature***. Indeed this is a sufficient, though not a necessary, condition for an open legislature.[11]

Second, consider what must be true when $S_3 \geq W/2$. This implies $S_2 \geq W/2$ and hence $S_2 + S_3 \geq W$ and $S_1 + S_3 \geq W$ and $S_1 + S_2 \geq W$ and $S_1 < W$. Thus we know that we cannot be, respectively, in Cases B, D, E or A. Since Cases A – E exclusively and exhaustively partition the universe, ***if*** $S_3 \geq W/2$***, then the legislature must be a Case C "top-three" system***. This is a sufficient, though clearly not a necessary, condition for a top-three legislature.[12]

Thus the individual sizes of both the largest and third-largest parties, relative to *W/2*, act as significant thresholds at the boundaries between legislative types. If the largest party drops below *W/2* in size, we are certain to be in an open system, while the third largest party being larger than *W/2* guarantees that we are in a top-three system. The impact of the size of the second largest party is less striking, although we know that we are not in a top-three system if $S_2 < W/2$, nor in an open system if $S_2 \geq W/2$. A wide range of sizes for the second largest party can be consistent with being in Cases A, B or D.

While it is conventional to analyze allocations of legislative seats between parties in terms of the sizes of individual parties or coalitions of parties relative to the winning threshold, it is less obvious that the sizes of the top three parties, relative to *half* the winning threshold, might have a major impact on the legislative decision-making environment. Yet, when we consider the simple arithmetic of legislative decision-making, *W/2* defines an important threshold between types of legislative decision-making environment. When one of the top three parties crosses this threshold, it is quite possible that the legislature will be flipped from one case to another and significant discontinuities in legislative decision-making will arise.

[11] To see it is not necessary, consider a 100-seat legislature with a simple majority rule, an open legislature would arise if the partition of seats between 5 parties was (30, 18, 18, 17, 17).

[12] An obvious example of a top-three 100-seat legislature with a simple majority rule would arise if the partition of seats between 3 parties was (49, 49, 2).

2.10: Legislative types and public policy decisions

So far we have considered no aspect of the substance of decisions that might interest legislative parties. It is easy to show, however, that the different legislative environments identified by the partitioning of legislatures set out in Figure 1 have an important bearing on decision-making on any one-dimensional policy issue – that is any issue to be decided for which legislative party preferences can be described in terms of positions on a single dimension. This is because the constraints defining the five cases set out in Figure 1 impose very different, and sometimes very striking, constraints on the identity of the party containing the “median” or “pivotal” legislator on an arbitrary policy dimension – this is a dimension for which we do not know *a priori* the ordering of party positions. Note that the notion of a “median”, as opposed to “pivotal”, legislator implies a simple-majority winning threshold. This section thus presents the stronger results that can be derived for legislative decision-making using a simple-majority winning threshold.

Obviously and trivially in Case A, with a single winning party, the median legislator on every conceivable policy dimension must be a member of the largest party.

Knowing that we have a Case B legislature with a strongly dominant party, however, we also know a great deal about the location of the median legislator on an arbitrary issue dimension. Most obviously, consider the subset of cases in Case B where there is a “system-dominant” party – a party that can form a winning coalition with every other party. It is clear that ***a system-dominant party must contain the median legislator on any issue dimension for which it does not occupy one of the two most extreme party positions***. The probability that a system dominant party is median on an arbitrary issue dimension is thus *(N-2)/N*. This follows directly from the definition of system dominance. A system dominant party can form a winning coalition with a party located on either side of it on an arbitrary issue dimension. Only for a dimension on which there is no party on both sides of it is a system dominant party not median. Furthermore we also know that, when there is a system dominant party located at one extreme of the issue dimension under consideration, the median legislator must belong to the party adjacent to the system dominant party. This follows directly from the fact that a coalition between a system dominant party and any other party is winning.

Thus ***the universe of all conceivable policy dimensions that could possibly be considered by a legislature with a system-dominant party can be partitioned into two***

subsets, defined by whether or not the system dominant party controls the median legislator. One subset, ***D***, comprises all conceivable issue dimensions on which the system dominant party is ***not*** first or last in the ordering of party positions on the dimension. ***A system dominant party controls the median legislator on every issue dimension in D. Thus, regardless of the issue positions of all other parties, a system dominant party is at the generalized median of every issue space constructed only from issues in D.***

The other subset, ***D'***, comprises all conceivable issue dimensions on which the system dominant party is either first or last in the ordering of party positions. In this case, as we have seen, the median legislator belongs to the party ranked next to the system dominant party on the dimension in question. Overall, even when there is an infinite or indefinite number of issue dimensions that might form the basis of legislative decisions, ***the median legislator must be controlled either by a system-dominant or by the party adjacent to it on the issue to be decided, regardless of the issue positions of all other parties***.

If a P_1 is strongly dominant in a Case B legislature, then by definition it can form majority coalitions with both P_2 and P_3. From this it follows that, ***if*** $\boldsymbol{P_2}$ ***and*** $\boldsymbol{P_3}$ ***are on opposite sides of a strongly dominant*** $\boldsymbol{P_1}$ ***on the policy dimension under consideration, then*** $\boldsymbol{P_1}$ ***contains the median legislator, regardless of the issue positions of all other parties***. If P_2 and P_3 are on the same side of P_1 on the policy dimension under consideration, then ***the median legislator will be located on the interval between*** $\boldsymbol{P_1}$ ***and either*** $\boldsymbol{P_2}$ ***or*** $\boldsymbol{P_3}$***, whichever is closest on the dimension in question to*** $\boldsymbol{P_1}$.

Again implies that ***the universe of all conceivable policy dimensions that could possibly be considered by a legislature with a strongly dominant party can be partitioned into three subsets***, defined by orderings of the top three parties. One subset, $\boldsymbol{D_1}$, comprises the set of issues for which a strongly dominant P_1 is located between P_2 and P_3. We know that $\boldsymbol{P_1}$ ***controls the median legislator on every conceivable issue dimension in*** $\boldsymbol{D_1}$***, and is thus at the generalized median of any issue space comprising only issues from*** $\boldsymbol{D_1}$***, regardless of the issue positions of all other parties.*** The second subset, $\boldsymbol{D_2}$, comprises the set of issues for which for which P_2 is located between P_1 and P_3, where we know the median legislator is on the interval P_1P_2. The final subset, $\boldsymbol{D_3}$, comprises the set of issues for which for which P_3 is located between P_1 and P_2, where we know the median legislator is on the interval P_1P_3.

This type of partitioning of the infinite universe of conceivable issue dimensions generalizes straightforwardly to the notion of *k*-dominance. The universe of all conceivable policy dimensions that could be considered by a legislature with a *k*-dominant party can be partitioned into *k* subsets, $\boldsymbol{K_1}$**,** $\boldsymbol{K_2}$ **…** $\boldsymbol{K_k}$. One subset, $\boldsymbol{K_1}$, comprises the set of issues for which a strongly dominant P_1 is located between some pair of members of $\{P_2 , P_3 \ldots P_k\}$. We know that P_1 controls the median legislator on every conceivable issue dimension in $\boldsymbol{K_1}$, and is thus at the generalized median of any issue space comprising only issues from $\boldsymbol{K_1}$**,** regardless of the issues positions of all other parties. Thus ***D*** for a system-dominant party is a special case of $\boldsymbol{K_1}$ where $k = n$, while $\boldsymbol{D_1}$ is a special case where $k=3$. Generally, the larger is *k*, the more likely, other things being equal, that a strongly dominant P_1 is located on an arbitrary dimension between some pair of members of $\{P_2 , P_3 \ldots P_k\}$.

The possibility of partitioning the universe of possible issue dimensions in this way also has implications for long-term coalitions between parties that might be formed to control decisions on a large set of issue dimensions that might arise during the lifetime of a legislature, for example government coalitions negotiated in an issue space of infinite or indeterminate dimension. Even an infinite number of actual or conceivable issue dimensions in a Case B legislature with a *k*-dominant party can be tied into just a few bundles, $\boldsymbol{K_1}$**,** $\boldsymbol{K_2}$ **…** $\boldsymbol{K_k}$. If the largest party is interested only in issues in $\boldsymbol{K_1}$**,** the relative size of which is likely to increase as *k* increases, then it has no need to form a long-term coalition with any other party since it controls the median legislator on all issues in $\boldsymbol{K_1}$**,** regardless of the issue positions of all other parties. Recall that $\boldsymbol{K_1}$ is the set of all issues for which a *k*-dominant party is not the most extreme of the set of parties with which it can form two-party winning coalitions. If the largest party is interested in issue dimensions on which it has a relatively extreme position in this sense, then these dimensions must be in $\boldsymbol{K_2}$ **…** $\boldsymbol{K_k}$. and the largest party will easily be able to identify the other party or parties with which it may wish to engage in logrolling. Depending on the precise model of government formation deployed, this logrolling may or may not manifest itself in a long-term government coalition.

We can also make quite strong statements about the location of the median legislator in a top-three legislature. Since no party outside the top three can be pivotal, we know that the median legislator on any conceivable policy dimension must belong to one of the top three parties. In fact we know much more than this. Ordering the preferences of just the top three parties on an arbitrary issue dimension, the inequalities defining a top-three legislature mean that ***the median legislator must belong to the most central of the three largest parties, regardless of the issue positions of all smaller parties.***. If the three largest parties are ordered P_a, P_b, P_c on some dimension, then we know from the Case C inequalities that $P_a + P_b \geq W$. Thus the median legislator cannot be either to the left, or to the right, of *both* P_a and P_b and must be on the interval P_aP_b. We also know that $P_b + P_c \geq W$ and thus that the median legislator is on the interval P_bP_c. Since the parties are ordered P_a, P_b, P_c, these intervals intersect only at P_b and it follows that P_b controls the median legislator. In a nutshell, if we have a Case C legislature, we know that the median legislator on any conceivable policy dimension will belong to the middle party of the top three on that dimension.

This further implies that the infinite universe of all conceivable issue dimensions that could possibly be considered by a top-three legislature can be partitioned into three subsets, $\boldsymbol{T_1}$, $\boldsymbol{T_2}$ and $\boldsymbol{T_3}$, defined by which of the top three parties (P_1, P_2 or P_3) controls the median legislator. This in turn means that ***P_1 in a top-three legislature controls the generalized median in any multidimensional issue space constructed solely from issue dimensions in T_1.*** *P_2 and P_3 are in equivalent positions with respect to* $\boldsymbol{T_2}$ *and* $\boldsymbol{T_3}$*, respectively.*

Knowing that we have a Case D, “top-two”, legislature, we know that ***the median legislator must be located on the interval between P_1 and P_2 on any conceivable issue dimension, regardless of the issue positions of smaller parties.*** This follows straightforwardly from the fact that P_1P_2 is a winning coalition. On any conceivable policy dimension, therefore, the median legislator cannot be either to the left or to the right of both P_1 and P_2. This is much less of a constraint on the location of the median legislator than in the previous three cases – indeed if P_1 and P_2 are at opposite ends of some issue dimension, it is no constraint at all.

Since the definition of a Case E legislature is that all two-party coalitions between the three largest parties are losing, this definition imposes no new constraint upon the location of the median legislator.[13]

Overall, the most striking pattern to emerge from this discussion is that the simple integer arithmetic of legislative voting increasingly constrains the location of the median legislator as we move “leftwards” from Case E to Case A. Obviously, in Case A with a single winning party, the legislative numbers absolutely constrain the location of the median legislator, on any arbitrary issue dimension, to be the location of the largest party. In Case B, a *k*-dominant party controls the median legislator on any multidimensional issue space comprising only issues from $\boldsymbol{K_1}$, the set of issues on which it is not at the extreme of

[13] Although further inequalities dealing with the sizes of parties outside the top three may impose such constraints.

the set of parties with which it can form two-party winning coalitions. This set is likely to increase in relative size as *k* increases. As we move through Cases C, D, and E, constraints on the location of the median legislator decrease. Only for the open legislature in Case E, however, does the legislative arithmetic, seen in this context as the sizes of the three largest parties relative to *W*, impose no constraint on the location of the median legislator.

3. CASE CLASSIFICATIONS IN THE UNIVERSE OF 100- AND 101-SEAT LEGISLATURES

It is clearly interesting to characterize the relative frequency of the different legislative types, as defined above, and the relative bargaining advantage of different types of party in different types of legislature. In the following section we investigate these matters for real legislatures. In this section we investigate them for an arbitrary (random) legislature. To do this, we generated the universe of possible non-equivalent integer partitions of 100 (and 101) legislative seats between up to ten parties. For a description of how we created this "metadata" set, see Laver and Benoit (2003).[14] Table 1 reports the classification of legislative cases in the universe of possible 100-seat legislatures with up to ten parties and a simple majority winning quota of 51 seats.

Table 1: Frequencies of legislative cases in universe of possible 100-seat legislatures

N. of parties	*Case A*	*Case B*	*Case C*	*Case D*	*Case E*	*Total*
3	600	25	208	0	0	833
4	3,364	2,136	1,628	24	1	7,153
5	11,222	18,314	4,950	2,990	749	38,225
6	25,949	68,681	8,778	26,558	13,281	143,247
7	45,958	160,331	11,094	103,132	86,739	407,254
8	66,877	275,477	11,272	251,095	326,191	930,912
9	84,074	383,847	9,941	451,678	856,988	1,786,528
10	94,760	461,217	7,993	659,581	1,754,315	2,977,866
Total	332,804	1,370,028	55,864	1,495,058	3,038,264	6,292,018

[14] We call it a metadata set because all logically possible data within its domain is contained within the database. This allows us to "smash" recalcitrant propositions in an (albeit ugly) way by demonstrating that there is no counter-example, as well as checking analytically demonstrated propositions. With enhanced computer firepower (and patience) it is trivial to increase the domain to include 100-seat legislatures with up to 100 parties, although most of the effects we observe appear to be reaching their limit as the number of parties approaches 10. What is also striking, however, is that Laver and Benoit (2003) found three- and four-party legislatures to be highly pathological, with the convergence of key effects toward their limit only beginning when the number of parties was five or more.

3	72.0%	3.0%	25.0%	0.0%	0.0%	100.0%
4	47.0%	29.9%	22.8%	0.3%	0.0%	100.0%
5	29.4%	47.9%	12.9%	7.8%	2.0%	100.0%
6	18.1%	47.9%	6.1%	18.5%	9.3%	100.0%
7	11.3%	39.4%	2.7%	25.3%	21.3%	100.0%
8	7.2%	29.6%	1.2%	27.0%	35.0%	100.0%
9	4.7%	21.5%	0.6%	25.3%	48.0%	100.0%
10	3.2%	15.5%	0.3%	22.1%	58.9%	100.0%

The winning quota is 51 seats

Table 1 shows that the most likely outcome of a random partition of seats between parties is a Case A (single party majority) legislature with a simple majority winning quota if there are three or four parties, a Case B legislature (with a strongly dominant largest party) if there are five, six, or seven parties, and a Case E (open) legislature if there are eight, nine or ten parties. Perhaps the most striking thing about these results is the frequency of strongly dominant largest parties (Case B) in random legislatures. Note also that the relative frequency of legislative types is very strongly conditioned by the number of parties. In particular, single party majority and top three legislatures become very infrequent in legislatures with large numbers of parties.

Table 2: Mean Shapley Values by Number of Parties and Legislative Csse

	A Majority P_1	B Strong Dominance	C Top 3	D Top 2	E Open System
***n*=4**					
E_1	1.00	0.51	0.33	0.33	0.25
E_2	0	0.17	0.33	0.33	0.25
E_3	0	0.17	0.33	0.17	0.25
E_4	0	0.15	0	0.17	0.25
***n*=5**					
E_1	1.00	0.50	0.33	0.32	0.22
E_2	0	0.17	0.33	0.29	0.21
E_3	0	0.17	0.33	0.15	0.19
E_4	0	0.11	0	0.12	0.19
E_5	0	0.05	0	0.12	0.19
***n*=6**					
E_1	1.00	0.51	0.33	0.35	0.27
E_2	0	0.16	0.33	0.26	0.20
E_3	0	0.16	0.33	0.16	0.18
E_4	0	0.09	0	0.10	0.15
E_5	0	0.05	0	0.09	0.13
***n*=7**					
E_1	1.00	0.52	0.33	0.36	0.28
E_2	0	0.15	0.33	0.24	0.20
E_3	0	0.15	0.33	0.15	0.17
E_4	0	0.08	0	0.10	0.13
E_5	0	0.05	0	0.08	0.11
***n*=8**					
E_1	1.00	0.54	0.33	0.37	0.28
E_2	0	0.14	0.33	0.23	0.19
E_3	0	0.14	0.33	0.15	0.16
E_4	0	0.08	0	0.09	0.12
E_5	0	0.04	0	0.07	0.10
***n*=9**					
E_1	1.00	0.56	0.33	0.38	0.29
E_2	0	0.13	0.33	0.22	0.19
E_3	0	0.13	0.33	0.14	0.15
E_4	0	0.07	0	0.08	0.11
E_5	0	0.04	0	0.06	0.09
***n*=10**					
E_1	1.00	0.57	0.33	0.39	0.28
E_2	0	0.12	0.33	0.21	0.18
E_3	0	0.12	0.33	0.13	0.14
E_4	0	0.07	0	0.08	0.11
E_5	0	0.04	0	0.06	0.09

Table 2 reports the mean value of the Shapley-Shubik index for parties of different rankings in random legislatures of different types. Since this index is normalized to sum to 1.0 across all parties, the mean value for all parties in a five-party legislature is 0.20, in an eight-party legislature 0.125, and so on.

Table 3: Median EWRs by Number of Parties and Party System Type

	A Majority P_1	B Strong Dominance	C Top 3	D Top 2	E Open System
***n*=3**					
E_1	1.54	1.33	0		
E_2	0	0.45	1.28		
E_3	0	0.93	1.59		
***n*=4**					
E_1	1.64	1.14	0.85	0.89	1.00
E_2	0	0.61	1.01	0.89	1.00
E_3	0	0.98	1.51	1.34	1.00
E_4	0	1.52	0	1.34	1.00
***n*=5**					
E_1	1.69	1.22	0.93	1.00	0.83
E_2	0	0.67	1.08	1.03	0.95
E_3	0	1.01	1.45	0.89	0.95
E_4	0	1.04	0	0.89	1.05
E_5	0	0.83	0	1.33	1.43
***n*=6**					
E_1	1.75	1.25	0.98	1.08	1.03
E_2	0	0.69	1.11	0.99	0.97
E_3	0	1.00	1.39	0.98	0.96
E_4	0	0.91	0	0.83	0.94
E_5	0	0.83	0	1.03	1.11
***n*=7**					
E_1	1.79	1.29	1.01	1.13	1.09
E_2	0	0.68	1.15	0.96	0.99
E_3	0	0.97	1.39	1.01	0.97
E_4	0	0.85	0	0.85	0.92
E_5	0	0.75	0	0.92	1.00
***n*=8**					
E_1	1.79	1.32	1.04	1.15	1.11
E_2	0	0.66	1.15	0.94	1.00
E_3	0	0.92	1.39	1.02	0.98
E_4	0	0.83	0	0.87	0.92
E_5	0	0.73	0	0.88	0.95
***n*=9**					
E_1	1.82	1.36	1.08	1.17	1.12
E_2	0	0.63	1.15	0.92	1.01
E_3	0	0.87	1.39	1.02	0.98
E_4	0	0.82	0	0.87	0.93
E_5	0	0.73	0	0.87	0.94
***n*=10**					
E_1	1.82	1.38	1.08	1.18	1.12
E_2	0	0.60	1.19	0.91	1.01
E_3	0	0.82	1.39	1.02	0.98
E_4	0	0.80	0	0.88	0.93
E_5	0	0.73	0	0.87	0.94

Table 3: Relative frequency of largest parties with different k-dominance levels

N of parties	**Not Dominant**	**Majority P_1**	**k=3**	**k=4**	**k=5**	**k=6**	**k=7**	**k=8**	**k=9**	**k=10**
***n*=3**	208	600	**25**							
***n*=4**	1,653	3,364	300	**1,836**						
***n*=5**	8,689	11,222	7,036	6,912	**4,366**					
***n*=6**	48,617	25,949	32,436	18,730	9,338	**8,177**				
***n*=7**	200,965	45,958	81,422	36,953	18,366	10,942	**12,648**			
***n*=8**	588,558	66,877	144,082	57,702	28,654	17,008	11,208	**16,823**		
***n*=9**	1,318,607	84,074	203,097	75,959	37,622	22,301	14,632	10,262	**19,974**	
***n*=10**	2,421,889	94,760	245,104	87,978	43,524	25,718	16,844	11,753	8,594	**21,702**
***n*=3**	25.0%	72.0%	3.0%							
***n*=4**	23.1%	47.0%	4.2%	25.7%						
***n*=5**	22.7%	29.4%	18.4%	18.1%	11.4%					
***n*=6**	33.9%	18.1%	22.6%	13.1%	6.5%	5.7%				
***n*=7**	49.3%	11.3%	20.0%	9.1%	4.5%	2.7%	3.1%			
***n*=8**	63.2%	7.2%	15.5%	6.2%	3.1%	1.8%	1.2%	1.8%		
***n*=9**	73.8%	4.7%	11.4%	4.3%	2.1%	1.2%	0.8%	0.6%	1.1%	
***n*=10**	81.3%	3.2%	8.2%	3.0%	1.5%	0.9%	0.6%	0.4%	0.3%	0.7%

Table 3 reports the relative frequency of largest parties with different *k*-dominance levels, with the numbers of system dominant parties highlight in bolds. When there are four parties, just over a quarter of all possible legislatures have a system dominant party, which can form a majority coalition with any other party in the system. This proportion declines steadily as the number of parties increases. Only when the number of parties exceeds seven is an arbitrary legislature more likely than not to have no majority or strongly dominant party.

4. CASE CLASSIFICATIONS AND OUTCOMES IN REAL LEGISLATURES

Relative frequencies generated for arbitrary legislatures may tell us nothing about the real world, since the processes of real party competition may results in real legislatures that fall into the case classifications in very particular ways. Here we report case classifications in a dataset of real legislatures assembled by McDonald and Mendes (2002), which lists complete legislative seat allocations and a range of other data for the set of countries listed in Table 4, for the years 1950-95. (These data to be extended and updated in future drafts).

Table 4: Classification of real legislatures 1950-95

Country	*Legislative case*					
	A	B	C	D	E	Total
Australia	18	0	0	0	0	18
Austria	3	1	8	0	0	12
Belgium	1	2	2	4	6	15
Canada	8	4	2	0	0	14
Denmark	0	11	0	5	2	18
Finland	0	1	0	6	6	13
France	2	8	0	0	2	12
Germany	1	5	6	0	0	12
Iceland	0	12	0	1	1	14
Ireland	3	9	2	0	0	14
Italy	0	8	0	1	2	11
Luxembourg	0	5	4	1	0	10
Netherlands	0	1	3	6	3	13
New Zealand	14	0	0	0	0	14
Norway	2	9	0	0	0	11
Portugal	4	4	0	0	0	8
Spain	2	4	0	0	0	6
Sweden	1	13	0	1	0	15
Switzerland	0	4	0	3	5	12
United Kingdom	12	0	0	1	0	13
United States	11	0	0	0	0	11
Total	82	101	27	29	27	266
Percentage	30.1	38.0	10.2	10.9	10.2	100.0

Table 4 reports our classification of the real legislatures in this dataset into the five types defined above.[15] Although we must very cautious about the inferences we draw given the inevitable potential for bias arising from the particular selection of cases and years in this dataset, Table 4 shows that the most common legislative types in the domain of cases under consideration are those with single majority party or a system dominant party. The group of countries with single party majorities (Case A) almost exclusively comprises Britain and former British colonies using disproportional electoral systems. The group of countries with strongly dominant parties (Case B) comprises a group of Scandinavian countries, plus Ireland, Italy and France. The Case C (top three) legislature so beloved by formal theorists modeling "multi-party" competition predominates only in Austria, and to a lesser extent Germany and Luxembourg. More open systems (Cases D and E), in which there or fewer arithmetic constraints on the formation of coalitions tend to be found in Belgium, Finland, the Netherlands and Switzerland.

The classification of legislatures into types was driven by relative bargaining advantage that different types appear on *a priori* grounds to give to different party rankings. One way of assessing this bargaining advantage is to look at distribution of political outcomes rather than *a priori* bargaining power. Here we use two clear-cut and self-evident measures of the successful exercise of power in parliamentary government systems – the ability of a party to control the position of prime minister and the ability of a party to govern alone as a minority government.

Table 5 reports the relative frequencies with which the first, second and third largest parties control the prime ministership. We classify all governments in the dataset, excluding all single party majority (Case A) legislatures.[16] The results show the striking impact of legislative type on who controls the PM position.

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[15] Note that, while we analyze *governments* in the tables that follow, this table classifies *legislatures* that come into being following legislative elections – for which there may be multiple governments.

[16] Switzerland is also excluded. This is because it is not a parliamentary government system in the strict sense in which the executive is responsible to, and can be dismissed by, the legislature; partly as a result of this, Swiss governments have tended to be "Magic Formula" coalitions that include most of the main parties and rotate the PM position between parties on an annual basis.

Table 5: Party control of the Prime Ministership by legislative type

Party 1 holds PM position?	Legislative case				
	B	C	D	E	Total
No	43	11	29	30	113
	23.37	26.19	58.00	51.72	33.83
Yes	141	31	21	28	221
	76.63	73.81	42.00	48.28	66.17
Total	184	42	50	58	334
	100.00	100.00	100.00	100.00	100.00

Pearson chi2(3) = 31.4328 Pr = 0.000

Party 2 holds PM position?	Legislative case				
	B	C	D	E	Total
No	163	32	28	55	278
	88.59	76.19	56.00	94.83	83.23
Yes	21	10	22	3	56
	11.41	23.81	44.00	5.17	16.77
Total	184	42	50	58	334
	100.00	100.00	100.00	100.00	100.00

Pearson chi2(3) = 37.4312 Pr = 0.000

Party 3 holds PM position?	Legislative case				
	B	C	D	E	Total
No	178	41	47	49	315
	96.74	97.62	94.00	84.48	94.31
Yes	6	1	3	9	19
	3.26	2.38	6.00	15.52	5.69
Total	184	42	50	58	334
	100.00	100.00	100.00	100.00	100.00

Pearson chi2(3) = 13.3303 Pr = 0.004

The largest party is much more likely to control the PM position in Case B or Case C legislatures. In relation to control of the PM position, the big distinction between vases comes between case B and C on the one hand, and cases D and E on the other. What seems to make a difference to the largest party's ability to control the PM positions is the inequality $S_1 + S_3 < W$, which cuts the cases between C and D. In other words, ***the largest party's ability to control the PM position declines sharply in legislatures where it is not member of at least two two-party winning coalitions – making it more difficult for the largest party to play other parties off against each other.***

The other striking feature of Table 5 is that, in line with expectations, the second largest party comes into strongest contention for the PM position in Case D "top two" legislatures.

Table 6: Largest party forms single-party minority government, by legislative type

Largest-party-only minority government?	Legislative case				
	B	C	D	E	Total
No	120	38	46	56	260
	65.22	90.48	92.00	96.55	77.84
Yes	64	4	4	2	74
	34.78	9.52	8.00	3.45	22.16
Total	184	42	50	58	334
	100.00	100.00	100.00	100.00	100.00

Pearson chi2(3) = 38.4739 Pr = 0.000

Table 6 shows the incident of single party majority governments comprising only the largest party, by legislative type. This again shows very strong case-based effects. As expected, such single party minority governments are very much more likely when the largest party is strongly dominant (Case B). Indeed almost all observed cases of single party minority governments (64 out of 74) arise in Case B legislatures. This does seem to be significant empirical conformation that the case classification defined in this paper is capturing something important about bargaining in real legislatures.

```
logit p1govt p1share p2share p3share if case != "A" & ISO !="CH" & p1p2p3 != 1
Logistic regression                               Number of obs   =        309
                                                  LR chi2(3)      =      28.89
                                                  Prob > chi2     =     0.0000
Log likelihood = -156.80126                       Pseudo R2       =     0.0843

------------------------------------------------------------------------------
      p1govt |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     p1share |   6.649337   2.016387     3.30   0.001      2.69729    10.60138
     p2share |   .9796586   2.600706     0.38   0.706    -4.117631    6.076948
     p3share |  -5.586833   3.651329    -1.53   0.126    -12.74331     1.56964
       _cons |  -.6489112   1.074665    -0.60   0.546    -2.755216    1.457394
------------------------------------------------------------------------------

logit p1pm p1share p2share p3share if case != "A" & ISO !="CH" & p1p2p3 != 1
Logistic regression                               Number of obs   =        309
                                                  LR chi2(3)      =      44.08
                                                  Prob > chi2     =     0.0000
Log likelihood = -177.92778                       Pseudo R2       =     0.1102

------------------------------------------------------------------------------
        p1pm |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     p1share |   9.097572   1.909538     4.76   0.000     5.354947     12.8402
     p2share |  -.4799261   2.306336    -0.21   0.835    -5.000261    4.040409
     p3share |  -3.553592   3.374232    -1.05   0.292    -10.16696     3.05978
       _cons |  -2.048423   1.033173    -1.98   0.047    -4.073405   -.0234415
------------------------------------------------------------------------------

logit OnePtyMinGov p1share p2share p3share if case != "A" & ISO !="CH" & p1p2p3 != > 1
Logistic regression                               Number of obs   =        309
                                                  LR chi2(3)      =      79.89
                                                  Prob > chi2     =     0.0000
Log likelihood =  -126.6071                       Pseudo R2       =     0.2398

------------------------------------------------------------------------------
OnePtyMinGov |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     p1share |   19.53964   3.037914     6.43   0.000     13.58544    25.49384
     p2share |  -9.274479   2.696253    -3.44   0.001    -14.55904   -3.989921
     p3share |  -5.710638   4.756513    -1.20   0.230    -15.03323    3.611956
       _cons |  -5.832804   1.714109    -3.40   0.001    -9.192396   -2.473211
------------------------------------------------------------------------------

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```

5. SUMMARY AND CONCLUSIONS

The regularities identified in the preceding discussion are to a large extent "model free", in the sense that they are no more than the logical implications of the simple integer arithmetic of voting in legislatures. These regularities and constraints apply to any decision making model that depends upon winning legislative votes, since any such model, whether it is concerned with policy-making or allocating private goods, must be constrained in the context of some decisive winning quota by the need for the legislative numbers to add up. We might take as an analogy a supermarket. Different people shopping in a supermarket load their baskets with different bundles of goods, according to their diverse and idiosyncratic tastes. We can analyze their behavior using 101 different theories. But, unless they are thieves, all shoppers are constrained by the fact that they may only walk out of the supermarket carrying goods whose total value is less than the money they have in their pocket. Despite the cornucopia of different models of shopping behavior in a supermarket, the same underlying arithmetic constrains them all.

The five legislative cases identified by the inequalities set out in Figure 1 are not in any sense special cases. They are an exclusive and exhaustive partition of the universe of possible legislatures. Furthermore, and of course we have *potential* models of decision-making in mind when we make this judgment, the different cases characterize different decision-making environments in terms of the options open to the largest parties in the system. Few would dispute this for Case A legislatures, with a single winning party. Indeed almost all published models of legislative decision making treat Case A legislatures completely differently and take the need to do this as self-evident. But is it important to remember that, when there is no single winning party, every possible legislature in which decisions are made according to some decisive rule must fall into one of the four cases B-E in Figure 1. The first three of these cases are distinguished by privileging the largest one (Case B), two (Case D) or three (Case C) parties in the system. Case E does not distinguish any of the three largest parties to the same extent.

Case B is easily identified for any legislature by the inequalities $S_1 < W$ and $S_1 + S_3 \geq W$ and $S_2 + S_3 < W$, with the implied constraints that $S_1 \geq W/2$ and $S_3 < W/2$. Here, the decision making environment is characterized by the facts that: the largest party and only the largest party is a member of every two-party winning coalition; that any winning coalition excluding the largest party must include the second and third largest parties; that any winning coalition excluding the largest party must contain at least one pivotal party that can form a two-party winning coalition with the largest party; and that the location of the median legislator on any policy dimension is constrained either to be a member of the largest party (when it is not the most extreme of the set of parties with which it can for two party winning coalitions) or to be on the interval between the largest party and the closest party with which it can form a two-party winning coalition. We observed that the largest party, strongly dominant in a Case B legislature, not only has super-proportional *a priori* bargaining expectations in an arbitrary Case B legislature, but in real legislatures is much more likely to control the PM position, and much more likely to be in position to govern alone as a minority government.

Case C is easily identified by the inequalities $S_1 < W$ and $S_2 + S_3 \geq W$, with the further implication that $S_2 \geq W/2$. The decision making environment is characterized by the facts that: no party outside the top three ever makes a difference to the winning or losing of any legislative vote; that the support of two of the top three parties is needed to win any legislative vote; and that the median legislator on any policy dimension under consideration must belong to the "middle" of the top three parties. We observed that Case C is not particularly probable in an arbitrary legislature, and not very commonly observed in the real world – the main exception being Austria. We also noted that, in the real world, a Case C legislature renders the largest party much more likely to control the PM position than Case D or Case E. It does not, however, enhance the likelihood that the largest party can govern alone in a minority administration.

Case D is easily identified by the inequalities $S_1 < W$ and $S_1 + S_2 \geq W$ and $S_1 + S_3 < W$, with the implied constraint that that $S_3 < W/2$. The decision making environment is characterized by the facts that: only the top two parties can form a two-party coalition excluding all others; one or both of the top two parties is essential to any winning coalition; that this is true only for the top two parties; and that location of the median legislator on any policy dimension is constrained to lie on the interval between the two largest parties. Here, we note that there must be at least five parties. This has the implication, significant in itself, that a model of legislative decision-making under a simple majority threshold that cannot encompass five parties cannot deal with every possible decision-making environment.

"Open" Case E legislatures are easily identified by the inequality $S_1 + S_2 < W$, with the implied constraint that that $S_2 < W/2$, and the sufficient condition that $S_1 < W/2$. Only here do we find ourselves in the "anything goes" decision-making environment that seems to be envisaged by many models of legislative decision-making, though such legislatures remain constrained by the fact that any winning coalition must include at least three members. For all of the other cases, what can in fact happen is constrained by the integer arithmetic of legislative decision-making.

We observe both in arbitrary legislature and the real world that Case D and Case E legislatures look rather similar to each other, with the exception that the second largest party comes into strongest contention for the PM position in Case D "top two" legislatures. It is also the case that Case D and Case E legislatures both tend to be found in the same countries.

To be written soon: a better conclusion later, dealing with the dynamics of all of this.

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APPENDIX: TOWARDS A DYNAMIC MODEL OF LEGISLATIVE DECISION-MAKING: TRANSITIONS BETWEEN LEGISLATIVE CASES (WORK IN PROGRESS)

Transitions from Case A (single party winning) legislatures

Any minimal integer repartition between parties in a Case A legislature with four or more parties either leaves the legislature in Case A or moves the legislature from Case A to Case B with P_1 system dominant. Such repartitions must be one of the following.

- A repartition of one seat within the set of losing parties. In a Case E legislature this means a repartition within $(P_1)'$, the complement of P_1. In this case S_1 is unchanged, $S_1 > W$. *The legislature remains in Case A.*

- A repartition of one seat from a party in $(P_1)'$ to P_1. In this case S_1 must increase, $S_1 > W$. *The legislature remains in Case A.*

- A repartition of one seat from P_1 to a party in $(P_1)'$. In this case, the situation before the repartition, since $S_1 \geq W$ in a Case E legislature, is either $S_1 > W$ or $S_1 = W$.
  - If $S_1 > W$ before repartition then, since S_1 and W are integers, $S_1 \geq W$ after the minimum possible integer repartition. *The legislature remains in Case A.*
  - If $S_1 = W$ before repartition then $S_1 = W - 1$ after the minimum possible integer repartition. For the smallest possible party in an n-party system after repartition, $S_n \geq 1$. (We know the smallest party must have at least one seat). Thus $S_1 + S_n \geq W$. In a legislature with four or more parties, this implies $S_1 + S_2 \geq W$ and $S_1 + S_n \geq W$. Since $S_1 + S_n \geq W$, this means $(P_1P_n)'$, the complement of P_1P_n, is losing. This means $S_2 + S_3 < W$, since S_2S_3 is in the complement of P_1P_n when $n \geq 4$. Thus the Case B inequalities are satisfied and P_1 is strongly dominant. Because $S_1 + S_n \geq W$, (P_1 can form a winning coalition with the smallest party in the system), P_1 *is* system

dominant*. The legislature moves from Case A to Case B, with P_1* system *dominant.*

This result is interesting as we move into a dynamic context because it provides an example of potential constraints on the ways in which legislatures can cross the thresholds between the five legislative cases as a result of seat repartitioning between parties (whether as the result of a bye-election, for example, or as the result of party-switching by legislators). It also shows how manipulation of the case-defining inequalities can lay bare these constraints. In this case, ***any conceivable one-seat repartition in a Case A legislature with four or more parties must at worst leave the largest party in what is in effect the next best position, that of a system dominant party.***

APPENDIX 2:
DOODLES TO BE INCORPORATED IN A COMPREHENSIVELY REVISED VERSION

Minimum size of system-dominant parties in N-party legislatures

The smallest possible seat total, s_1, for the largest party, P_1, that can leave it system dominant in an n-party legislature with M seats arises when s_n is as large as possible, allowing s_1 to be as small as possible, with P_n being the smallest party and $s_1 + s_n = W$. This happens when the seats not allocated to P_1 are shared as equally as possible between all other parties, subject to integer seat allocations. That is, it occurs when:

$$s_n = \frac{M - s_1}{n-1}$$

Thus s_1 is minimized while P_1 remains system-dominant when:

$$s_1 + \frac{M - s_1}{n-1} = W$$

Bearing in mind that P_1 can only be system dominant if $n > 3$, this rearranges to give a theoretical minimum size for P_1 system dominant of:

$$s_1 = \frac{W(n-1) - M}{n-2} + \partial$$

Thus the theoretical threshold seat share for a system dominant party in 4- to 10-party systems is greater than:

N of parties	Threshold for system dominance
4	0.250
5	0.333
6	0.375
7	0.400
8	0.417
9	0.429
10	0.438

Note that parties with a higher seat share with this may well not be system dominant, since system dominance depends upon the seat shares of all parties.

Shapley-Shubik expectations of system dominant parties

Note that the Shapley-Shubik expectation (SSE) of a system dominant party in an n-party system is always:

$$SSE = \frac{n-2}{n}$$

This is because a system dominant party is pivotal in every position of every ordering of parties except the first and last position.

Thus a system dominant party will have a super-proportional SSE whenever:

$$\frac{s_1}{M} < \frac{n-2}{n}$$

Since a party can only be system dominant when *n > 3* and since we are not treating majority parties as system dominant, this is sufficient to show that system dominant parties always have super-proportional SSEs. Majority parties, of course, always have super-proportional SSEs, unless they control 100 percent of the seats.

Super-proportional Baron-Ferejohn expectation of system-dominant parties

Snyder et al. have developed the seminal Baron-Ferejohn model of bargaining in legislatures to accommodate weighted voting (Baron and Ferejohn 1989; Ansolabehere, Snyder, Strauss et al. 2005; Snyder, Ting and Ansolabehere 2005). They argue that the bargaining expectation of each party in a legislature is proportional to its “minimum integer weight” (MIW). The minimum integer weights for a set of parties in a legislature are significant because many models of legislative bargaining depend, not on the precise allocation of seats between parties, but rather on the set on winning coalitions that this seat allocation generates. In this sense, many different seat allocations all generate the “same” bargaining game. For example, consider the seat allocations and assume a simple majority voting rule: (31, 30, 29, 10); (28, 28, 23, 21); (48, 26, 25, 1). In each case, the set of winning coalitions is identical. Any two of the three largest parties can form a winning coalition, while the smallest party is essential to no winning coalition. The set of minimum integer weights for this legislature is generated by finding the smallest integer weight that can be allocated to each party while still generating precisely the same set of winning coalitions. Thus, in the above examples, the fact that the set of winning coalitions is identical for all cases is captured by the fact that they can all be represented by giving the parties minimal integer weights (1, 1, 1, 0), with a winning threshold of 2. Note that the mapping of seat shares into MIWs for a given bargaining game depends crucially in the winning threshold. Thus in the above example, if the winning threshold was 60 seats, then the set of winning coalitions changes and all three legislatures would have different minimum integer weights: (1,1,1,0); (1,1,1,1) and (2,1,1,0).

MIWs are a convenient way of grouping games with identical sets of winning coalitions and, to the extent that the predicted outcome of some bargaining situation is entirely determined by its set of winning coalitions, all bargaining situations with the same MIWs have the same predicted outcomes. Note that, for all Case C “Top Three” legislatures in the above typology, MIWs take the form (1, 1, 1, 0 , ……0). Note also that, when there is a system dominant party in an *n*-party legislature, then the MIWs take the form (*n-2*, 1, 1, …1), with a winning threshold of *n-1* and a total weight of *(n-2)* + *(n-1) = 2n - 3*. In other words, all of the *n-1* smaller parties have equal (unit) weight and can form a winning coalition by combining, while the largest party can form a winning coalition with any one of them. Thus a system dominant party may have super-proportional bargaining expectations under the Baron-Ferejohn model. This will happen when

$$\frac{s_1}{M} < \frac{n-2}{2n-3}$$

This inequality means that a system dominant party may have super-proportional Baron-Ferejohn Expectations (BFEs) if its seat share is below the following thresholds:

N of parties	Max seat share for super-proportional BFEs
4	0.400
5	0.429
6	0.444
7	0.455
8	0.462
9	0.467
10	0.471

Combining the two tables above, we see that the range of seat seat for which the largest party can be both system dominant and have super-proportional BFEs narrows steadily as the number of parties increases. This range is quite wise for four-party systems (0.250< s_1 < 0.400) and quite narrow for 10-party systems (0.438 < s_1 < 0.471).

Snyder et al. go one stage further than this, however, extending the Baron-Ferejohn bargaining model to argue that the *a priori* bargaining expectations of the actors are proportional to their MIWs. Part of their proof of this proposition depends upon the assumption that the probabilities with which actors are recognized to make proposals are ***proportional to their MIWs***, an assumption that may be somewhat dubious. Think of a 100-seat legislature such as (43,23,15,8,7,4), with minimum integer weights (8,3,3,3,2,1). We need a story telling us why recognition probabilities and payoffs will be proportional to the MIWs in this case, which bear only an obscure relationship to seat shares. In particular, it is difficult to think of a ***recognition rule*** under which the probability a legislator is recognized to make a proposal is proportional to the MIW of the party to which s/he belongs.

Setting these concerns on one side for a moment, the Baron-Ferejohn interpretation does nonetheless give us a clear intuition a good reason why the Shapley-Shubik index may indeed over-weight the bargaining expectations of dominant parties and thus be suspect in this context. Consider a legislature such as (35, 17, 16, 16, 16) for which the largest party is system dominant under simple majority rule and for which the minimum integer weights are (3,1,1,1,1). If *a priori* bargaining expectations are proportional to MIWs, as Snyder et al. claim, then the *a priori* expectation of the largest party is (4/7 =) .429, and the expectation vector for this legislature is (.429, .143, .143, .143, .143). Since the largest party is pivotal at every position in the ordering except first and last, the Shapley vector for this legislature is (.6, .1, .1, .1, .1). The Baron-Ferejohn model suggests that the largest party can expect three times the payoff of the other parties; the Shapley value suggests that the largest party can expect six times the payoffs of the other parties.

The intuition behind the Baron-Ferejohn predictions is that, assuming no time discounting and a non-zero recognition probabilities for all parties, then the structure of the set of winning coalitions means that each of the smaller parties has a “reservation price” of .25 of the total payoff when made an offer by any party. This is because each knows the worst case is that it can wait to be recognized and then propose a winning coalition of the four smallest parties, sharing the payoff equally between them. Since this is common knowledge, the largest party, if recognized first, will offer .25 to an arbitrary one of the smallest parties. Similarly, any one of the smaller parties, if recognized first, will either offer: .75 to the largest party, keeping .25 for themselves; or .25 to each of the other smaller parties, keeping .25 for themselves; or will use a strategy that mixes these two options. Thus, in any given coalition in which it participates with one of the smaller parties, the largest party can expect three times as much as its coalition partner, while each of the smaller parties has the same expectation as each of the others.

The reason, on the Baron-Ferejohn logic, that the Shapley vector “over-weights” the largest party is that that Shapley-Shubik index increases the bargaining power of the largest party for every winning coalition in which it is pivotal. However some of these coalitions – in particular in this example the set of coalitions between the largest party and ***two*** of the smaller parties – are predicted by the Baron-Ferejohn model never to form since, containing at least one surplus party, no rational actor would ever propose such a coalition. In effect the Baron-Ferejohn model predicts that only minimal winning coalitions will form, and thus implies parties cannot enhance their bargaining power, as they do on the Shapley-Shubik argument, by being pivotal in non-minimal winning coalitions.

The crucial substantive "new institutionalist" insight from the Baron-Ferejohn model is that bargaining expectations are conditioned by the institutional structure that determines the probabilities with which actors are recognized to make proposals – the recognition probabilities. Thus proofs about *a priori* (as opposed to *ex post*) bargaining expectations depend critically on assumptions about recognition probabilities and sequencing. Snyder et al. assume sequencing is random, subject to recognition probabilities that are proportional either to MIWs or are equal for all parties – assumptions that are contradicted by empirical evidence offered by Diermeier and Merlo, who show the assumption with the best empirical fit is that recognition probabilities are proportional to ***seat shares***. (Diermeier and Merlo 2004).

To see why both the sequencing and probabilities of recognition are important, imagine in our running example that, whatever the recognition probabilities, ***we knew for certain*** (perhaps because of some procedural rule) ***that the largest party would be recognized first to make a proposal***. In the Baron-Ferejohn model, the first proposer always makes the Nash equilibrium proposal and the game ends, so the largest party would offer an arbitrary one of the smaller parties .25, keeping .75 for itself, and the game would end. If this proposal sequence was an inflexible rule, then the largest party would have an expectation of 0.75, and the four smaller parties would share the remaining expectation of 0.25 between the four of them. The *a priori* BFE would thus be (3/4, 1/16, 1/16, 1/16, 1/16) and the largest party would expect ***twelve times*** as much as any other the smaller parties.

Adding time discounting and thereby opening up the possibility wars of attrition, etc., between prospective government partners would also change everything. So things are not as simple as they might look!

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