Introduction to Engineering I
Lecture 6

Dimensional Analysis
Scaling
Studio Problems
Dimensional Analysis

- Fundamental Dimensions
  - Length L
  - Time T
  - Mass M
  - Absolute Temperature $\Theta$
  - Charge Q

- Units
  - SI (Metric) Units: m, s, kg, K
  - Non-SI Units: ft, hr, lbm, R, esu

- Dimensions ≠ Units!
Dimensions of Physical Quantities

• Mass
  • Unit is kilogram (kg)
  • Dimension is M

• Length
  • Unit is meter (m)
  • Dimension is L

• Time
  • Unit is seconds (s)
  • Dimension is T

• Charge
  • Unit is coulombs (C)
  • Dimension is Q

• Temperature
  • Unit is Kelvin (K)
  • Dimension is Θ
Dimensions of Physical Quantities

- **Velocity**
  - Unit is m/s or m \cdot s^{-1}
  - Dimension is L\text{T}^{-1}

- **Momentum**
  - Unit is kg \cdot m \cdot s^{-1}
  - Dimension is M\text{L}\text{T}^{-1}

- **Force**
  - Unit is N = kg \cdot m \cdot s^{-2}
  - Dimension is M\text{L}\text{T}^{-2}

- **Energy**
  - Unit is J = N \cdot m = kg \cdot m^2 \cdot s^{-2}
  - Dimension is M\text{L}^2\text{T}^{-2}

- **Pressure**
  - Unit is Pa = N/m^2 = kg \cdot m^{-1} \cdot s^{-2}
  - Dimension is M\text{L}^{-1}\text{T}^{-2}

- **Power**
  - Unit is W = J/s = kg \cdot m^2 \cdot s^{-3}
  - Dimension is M\text{L}^2\text{T}^{-3}

- **Density**
  - Unit is kg/m^3 = kg \cdot m^{-3}
  - Dimension is M\text{L}^{-3}

- **Torque**
  - Unit is N \cdot m = kg \cdot m^2 \cdot s^{-2}
  - Dimension is M\text{L}^2\text{T}^{-2}
Using Dimensional Analysis

• Dimensions on both sides of an equation must match
• Useful in determining functional relationships
• Idea:
  • \( X = f(M, L, T, \text{ etc.}) \)
  • Dimensions of \( X \) = Dimensions of \( f(M, L, T) \)
  • By matching the dimensions, we can determine the form of \( f \)
• \( X \propto T \)
  • Read as "\( X \) is proportional to \( T \)"
  • This means \( X = kT \), where \( k \) is a constant
Example: Force = f(?)

- Plausible variables:
  - Mass
  - Velocity
  - Acceleration

- Unlikely variables
  - Time
  - Temperature

- Hypothesis: Force = f(mass, velocity, acceleration)
- Use dimensional analysis to determine the relationship
Example: Force = f(?)

- Dimensions of variables
  - Force = F; dimensions are MLT\(^{-2}\)
  - Mass = m; dimension is M
  - Velocity = v; dimensions are LT\(^{-1}\)
  - Acceleration = a; dimensions are LT\(^{-2}\)

- Suppose \( F \propto m^\alpha v^\beta a^\gamma \)

- Dimensions must match:

\[
MLT^{-2} = (M)^\alpha (LT^{-1})^\beta (LT^{-2})^\gamma
\]
\[ F \propto m^{\alpha} v^{\beta} a^{\gamma} = MLT^{-2} = (M)^{\alpha}(LT^{-1})^{\beta}(LT^{-2})^{\gamma} \]

- Dimensions on each side must match:
  - \( M^1 = M^\alpha \quad \Rightarrow \quad \alpha = 1 \)
  - \( L^1 = L^\beta L^\gamma \quad \Rightarrow \quad \beta + \gamma = 1 \)
  - \( T^{-2} = T^{-\beta} T^{-2\gamma} \quad \Rightarrow \quad \beta + 2\gamma = 2 \)

- Solution: \( \alpha = 1, \beta = 0, \gamma = 1 \)

- \( F \propto m^1 v^0 a^1 = ma \)
Example: Period of Pendulum

• Plausible variables:
  • Mass $m$
  • Length $l$
  • Acceleration of gravity $g$

• Unlikely variables
  • Time
  • Temperature

• Hypothesis: Period = $f(\text{mass, length, acceleration})$
• Use dimensional analysis to determine the relationship
Example: Period of Pendulum

- Dimensions of variables
  - Period = \( t \); dimension is \( T \)
  - Mass = \( m \); dimension is \( M \)
  - Length = \( l \); dimension is \( L \)
  - Acceleration = \( g \); dimensions are \( LT^{-2} \)

- Suppose \( t \propto m^\alpha l^\beta g^\gamma \)

- Dimensions must match:
  \[
  T = (M)^\alpha (L)^\beta (LT^{-2})^\gamma
  \]
\( t \propto m^{\alpha}L^{\beta}g^{\gamma} = T = (M)^{\alpha}(L)^{\beta}(LT^{-2})^{\gamma} \)

- Dimensions on each side must match:
  - \( M^{0} = M^{\alpha} \Rightarrow \alpha = 0 \)
  - \( L^{0} = L^{\beta}L^{\gamma} \Rightarrow \beta + \gamma = 0 \)
  - \( T^{1} = T^{-2\gamma} \Rightarrow -2\gamma = 1 \)

- Solution: \( \alpha = 0, \beta = 1/2, \gamma = -1/2 \)

- \( t \propto m^{0}L^{0.5}g^{-0.5} \)

\[ t \propto \sqrt{\frac{l}{g}} \]
Scaling

Objects have a characteristic linear dimension
Scaling – shape is preserved

Surface area $\propto L^2$
Volume $\propto L^3$

Area $A$
Volume $V$

$L' = \alpha L$
$A' = \alpha^2 A$
$V' = \alpha^3 V$

The symbol $\propto$ means “proportional to”
Scaling

Area $A$
Volume $V$

Area $9A$
Volume $27V$
Scaling

Linear dimensions $\propto L$ (shape is preserved after scaling)
Surface area $\propto L^2$
Volume $\propto L^3$
Importance: some properties $\propto$ area, others are $\propto$ volume

Proportional to Area:
  - Strength
  - Cooling rate

Proportional to Volume:
  - Mass
  - Energy production in combustion/metabolism
Consequences

• One large ice cube take longer to melt than the same mass divided among smaller cubes

• Animals cannot be arbitrarily large
  • If you scale $L$ by 10 $\times$, bone and muscle strength goes up by 100 $\times$, but weight goes up by 1000 $\times$
  • Ratio of mass to strength increases by 10 $\times$
  • Imagine carrying nine people your size and weight
  • Conversely, insects can carry large loads in comparison to their weight

• Warm-blooded animals cannot be arbitrarily small
  • If you scale $L$ by 0.1 $\times$, cooling rate ($\propto$ area) goes down by 0.01 $\times$ and ability to heat body ($\propto$ mass) goes down by 0.001 $\times$
  • This is why small mammals and birds must eat all the time