

# Monticello Transient Conduction Problem

## Introduction

Even though their “R” value  $\left( = \frac{L}{k} \left[ \frac{\text{hr } ^\circ\text{F}}{\text{Btu}} \right] \right)$  is quite low, the thick masonry walls of an old structure like Monticello have an enormous effect on the heating and cooling load of the building. Mr. Jefferson and his contemporaries in the early nineteenth century didn't understand the insulating value of air immobilized in fibrous materials. We exploit that phenomenon today in the form of fiberglass, foams and other lightweight insulations. Nevertheless they, and earlier civilizations all over the world, certainly knew about and exploited the effects of thermal mass. In more recent years manufacturers of fiberglass insulation have compared the “R” value of their product with that of brick and masonry and have claimed correctly that it is vastly superior. The masonry industry has countered (again correctly) that in assessing the value of its products, you



Figure 1. Monticello, Photo courtesy J.H. Arthur

must consider transient behavior, that is, the thermal mass, and that the “R” value, a steady-state property, is therefore, not the appropriate measure. We will study the matter here.

At Monticello the walls are made of 16" of solid brick covered directly with 1" of plaster on the inside. You can see this same construction “au naturel” at Mr. Jefferson’s retreat, Poplar Forest, which is currently under restoration outside Lynchburg, Virginia. This is certainly not a construction you would find in most contemporary residences, but some modern homes do incorporate thermal energy storage in the form of poured concrete or rocks. In this problem we will study the effect of this large thermal mass by numerically solving a simple 1-D, transient conduction problem with appropriate property values and boundary conditions.

This problem is a relatively simple example of how the textbook charts based on analytical solutions to the transient, one-dimensional conduction equation are often not applicable. While the analytical solutions on which the charts are based are only appropriate for a sudden, step change in thermal boundary conditions, in this case one of the boundary conditions we wish to apply is a periodic function of time of day. Also, the two sides of the slab are exposed to different convection conditions, a scenario that once again is not covered by the standard “chart” solutions.

## Learning Objectives

When you have finished this project you should be able to:

1. “Discretize” a one-dimensional, transient problem,
2. Write appropriate heat balance equations for both internal and surface cells,
3. Be able to express the heat balance equations in both explicit and implicit form and be able to recommend which to use for a particular problem,
4. Set up and solve the appropriate heat balance equations using a spreadsheet,
5. Compute surface heat fluxes from computed results and explain how this calculation can help in verifying your solution,
6. Present, justify and explain computed results,
7. Be able to explain the difference between thermal resistance and thermal capacitance.

## Implementation

In this project we will develop a 1-D, finite-volume model to study the transient storage effects in Monticello’s walls. For the wall itself we will use a grid spacing of .0432m (1.7”), giving nine full-sized nodes plus half-sized nodes at each end (See sketch). That setup means 11 unknown temperatures and three separate predictive equations that you will have to derive. We will use the thermal properties of face brick (the ones without the holes) for the whole thickness. (You will notice in the property tables that face brick has a higher density and higher thermal conductivity because it lacks the dead air space in the holes. You can use the specific heat for common brick.)

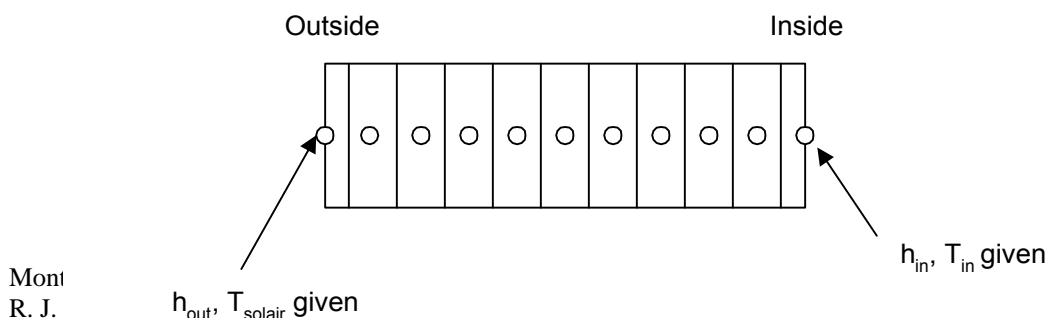


Figure 2. Nodalization of Wall

At the inside surface let us assume  $h = 8 \text{ W/m}^2 \text{ K}$  with a constant air temperature of  $21^\circ\text{C}$ . (We are assuming that Mr. Jefferson keeps his air conditioning set at this value.) For the outside we must account for both radiative and convective gains and losses. Rather than treat these gains and losses separately, it is common (and vastly easier) to use an effective *Sol-Air* temperature, which, when put into the standard Newton's law of cooling ( $q = h A \Delta T$ ), has the equivalent effect. The Sol-Air temperature is a function of the date, hour, latitude, and the radiative properties and orientation of the wall with respect to the sun. We will use values for a dark, west-facing wall at  $40^\circ \text{ N}$  latitude on July 21. Hour-by-hour values for these conditions as given in the ASHRAE *Handbook of Fundamentals* are as follows:

24.4, 24.4, 23.8, 23.3, 23.3, 25.0, 27.7, 30.0, 32.7, 35.0, 37.7, 40.0,  
53.3, 64.4, 72.7, 75.5, 72.2, 58.8, 30.5, 29.4, 28.3, 27.2, 26.1, 25.0

Here the first value ( $24.4^\circ\text{C}$ ) corresponds to 1:00 a.m.; the last value ( $25.0^\circ\text{C}$ ) corresponds to midnight. Note that the effective Sol-Air temperature rises to nearly  $170^\circ \text{ F}$  ( $75.5^\circ\text{C}$ ) in the afternoon. (We are assuming there are no trees shading this wall.) If you wind up using a timestep of less than an hour, it would be advisable to fill in the intermediate values using linear interpolation. Otherwise the heat flux vs. time plot graph you eventually plot will exhibit some non-physical "kinks" in response to the step change in boundary condition every hour.

Use a convection coefficient  $h = 18 \text{ W/m}^2 \text{ K}$  at the outside surface. (The higher outdoor value reflects the effect of breezes that are presumably not present indoors.) You can start your simulation from arbitrary initial conditions, but make a reasonable guess. Because you don't know the actual temperature distribution at the time you begin your simulation, you will have to run your simulation through several days to obtain a steady, periodic condition. (Hint, determine the characteristic conduction time of the wall  $\left(\frac{L^2}{\alpha}\right)$  to give you an estimate of that time. This estimate can be found by remembering that the time constant for an electrical circuit consisting of a resistor and a capacitor is given by  $t = RC$ . In our wall the thermal resistance is  $R = L/Ak$  and the thermal capacitance is  $C = AL\rho c_p$  and, of course, they are distributed and commingled, not lumped.) For Mr. Jefferson's wall  $L = .432 \text{ m}$  (17") and you can compute  $\alpha$ , the thermal diffusivity, from the thermal conductivity ( $k$ ), density ( $\rho$ ) and specific heat ( $c_p$ ) values you find for brick. (Assume the 1" plaster layer on the inside has thermal properties identical to those of brick.)

After you have completed the calculation, plot for one full day the heat transfer at both the inner and outer surface as a function of the time of day. This means that in order to make the plot, you will have to save at least the predicted temperature of the outside surface and that of the inside surface for each hour of the last day of your simulation. The heat transferred at the inner wall divided by the overall heat transfer coefficient for the wall will give you the hourly "cooling load temperature difference" (CLTD) commonly used in load estimating.

In this problem you have convective boundary conditions at both sides and a time-dependent  $T_\infty$  at the outside. If you were doing this project in a high-level programming language, you would need an outer repetition structure ("Do Loop") covering the number of days you need to run. The next Do Loop in will cover the 24 hours of the day. Whether or not there is yet another one inside this depends on whether you decide to implement an *explicit* or *implicit* solution. When you compute the allowable explicit timestep for the given material properties and

grid spacing – and you do need to check it both for internal cells and for the surface cells – you will find it to be less than one hour. If you do decide to implement an explicit solution, then pick a timestep that gives an integral number of steps in each hour and is stable. Inside this loop will be another Do Loop to cover the nine internal nodes, plus the special equations needed at each end. Your text includes the derivations of the explicit forms of the appropriate predictive equations for both internal cells and for the half-cells at the two surfaces exposed to convection and discusses the time step limitation. Your report must include the derivations of these equations.

If you want to see the enormous effect of the direct solar gain, you may want to try using Sol-Air data for a dark, north wall in July under the same conditions as a comparison:

24.4, 24.4, 23.9, 23.3, 23.3, 32.3, 32.2, 30.6, 32.8, 35.0, 37.8, 39.4,  
41.1, 41.1, 41.1, 40.0, 42.2, 41.7, 30.6, 29.4, 28.3, 27.2, 26.1, 25.0

## Reference

2001 ASHRAE Handbook - Fundamentals, American Society of Heating, Refrigerating and Air-Conditioning Engineers, Inc., Atlanta (2001).

## Appendix I - Some Specific Hints

### Spreadsheet with *Explicit* Differencing

This problem is very easy to set up on a spreadsheet. Set up 11 columns for the nodal temperatures plus a column for the  $T_{\infty}$  specified earlier as the  $T_{\text{solair}}$  outside, and another for the (constant) indoor temperature. Each row (and there will be a whole lot of them!) will represent a different time. Essentially, you will need a row for each time step, and that will mean several days times 24 hours per day times the number of time steps you need per hour. At the top of your spreadsheet, you probably will want to input certain parameters, e.g., the grid Fourier number, the grid Biot number at the outside and that at the inside. When you use those input parameters in the cell formulae, don't forget to use absolute, as opposed to relative, addresses, or better yet, "name" the cells containing these parameters. You will be duplicating the predictive formulae many, many times. This approach isn't very elegant, but it works, and it is easy to generate the necessary plots at the end from the final day's results.

As a first step in your development process you may want to just set the temperature of the cell corresponding to the surface cell to the current "Sol-air" temperature and that corresponding to the inside surface cell to the interior temperature. Then once you are sure you have the formulae for the interior cells (2-10) correct, you can implement the special forms you have derived for the surface cells. (Essentially you will be temporarily substituting BC's of the 1<sup>st</sup> kind for BC's of the 3<sup>rd</sup> kind.)

One way to check that your calculation has converged to a steady, periodic condition is to calculate the surface heat flow at both surfaces at each time step and sum each of them over every day. The daily sum at the inside and that at the outside will not match well at all for a couple of days, but eventually they should match very well.

### Spreadsheet with *Implicit* or *Semi-implicit* Differencing

You can implement an implicit or semi-implicit scheme on a spreadsheet also. You will probably pick a timestep of an hour, since that is the frequency at which the  $T_{\text{solair}}$  values are provided, rather than the fraction of an hour needed if you use an explicit method. Write out the heat balance equations (again one for the interior points and special forms at the half-cells at the surfaces) and use a weighted average of the heat fluxes at the beginning of the timestep and those at the end. Equal weighting ( $\gamma = 0.5$ ) is fine. Solve those equations for the new value of the temperature at the point where you did the heat balance, making sure to include on the left side, both the term coming from the time derivative and the term arising from the flux terms evaluated at the new time level. Then divide through by the resulting coefficient to give your predictive equation. Input the governing equations (internal and special forms at the ends) for the unknown temperatures. The predictive equation for the internal points each will involve its two unknown neighboring temperatures plus old (known) values at itself and (if you are using a semi-implicit formulation) the same two old neighbors. (We say this semi-implicit formulation involves a 6-point *stencil*.) Your spreadsheet will balk at these forms since they involve unknown values - and maybe put up a flag that says “circular reference,” but all spreadsheets have a recalculation or iteration option to handle situations like this. (This option can be used gainfully for elliptic (boundary-value) equations also - like those that describe 2-D steady-state conduction.)

Some spreadsheets allow the user to specify the desired sweep direction for the iteration – across, up and down, or both. If yours does have that option, you should sweep across only. You are solving a “parabolic” partial differential equation here. Time is a “one-way” coordinate and once you have gotten a solution at a particular point in time, there is no justification for repeatedly recalculating it.

As noted above for the explicit formulation, you may want to just set the surface temperatures to the fluid temperatures ( $T_{\text{solair}}$  on the outside and the fixed interior temperature on the inside) until you have the calculation for the internal cells working properly. You might also want to implement the daily heat balance calculation discussed above to help you gain confidence in your algorithm.

Even with your semi-implicit formulation, you will still wind up with a lot of rows – 24 x the number of days you determine it takes to get to a steady, periodic condition. Another approach is to use only 25 rows and automatically transfer the end-of-day results back up to the row representing the beginning of the day. With this scheme you will obtain the steady, periodic condition, but will not know how many days it took to achieve it.

## Appendix II – Periodic Conduction in a Semi-infinite Body

You can gain some additional physical feeling for the effects of thermal mass by running the HTTsemicalc.xls spreadsheet for Periodic Conduction in a Semi-infinite body. A “semi-infinite” body is one thick enough that any change in the thermal boundary condition

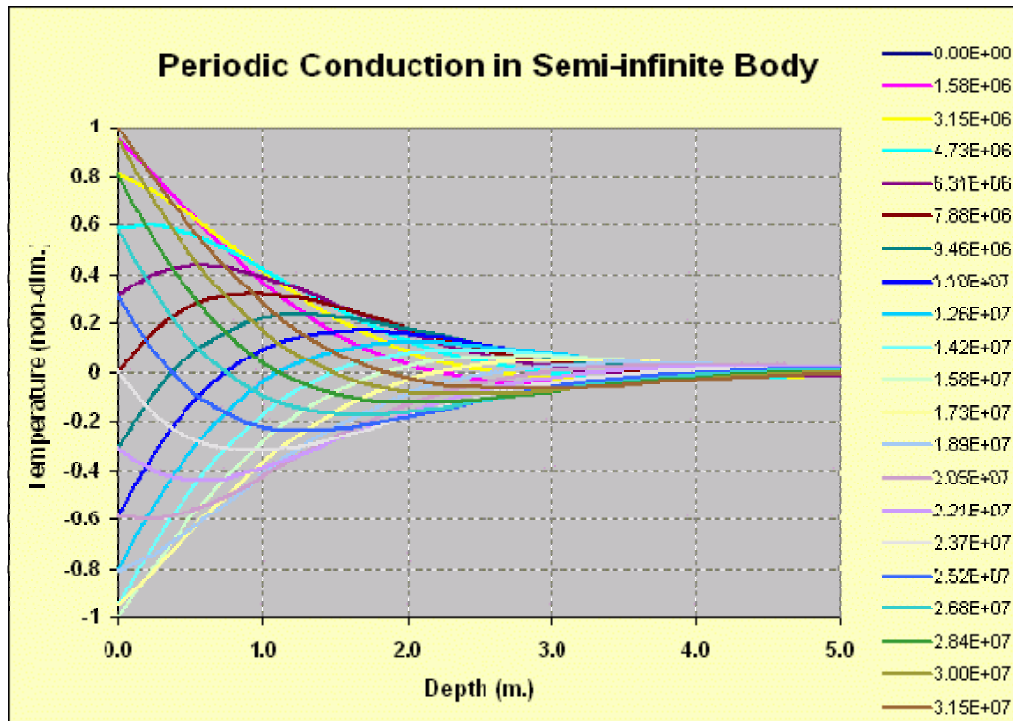


Figure 3. Sample Output from Periodic Conduction Spreadsheet - One year ( $31.5 \times 10^6$  seconds) is covered by the calculation. Thermal properties typical of soils are used.

implemented on one side never penetrates to the other boundary. This workbook, an implementation of an analytical solution [2], allows you to change the thermal properties of the material, the period of the sinusoidal surface temperature variation, etc. Knowing the amplitude of the annual surface temperature variation and the thermal properties of the soil, one can in theory determine how deeply a water pipe should be buried to lie safely below the frost line (but consult your local building code and your local utility before you dig!).

### Reference

Gebhart, B., *Heat Transfer*, McGraw-Hill, New York (1971).