Integrability and τ-functions on
Random Walkers & Isomonodromy Deformation Systems

By

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B.S. (Columbia University) 2007

DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

in the

OFFICE OF GRADUATE STUDIES

of the

UNIVERSITY OF CALIFORNIA

DAVIS

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2016

A
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Axel I. Sáenz Rodríguez
June 2016
Mathematics

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Abstract

We consider three types of integrable models which happen to lie under the classification of random walkers or isomonodromic deformations. We deal with an infinite queue that moves forward at random times, indecisive walkers on a circle walking left with probability p and walking right with probability $1-p$ at any given time, and a deformation of a matrix multi-valued holomorphic function on the projective line which preserves the monodromy structure of the function. Particularly, we work with the *Plancherel growth process* where we write the Schur generating function and show that it is a KP τ-function. Also, we work with the *asymmetric simple exclusion process (ASEP)* on a finite ring lattice and prove that the Bethe ansatz gives a complete set of eigenfunctions for generic parameters. Lastly, we write a solution and the τ-function to the *Painlevé 1* isomonodromy system using the Eynard-Orantin topological recursion. Additionally, we include an introductory background for integrable systems and an appendix that gives a resolution of a family of singular maps through blow-ups, which is used for the proof on the Bethe ansatz.

Acknowledgments

Thank you. Gracias.

Le doy las gracias a mi familia que siempre me han apoyado intelectual, espiritual, y fsicamente . Papi, Mami, Kris, y Cici, gracias.

I'll like to thank both Motohico Mulase and Craig Tracy who have introduced me to the research area in this work. Along with Dmitry Fuch and Albert Schwarz, they have always had their doors open to me when I wanted to discuss math. The time with them was instrumental in my research. I'm also thankful for great and stimulating conversations with my fellow grad students: Henry Kvinge, Swati Patel, Robert "Spiggets" Beck, Calina Copos, David Stein, Carson Rogers, George Mossesian, Gustavo Navarro Sepulveda, David Renfrew, and Federico Castillo.

In my work, I was fortunate to collaborate with Norma Do, Eric Brattain, and Kohei Iwaki. Work with them was loads of fun. Thank you, guys.

While writing my thesis, I consulted with Li-Cheng Tsai, Sophie Renner, and David Sutherland. I met them all at the Kavli Institute of Theoretical Physics (KITP) where I spent some time in the last year of my Ph.D. I like to thank them for clarifying many unfamiliar concepts in math and physics. Also, this research was supported in part by the National Science Foundation under Grant No. NSF PHY11-25915 in my stay at KITP. My time in KITP was wonderful.

Throughout my years in graduate school, I've been fortunate to have myriad sources of funding: the GAAN Fellowship, the National Science Foundation under grant DMS-0955584 (from Dan Romik), UC Davis Graduate Research Mentorship Fellowship, and UC Davis Dissertation Year Fellowship. This work would not have been possible without such support and for that I am thankful.

Last but not least, everyone knows that hard work can't get done without hard partying. I'd like to thank my friends whom helped me party: Arib Rahman, Prunil Patel, Dan Lee, Blake "Tito" Pollard, Alex "Bogey" Boger, Raquel Araujo, Rose Levenson-Palmer, Nicanor Rocha, Bogdan Copos, Ramsey Shadfan, Sophia Benkiran, Linda Tran.

CHAPTER 1

Introduction

We work with the evolution of three systems: the Plancherel growth process, the asymmetric simple exclusion process (ASEP) on a lattice ring, and the Painlevé 1 isomonodromy system. In particular, we are interested in the dynamics of these models and we consider in detail the evolution/deformation equations for each model. In this chapter, a short introduction to all of these models is given along with the statement of the results given in the subsequent chapters. The aim of this thesis is to give a self-contained exposition of the results stated in this introduction starting from the theory of *Integrable Systems*. Moreover, we place a spotlight on τ-functions, a type of generating function for an integrable system (see ch. 3).

1.1. Results

1.1.1. Plancherel Growth Model. The Plancherel growth model is a Markov process of random walkers on an infinite lattice line, which we take as the integers $\mathbb{Z}$. In particular, we have that at each location there can be at most one walker, each walker (after an "exponential clock" goes off) only moves only one spot to the right, and the walkers can't jump over another walker. Also, for the dynamics, it is important the starting position of the walkers (see [**Cor12**] for a review), and we consider the case when all the walkers start left of the origin. Moreover, we note that the Plancherel growth process is also known in the literature as a $2D$ interface growth model where boxes are randomly stacked on an infinite wedge [**KC03**, **Ker99**, **BOO00**]. More precisely, the process is defined on the irreducible representations of the symmetric group on n elements, S_n. Indeed, it a classical result by Schur and Young that the irreducible representations of S_n are indexed by Young diagrams (i.e. partitions of n) [**FH91**]. Then, the process of going from an irreducible representation of S_n to an irreducible representation of S_{n+1} is the Plancherel growth process, and the exact bijection between the random walkers and the $2D$ interface is given by a bijection between the configuration of the walkers and the Young diagrams.

In ch. 3, we construct the Plancherel growth process using an infinite fermionic Fock space with an action on it by an infinite Clifford algebra. The infinite fermionic Fock space corresponds to the limit of an infinite wedge product of an infinite dimensional vector space, and from the same infinite dimensional vector space we define the Clifford group that acts on the infinite fermionic space. Moreover, along with this construction we can define an action of a central extension of $GL(\infty)$ on the infinite fermionic Fock space, which allows us to define the Sato Grassmanian, a variety that parametrizes infinite subspaces of an infinite vector space. As such, we gain a rich geometric structure, and we define the KP τ-function, a function that was first studied in the context of soliton theory in the seventies and eighties [**AS81**, **Hir04**, **SW85**, **ZK65**, **Sat81**]. In our case, we show that the Schur generating function for the configuration of the Plancherel growth process at time t is a τ-function. Thus,

THEOREM 1.1.1. *The Plancherel growth process is KP integrable.*

Again, we note that in the past there has been work on the Plancherel growth process focusing on representation theory, analysis, and probability. In particular, we like to note some of the previous work on this model by Vershik and Kerov [**KC03**] and, independently, by Logan and Shepp [**LS77**] in the seventies. Namely, we have that the Plancherel growth process obeys the arcsine law. This a result in probability theory that show the convergence of the measure on the irreducible representation of S_n, after a rescaling, to a delta function, or rather that, under this process, the limit shape of the Young diagrams, after a rescaling, is given by an arcsine function. This a result analogous to central limit theorem, and it has caused vast research which (which we don't mention here) in the area since then.

1.1.2. ASEP Model. The ASEP model is a continuous time Markov process. Particles lie on a lattice line, no two of which lie on the same spot at the same time. After an "exponential clock" goes off, a particle decides to jump left or right with probability p or $1-p$, respectively. If the adjacent spot to the particle is occupied, said particle will remain in its place. In [**TW09**], it is shown that, given the initial conditions of an infinite line with all the particles to the right of the origin occupied at time $t=0$, the m^{th} particle from the left will have the Tracy-Widom distribution with the $t^{1/3}$ scaling for $\frac{m}{t} \in (0,1)$ as $t \to \infty$; one of the first results, in a general setting, establishing the KPZ universality class, a major area of ongoing research which describes the limiting behavior of many models further discussed in ch. 3.

In ch. 4, based on work with E. Brattain and N. Do [**BDS15**], we consider the ASEP model with periodic boundary conditions. One can think of this as the ASEP model on a ring lattice of size L and N particles, and one wishes to consider the asymptotics, as $L \to \infty$ with N/L fixed, to uncover the scaling limits. The first step to compute the asymptotics is to diagonalize the Hamiltonian of the model, which has been done for some finite values of L [**HNS13**] by the use of the *Bethe ansatz*, but this is not sufficient in the limit $L \to \infty$.

The Bethe ansatz, established in 1931 [**Bet31**], diagonalizes a Hamiltonian by "guessing" the eigenfunctions of the model. In the ansatz, one assumes that all the particles in the model are far enough from each other so that there is no interaction among the particles, and thus, one solves the non-interacting model and accounts for the interaction through boundary conditions. The Bethe ansatz approach has proven to be useful in many other models such as XXZ chain, the eight and six-vertex model. Despite its broad applications, it was not known whether the Bethe ansatz provides a complete eigen-basis for a general Hamiltonian. In our work, we settle this issue:

THEOREM 1.1.2 (Brattin-Do-Saenz [**BDS15**]). *For any L, N, and generic p, the Bethe ansatz gives a complete basis for the periodic ASEP model, diagonalizing the Hamiltonian for the model.*

Prior to our work, there was progress on this area by R. Baxter [**Bax07**], R. Langlands [**LSA95**], and many others which we couldn't mention here. Our work was heavily influenced by the work of R. Langlands in [**LSA95**]; we use Blow-Ups to desingularize the maps counting the number of basis elements given by the Bethe ansatz, the Lefschetz Theorem from algebraic topology to establish a combinatorial equation counting the basis elements, and the combinatorics of weighted planted forests to get the final count to prove our result.

1.1.3. The First Painlevé Equation and Quantum Curves. Given a matrix multi-valued holomorphic function on a punctured sphere $Y : \Sigma \to M(n, \mathbb{C})$, one may define a representation of the fundamental group of the puncture sphere Σ to the matrix algebra $M(n, \mathbb{C})$ which describes the multi-value of the function Y (i.e. how the function changes as one follows the function around a puncture), and we call this representation the *monodromy representation*. In the work of [**JMU81**,

JM81], the authors considered the moduli space of such monodromy representations (i.e. the space that parametrizes the monodromy representations), and determined conditions/equations that keep the monodromy representation fixed, which is called an isomonodromic deformation. Moreover, the authors showed that isomonodromic systems are intregrable and also defined the τ-function in this context.

In ch. 5, we focus on the first Painlevé equation [**Pai02**] given by

$$\frac{d^2q}{dt^2} = 6q^2 + t,$$

where a solution, $q(t)$, is called a *Painlevé Transcendent*. This is a consistency equation for an isomonodromic deformation system [**JMU81**, **IN86**] with Lax pair equations

$$\frac{\partial \Psi}{\partial x} = \begin{pmatrix} p & 4(x-q) \\ x^2 + qx + q^2 + \frac{t}{2} & -p \end{pmatrix} \Psi \overset{def}{=} A(x,t)\Psi \tag{1.1.1}$$

$$\frac{\partial \Psi}{\partial t} = \begin{pmatrix} 0 & 2 \\ \frac{x}{2} + q & 0 \end{pmatrix} \Psi \overset{def}{=} B(x,t)\Psi. \tag{1.1.2}$$

There are in total six Painlevé equations, which have gained popularity in the mathematics and physics communities [**MTW77**, **JMU81**, **Tra78**], with applications to *Statistical Mechanics* and *Interacting Particle Systems*. Based on work with K. Iwaki, we explicitly apply the Eynard-Orantin topological recursion (see ch. sec. 5.3 or [**EO07**] for further detail) for the Lax pair and construct a solution for the Lax pair. In particular, the Eynard-Orantin topological recursion computes a "partition function" given a Riemann surface, called a spectral curve, along with a distinguished 1-form. Moreover, in many cases, [**DM14b**, **DM14a**, **DM13**, **MS12**, **Sch15**, **DBMN⁺13**, **LMS13**, **Zho12**], the partition function computed by the Eynard-Orantin topological recursion (formally) satisfies a differential equation which is related to the spectral curve through the semi-classical limit in WKB-analysis. This differential equation arising from the spectral curve has been coined as *quantum curve*. Then, in work along K. Iwaki [**IS15**], we compute the quantum curve for an $\hbar$-deformed isomonodromic system for the Painlev'e 1 equation. Specifically, we proved:

THEOREM 1.1.3 (Iwaki-Saenz [**IS15**]). *Given the Spectral Curve*

$$y^2 = 4(x - q_0)^2(x + 2q_0). \tag{1.1.3}$$

One defines,

$$\psi(x(z), t; \hbar) := \exp\left(\sum_{g \geq 1, n \geq 1} \frac{\hbar^{2g-2+n}}{n! 2^n} \int_{-z}^{z} \cdots \int_{-z}^{z} W_{g,n}(z_1, \ldots, z_n) \right)$$

*via the Eynard-Orantin Topological Recursion (see ch. sec. 5.3) with the $W_{g,n}(z_1, \cdots, z_n)$'s, the symmetric n-linear forms on n-copies of the Spectral Curve defined by the recursion. Then, $\psi(x, t; \hbar)$ is a solution to the $\hbar$-deformed isomonodromic system. Also, we have that τ-function for Painlev'e 1 ismonodromy system is given by the generating function of the free energies $F_g = \frac{1}{2\pi i(2-2g)} \oint_{\gamma_0} \Phi(z) W_{g,1}(z)$, for $g \geq 2$ (with F_0 and F_1 given by special definition see [**EO07**]), where $\Phi(z) = \int_{z_o}^{z} y(z) dx(z)$ with z_o a generic point. That is,*

$$\log \tau(t, \hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(t).$$

1.2. Outline

It is our goal to reach our result in most logically efficient way starting from the theory of integrable systems while exposing many of the interesting aspects of the τ-function and our methods.

In Chapter 2, we begin by introducing the theory of integrable systems. We focus on two types of integrability Liouville integrability and Frobenious integrability, noting the role of Liouville integrability in classical mechanics. Also, we take a quick detour into quantum mechanics, where there is no good definition of integrability, but instead one considers exactly solvable models, which roughly speaking are models where the dynamics equations can be explicitly written down. In the last two sections, we introduce isospectral deformations and ismodromic deformations. These are systems of consistency equations that follow from integrability. The Planchere growth process is related to isospectral deformations, the ASEP model on a lattice ring is related to exactly solvable models, and the Painlevé model is related to isomonodromic deformations.

In Chapter 3, we introduce the theory of infinite fermionic Fock spaces. In this setting, we are able to define the Plancherel growth process and also give a definition of the KP τ-function. The majority of the chapter is dedicated to setting up the theory and many interesting and characteristic results from the theory are given. In particular, we set up the process of random walker on a graph with a exclusion rule (i.e. no two walkers may be on the same vertex at the same time). The Plancherel growth process, that we consider, is a particular case when such process is also a Markov process.

In Chapter 4, we introduce the ASEP model on a ring and the Bethe ansatz. We show that the Hamiltonian can be diagonalized (for generic parameters) through the Bethe ansatz. This is based on work done in conjunction with E. Brattin and N. Do [**BDS15**]. We use a versioon of the Lefschetz fixed point theorem from algegraic topology to turn this problem into a counting problem in combinatorics, where we end up counting weighted planted forest. This chapter is mostly self-contained and it doesn't require any results previously presented. Instead, we use techniques from algebraic topology, complex geometry, algebraic geometry, and combinatorics that we develop within the chapter except for the algebraic geometry, which we leave for the appendix.

In Chapter 5, we introduce the Painlevé 1 isomonodromy system along with the Eynard-Orantin topological recursion. We use techniques from WKB-analysis to match the "partition function" from the Eynard-Orantin topological recursion to the solution of the Painlevé 1 isomonodromic system. This is based on work done in conjunction with K. Iwaki [**IS15**]. This chapter is also mostly self-contained.

In the Appendix, we give a brief introduction to blow-ups in algebraic geometry. This material is most pertinent to our work on the Bethe ansatz on the periodic ASEP model. Indeed, we use the blow-ups to desingularize certain maps that are relevant in our results. Thus, we end up blowing-up, in a specific order, a family of subvarities of $(\mathbb{P}^1)^N$ for all $N \in \mathbb{N}$.

CHAPTER 2

Integrability

In this chapter, we begin the background/context of our work. Our results lie at the heart of "Integrable Systems", a vast area of mathematics that means a different thing to each person. Ask a geometer, and they will tell you about Lagrangian submanifolds on a symplectic manifold. Ask an analyst, and they will tell you about determinental formulas. Ask an algebraist, and they will tell you about representation theory and exactly solvable systems. The fact is that there are many notions of integrability out there, and even though this might seem like fault at first, it just goes to show the depth and interest in this field. We take a rather physical approach to this subject (i.e. from a physics point of view). We introduce two notions of "integrability": Liouville integrability and Frobenious integrability. Also, we take a detour and discuss similar ideas that arise in quantum mechanics. Although we will mainly be interested with Frobenius integrability, many ideas from the other concepts seem to be closely related to our work/results. The exact connection between these different notions of integrable systems seem to be considered case by case in the literature, and unfortunately we will not discuss this here in full detail. (See [**BBT03**, **Sut04**].) Instead, we aim to take the most direct path from what we consider to be the starting point of the ideas at play in our results to the results themselves. Thus, we quickly move from generalities to more and more technical ideas. We end this section with the integrability of isospectral and isomonodromic deformations, from our point of view, this is still quite general setting. Enjoy!

2.1. Integrable Systems

In this section, we introduce two notions of integrability that have spawned form physics. We begin with Liouville integrability, discuss related ideas in quantum mechanics, and finish with Frobenious integrability.

2.1.1. Liouville Integrable System. In the literature, integrable systems related to classical mechanics often lie in the scope of *Liouville integable system.* We use differential geometry to give a proper definition in this case. (We suggest the unfamiliar reader to read either [**GP10**, **Lee03**] for basic definitions of manifolds.) We review this subject following [**Lee03**]. Let us begin, with the following definition:

DEFINITION 2.1.1. *A symplectic manifold is a pair* (M, ω) *where* M *is an even dimensional smooth manifold and* $\omega \in \Omega^2(M)$ *is a closed non-degenerate differential 2-form.*

In particular, from the conditions on ω, we have that a symplectic manifold is orientable and that the volume form of the manifold is given by the n^{th} exterior product $\omega \wedge \cdots \wedge \omega$ where $2n$ is the dimension of M. Moreover, one of the most important and distinguishing results of symplectic manifolds is the Darboux theorem.

THEOREM 2.1.1 ([**Lee03**]). *Let* (M, ω) *be a* $2n$*-dimensional symplectic manifold. Then, for any* $p \in M$*, there are smooth coordinates* $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ *centered at* p *in which* ω *has the coordinate representation*

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i.$$

This result states that locally all symplectic manifolds are the same. We omit the proof, which can be found in most textbooks that deal with the subject (e.g. [**Lee03**]). In particular, we call the coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ the *canonical coordinates* or *Darboux coordinates*, even though they are not unique. Also, we note that in classical mechanics we have that the coordinates $(q_1, \ldots, q_n)$ represent the position coordinates, the coordinates $(p_1, \ldots, p_n)$ represent the momentum coordinates, and the symplectic manifold represent the "phase space" (i.e. the space of all configurations). Moreover, we are interested in the equations of motions of this system. First let us define, vector fields given any smooth function $f \in C^\infty(M)$ using the non-degenerate 2-form ω. Indeed, since ω is non-degenerate, it provides a bundle isomorphism $\hat{\omega} : TM \to T^*M$, from the tangent bundle to the cotangent bundle, and for any $f \in C^\infty(M)$ we define a vector field $X_f \in \Gamma(M, \Omega^\vee)$ such that

$$\omega(X_f, Y) = df(Y) = Yf,$$

called the *Hamiltonian vector field* of f. In the canonical coordinates, the vector filed is given by

$$X_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right)$$

A trivial, but important result, is that f is constant for any integral curve of X_f. That is, for any curve $\alpha : [0,1] \to M$ such that $X_f(\alpha(t)) = \frac{d\alpha}{dt}(t)$, then $f \circ \alpha : [0,1] \to \mathbb{R}$ is constant. The result follows from the observation that

$$X_f f = df(X_f) = \omega(X_f, X_f) = 0$$

In particular, a triplet (M, ω, H) where (M, ω) is a symplectic manifold and $H \in C^\infty(M)$ is a smooth function is called a *Hamiltonian system* where the *flow* of the system is the vector field $X_H \in \Gamma(M, \Omega^\vee)$ and the integral curves of X_H are the *trajectories*. Moreover, in the canonical coordinates, we have that any trajectory $\alpha(t) = (x_i(t), y_i(t))$ satisfies the differential equations

$$\frac{dx_i}{dt}(t) = \frac{\partial H}{\partial p_i}(x(t), y(t))$$
$$\frac{dy_i}{dt}(t) = -\frac{\partial H}{\partial q_i}(x(t), y(t)),$$

which are called *Hamilton's equation of motion.*

Now, let us consider functions that are constant on any trajectory of our Hamiltonian system, which we call a *conserved quantity.* We can characterize such function with the aid of the *Poisson bracket*, $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, which is defined by

$$\{f, g\} := \omega(X_f, X_g) \qquad f, g \in C^\infty(M).$$

In canonical coordinates, the Poisson bracket is given by

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}.$$

Thus, we have that $f \in C^\infty(M)$ is a conserved quantity if $\{f, H\} = 0$ since

$$0 = \omega(X_f, X_H) = df(X_H).$$

It turns out that such conserved quantities are deeply related to symmetries of the symplectic manifold. More precisely, we say that a smooth vector field $Y \in \Gamma(M, \Omega^\vee)$ is an *infinitesimal symmetry* of the Hamiltonian system (M, ω, H) if both ω and H are invariant under the flow of

Y. That is, both H and ω remain fixed under the pull-back of the infinitesimal automorphism $e^{Yt} : M \to M$ (for t small enough). Then, we have the celebrated theorem by Emmy Noether

THEOREM 2.1.2 ([**Lee03**]). *Let (M, w, H) be a Hamiltonian system. If f is a conserved quantity, then X_f is an infinitesimal symmetry. Conversely, if the de-Rahm cohomology group $H^1_{dR}(M) = 0$, then each infinitesimal symmetry is the Hamiltonian vector field of a conserved quantity, which is unique up to addition of a function that is constant on each component of M.*

Now, we say that two conserved quantities f and g are independent if their corresponding Hamiltonian vector fields X_f and X_g are linearly independent over $C^\infty(M)$. Thus, we have our sought-after definition:

DEFINITION 2.1.2. *A classical integrable system is a symplectic manifold (M, ω) together with a maximal independent set of conserved quantities.*

It is not too, hard to show that the maximal set of conserved quantities will always of size n where $2n$ is the dimension of M. One easy example, which generalizes, is the case of the symplectic manifold M with global canonical coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$. One can take $H := q_1$ and then a maximal independent set of conserved quantities could be $(q_1, \dots, q_n)$, but note there are multiple maximal independent set of conserved quantities. Indeed, by the Darboux theorem we can work locally and then the result follows from the previous example and linear algebra.

Lastly, we would like to point out that given a maximal set of conserved quantities, say $(f_1, \dots, f_n)$, that trajectories of the Hamiltonian system will lie in a submanifold N defined by the equations

$$f_1 - c_1 = \cdots = f_n - c_n = 0$$

for some set of constants $c_i \in \mathbb{R}$. This submanifold has the special property that $\omega|_N = 0$. In fact, any submanifod N of a symplectic manifold (M, ω) such that $\omega|_N = 0$ is called an *isotropic submanifold*, and if dim N =dim $\frac{1}{2}M$, the submanifold N is called a *Lagrangian submanifold*.

2.1.2. Quantum Integrable Systems. In the previous section, we saw a coarse outline of Liouville integrable systems. Now, we wish to introduce similar ideas in the context of quantum mechanics by keeping in mind some of the key features from the classical case: conserved quantities, Poisson bracket, and the Hamiltonian function. We note that in this case there is no good definition of integrable systems. Instead, one talks about *exactly soulvable systems* (see [**Sut04**] for further discussion). Also, let us note that now we change the focus from differential geometry to algebra, and the main difficulty now is that "objects" in this theory don't necessarily commute. In the end, the goal is to have a theory that describe some "dynamics" as in the previous sub-section. So, we begin with some definitions.

Definition 2.1.3. *A Hermitian vector space is a pair* $(V, \langle , \rangle)$ *where* V *is a complex vector space and* $\langle , \rangle : V \times V \to \mathbb{C}$ *is a positive definite skew-symmetric bilinear form that is anti-linear on the first argument and linear on the second argument. That is, for any* $\vec{u}, \vec{v} \in V$ *and* $\alpha, \beta \in \mathbb{C}$

$$\langle \vec{w}, \alpha\vec{u} + \beta\vec{v} \rangle = \alpha\langle \vec{w}, \vec{u} \rangle + \beta\langle \vec{w}, \vec{v} \rangle$$

$$\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$$

$$\langle \vec{v}, \vec{v} \rangle \geq 0,$$

where the bar is the complex conjugate and the last equation is only an equality if $\vec{v} = 0$.

Then, we have that our "state space" is a Hermitian vector space V, and now our main interest is the action of linear operators on V. In quantum mechanics, one is mostly concerned with operators that have a real spectrum (i.e. real eigenvalue) due to physical interpretation, and such operators are called *observables*. Instead, we will drop the conditions of the spectrum of the operator and focus on understanding the evolution/deformation equation

$$\frac{d}{dt}\vec{v} = H\vec{v} \tag{2.1.1}$$

where t is an evolution/deformation parameter, which can be thought of as a time parameter, and H is a linear operator on V. In physics, the operator H is called the "Hamiltonian", but we reserve ourselves from using such a name since we don't use the proper normailzation in our formulas. We do like to point out that this operator plays a similar role to the Hamiltonian function on a Hamiltonian system from the previous subsection as in both cases it gives the "equations of motion".

Solving the evolution/deformation equation (2.1.1) may be very easy in some cases. For example, if $\vec{v}$ is an eigenvector of H with eigenvalue λ, then the evolution/deformation equation becomes

$$\frac{d}{dt}\vec{v} = H\vec{v} = \lambda\vec{v}$$

and it has a solution $\vec{v}(t) = e^{\lambda t}\vec{v}_o$ where $\vec{v}_o$ is a given initial value. Moreover, we have that

$$H\vec{v}(t) = He^{\lambda t}\vec{v}_o = e^{\lambda t}H\vec{v}_o = e^{\lambda t}\lambda\vec{v}_o = \lambda\vec{v}(t),$$

and thus $\vec{v}(t)$ remains an eigenvector of H with an eigenvalue λ. Therefore, we have that λ is a conserved quantity in this system. Unfortunately, it is not always possible to find eigenvectors for any pair of a Hermitian vector space and a linear operator. Consider, for example, $V = L^2([0,1])$ the space of square integrable functions on the unit interval with the bilinear form

$$\langle f, g\rangle := \int_{[0,1]} \overline{f(x)}g(x)dx, \qquad f, g \in L^2([0,1])$$

and the linear operator $H = x\cdot$ (i.e. multiplication by x). One can check that there is no solution to the eigenvalue equation since $Hf(x) = \lambda f(x)$ implies that $(x-\lambda)f(x) = 0$, which means that $f = 0$ almost everywhere. Still, one can solve the evolution/deformation equation (2.1.1) so that $f(x,t) = e^{xt}f(x)$, but the problem is that there is no conserved quantities in this system. We make this precise

Definition 2.1.4. *The spectrum of a bounded linear operator $H \in \mathcal{B}(V)$, denoted $\sigma(H)$, is the set of complex numbers λ such that $(H - \lambda I) : V \to V$ is not one-to-one or onto, where I is the identity operator. In particular,*

(1) *The point spectrum of H, denoted by $\sigma_p(H)$, consist of all $\lambda \in \sigma(H)$ such that $H - \lambda I$ is not one-to-one. In this case λ is called an eigenvalue of H.*
(2) *The continuous spectrum of H, denoted by $\sigma_c(H)$, consists of all $\lambda \in \sigma(H)$ such that $H - \lambda I$ is one-to-one but not onto, and the range of $(H - \lambda I)$ is dense in V.*
(3) *The residual spectrum of H, denoted by $\sigma_r(H)$, consists of all $\lambda \in \sigma(H)$ such that $H - \lambda I$ is one-to-one but not onto, and the range of $(H - \lambda I)$ is not dense in V.*

Therefore, the elements of the point spectrum $\sigma_p(H)$ are the conserved quantities of a bounded linear operator H. We note that an eigenvalue $\lambda \in \sigma_p(H)$ might have non-trivial multiplicity (i.e. there might be multiple eigenvectors with eigenvalue λ). In particular, if H is a compact (i.e. maps pre-compact sets to compact set) self-adjoint operator, one has that $\sigma(H) = \sigma_p(H)$ and each eigenvalue only has finite multiplicity. This is part of the spectral theorem (see [**RS72**, **HN01**]) for compact-self adjoint operators. In this, case one may introduce more operators to resolve the multiplicity of the eigenvalues. That is, there exist a set of self-adjoint operators $\{H, H_1, H_2, \dots\}$ with a basis $\{e_n\}$ where each basis element is an eigenvector of each H_i simultaneously and for every $n \neq m$ there exist an operator H_j such that e_n and e_m have distinct eigenvalues. In fact, any set of bounded linear operators that satisfy the previous conditions are said to be a *complete set of operators* [**Dir30**]. Also, one can show that if two operators belong to a complete set of operator, then the operators must commute. There are generalizations of the spectral theorem which relax condition on the operator (see [**RS72**]). Instead, we cosider the evolution/deformation of operator so that their spectrum remains fixed where we denote $\mathcal{B}(V)$ as the set of bounded operators on V. These equations will play and important role when we discuss the KP hierarchy in the next section.

First, note that elements of $\sigma_p(H_i)$ are also conserved quantities for any i. Indeed, take any basis vector e_n with $H_i e_n = \lambda e_n$ for some $\lambda \in \mathbb{C}$. Since H_i and H commute, we have that

$$H_i \frac{d}{dt} e_n = H_i H e_n = H H_i e_n = H \lambda e_n = \lambda H e_n = \lambda \frac{d}{dt} e_n.$$

Thus, we have that e_n remains an eigenvector of H_i with eigenvalue as it evolves/deforms with relation to equation (2.1.1). In fact, the eigenvectors and eigenvalues of any bounded operator $F \in \mathcal{B}(V)$ will not be constant unless F commutes with H (i.e $[H, F] := HF - FH = 0$). On the other hand, we may deform a bounded operator $F \in \mathcal{B}(V)$ so that its eigenvalues remain constant. First, consider an eigenvector e_n of F with an eigenvalue λ, then

$$HFe_n = \frac{d}{dt} F e_n = \frac{d}{dt} \lambda e_n = \frac{d\lambda}{dt} e_n + \lambda \frac{de_n}{dt} = \frac{d\lambda}{dt} e_n + \lambda H e_n = \frac{d\lambda}{dt} e_n + H \lambda e_n = \frac{d\lambda}{dt} e_n + HFe_n,$$

which means that $\frac{d\lambda}{dt} = 0$. So, we have that λ is a constant without seemingly any conditions on F, but there is a subtle issue as e_n will not remain an eigenvector of F as it evolves. We can see this from the infinitesimal calculation

$$F(e_n + \epsilon H e_n) = F e_n + \epsilon F H e_n = \lambda e_n + \epsilon F H e_n \neq \lambda(e_n + \epsilon H e_n).$$

In fact, we must require that F also evolves/deforms with the parameter. For any $\vec{v} \in V$, we have

$$HF\vec{v} = \frac{d}{dt} F\vec{v} = \frac{dF}{dt}\vec{v} + F\frac{d\vec{v}}{dt} = \frac{dF}{dt}\vec{v} + FH\vec{v}.$$

So, the deformation/evolution equation for the operators is

$$\frac{dF}{dt} = [H, F], \tag{2.1.2}$$

which is called a *Lax equation* and the pair of operators (H, F) are called a *Lax Pair*. Now, a quick infinitesimal calculation,

$$(F + \epsilon[H, F])(e_n + \epsilon H e_n) = F e_n + \epsilon(HF - FH)e_n + \epsilon F H e_n = \lambda(e_n + \epsilon H e_n),$$

gives us the desired result. Therefore, by equations (2.1.1) and (2.1.2) we know how vectors and operators deform/evolve, respectively. In particular, we note that a deformation given by (2.1.2) is often called an *iso-spectral deformation* as it deforms an operator leaving its spectrum fixed. In the next section, we will consider the Lax equation (2.1.2) in far greater detail.

2.1.3. Frobenious Integrability. In the previous subsection, we introduced the notion of integrability as it arose in classical mechanics. Now, we consider one more type of integrability: Frobenius integrability. In the process, we see that this is in fact a different type of integrability as we have discussed in the previous two subsections. (Well, in the qunatum case there is no good definition, but we know that we are in a different setting as we will now deal with differential geometry rather than functional analysis.) More telling, we'll see that a symplectic manifold is only Frobenius integrable when the dimension of the symplectic manifold is 2, showing that the two types of integrability must be different. In the rest of the subsection we will be working primarily with Frobenius integrability and all of our results will deal with Frobenious integrability, but we include the other types of integrability as much of the subsequent work resembles that of classical and quantum mechanics.

Let us start with a simple example from differential equations and consider an equation

$$\frac{dy}{dx} = f(x, y)$$

for $x, y \in \mathbb{C}$ and smooth function $f : \mathbb{C}^2 \to \mathbb{C}$. Then, in solving the differential equation, one wishes to find a function $y(t)$ and $x(t)$ such that

$$\frac{dy(t)}{dx(t)} = f(x(t), y(t)),$$

and one can write

$$y(t) = \int f(x(t), y(t)) \frac{dx(t)}{dt} dt.$$

Note that we can define a 1-dimensional submanifold $N = \{(x(t), y(t)) | t \in I\} \subset \mathbb{C}^2$ for some domain $I \subset \mathbb{C}$, and by construction we have that the differential equation vanishes on N. That is, we have that the differential form $dy + f(x,y)dx$ vanishes on N, or more precisely, $(dy + f(x,y)dx)\,|_N = 0$. In the rest of this subsection, we will generalize these ideas. Namely, we let the domain of the smooth functions to be any smooth manifold, we use differential forms instead of differential equations, and the solutions to the system of differential forms is given by submanifolds of the given manifold. One advantage of this approach is that it is coordinate free and thus is purely geometric approach.

Take a smooth manifold M, and let $\Omega^1(M)$ be the space of smooth differential 1-forms on M (i.e. global sections of the cotangent bundle T^*M). We have

Definition 2.1.5. *A Pfaffian differential system is a collection of* 1*-forms* $\omega_1, \ldots, \omega_k \in \Omega^1(M)$. *A submanifold* $i : N \hookrightarrow M$ *is an integral manifold if* $i^*(\omega_j) = 0$ *for* $j = 1, \ldots, k$ *, where* $i^* : \Omega^1(M) \to \Omega^1(N)$ *is the induced map. Moreover, we say that* N *is maximal if ker* i^* *is generated by the differential forms* $\omega_1, \ldots, \omega_k$.

In the previous example, we have that our Pfaffian system is given by a single 1-form, $dy + f(x,y)dx$ and the maximal integral submanifold is given by N. Now, let's consider a case that doesn't have a maximal integral submanifold. Take a symplectic manifold (M, ω), we can write a $\omega = d\eta$ (at least locally since ω is closed) where the 1-form is called the *Liouville* 1*-form*. Then, the Lagrangian submanifolds are integral submanifolds. Indeed, in local coordinates we have that

$$\eta = -\sum_{i=i}^{n} p_i dq_i,$$

and the Lagrangian submanifold is given by

$$N = \{(q_1, \dots, q_n, p_1, \dots, p_n) | q_1 = c_1, q_2 = c_2, \dots, q_n = c_n\} \hookrightarrow \{(q_1, \dots, q_n, p_1, \dots, p_n)\},$$

where $c_1, \dots, c_n \in \mathbb{C}$ are any constants. In this example, we have that ker $i^* = \langle dq_1, dq_2, \dots, dq_n \rangle \ni \eta$, and clearly, the kernel is not generated by η. Thus, we have that N is not a maximal integral manifold with respect to η. In fact, there is no maximal integral manifold with respect to η (except for $n = 1$). In general, it is the case that a generic set of differential form will not have a maximal integral manifold.

Definition 2.1.6. *A system of differential forms $\omega_1, \dots, \omega_k \in \Omega^1(M)$ is Frobenius integrable if there exist a foilation (see [**Lee03**] for definition) of M by maximal integrable manifolds.*

This condition is stronger than just the existence of an integrable manifold. In fact, we require that each point in M belongs to a unique maximal integrable manifold. Then, we have the flows given by the differential forms are completely determined by the initial conditions.

The stark differences are the following: in Liouville integrability the flow/dynamics of our system is given a by a Hamiltonian function (which represents the energy of our system), whereas in Frobenius integrability the flow/dynamics by a system of 1-forms. Thus, there is no reason to expect any or all symplectic manifolds with the symplectic 2-form to be Frobenius integrable for in the case of Liouville integrability we required the extra information of a Hamiltonian function. In any case, one can think of H, a Hamiltonian function, and ω, the symplectic 2-form, to be a differential system consisting of 0-form and a 2-form, respectively. We note that both H and ω lie in the kernel of the map induced by the inclusion of the Lagrangian submanifolds, but such a differential system doesn't lie within our definition of a Pfaffian system. We merely point out that there are further generalizations of differential systems that can be defined and we suggest the interested reader to consult [**BCG+13**]. Instead, let us note the fundamental theorem on Pfaffian systems:

THEOREM 2.1.3 (Frobenius). *Let $\mathcal{I}$ be a differential ideal having as generators the (linear independent) 1-forms $\omega_1, \ldots, \omega_k$ so that it is closed under exterior derivative, i.e.*

$$d\mathcal{I} \subset \mathcal{I}.$$

In a sufficiently small neighborhood there is a coordinate system $y_1, \ldots, y_n$ such that $\mathcal{I}$ is generated by $dy_{n-k+1}, \ldots, dy_n$.

PROOF. The proof of this theorem is readily available in many of the classic text on integrable systems [**BCG⁺13**]. □

2.2. Isospectral Deformations

Given a Lax equation (2.1.2), as given in the last subsection, one may derive, under certain conditions and assumptions, a family of differential equations. This family of differential equations are called the *KP hierarchy*. In this section, we will derive this hierarchy following the ideas of [**Sat81**], but first we take a historical detour. As it is often the case, some of the equations in the KP hierarchy appeared much earlier in time from heuristic derivations and later on were derived precisely from simple mathematical principles. In particular, we first consider the Korteweg-de Vries (KdV) equation, which first appeared in the late 19^{th} century, and we describe some of its deep applications since then.

2.2.1. The KdV Equation. This is mostly historical and a review from [**Hir04**, **AS81**]. The KdV equation,

$$u_t + uu_x + u_{xxx} = 0 \tag{2.2.1}$$

is a partial differential equation which describes shallow waves. In 1832, John Scott Russell wished to describe a solitary wave he observed along the Edinburgh-Glasgow canal and obtained an equation of the wave speed $c = \sqrt{g(h+a)}$, where h is the depth of the channel, g is the acceleration due to gravity, and a is the amplitude of the wave. Then, in the 1870's, Boussinesq and Lord Rayleigh independently gave an equation for the shape of the wave

$$a(x,t) = \alpha \operatorname{sech}^2 \beta(x - ct),$$

which is a solution of the evolution equation (2.2.1) derived by Korteweg and de Vries in 1895 with $\alpha = 12\beta^2$ and $c = 4\beta^2$ for arbitrary $\beta \in \mathbb{R}$.

It wasn't until the 1960's that the KdV equation regained popularity under the work of Kruskal and Zabusky [**ZK65**] in 1965, where the authors performed numerical calculations exploring the interactions of waves given by the KdV equation. In particular, one should note that the waves described by the KdV are non-linear and dispersive meaning that superposition doesn't apply and that the profile wave changes over time, respectively, and in the numerical calculations of [**ZK65**] the authors found that solutions to the KdV equation were made up of "solitary-wave pulses" whose profile remained unaffected after an interaction with another "solitary-wave pulse". The term *solitons*, for this type of wave as they behave "like" particles, was introduced and a rich mathematical theory was developed by Date, Hirota, Jimbo, Kashiwara, Miwa, Sato, Segal, and Wilson (among many) to describe this phenomenon (see [**Hir04**]).

Most surprising and inspiring is the result of Kontsevich [**Kon92**], based on Witten's conjecture [**Wit91**], where he showed that the generating function for the intersection numbers of the moduli space of curves (of a point) satisfy the KdV equation using techniques from matrix models. This is a deep result as the intersection numbers are defined through sophisticated objects in algebraic geometry and are quite difficult to compute by straight forward methods. Nonetheless, using the fact the that the generating function of the intersection numbers satisfies the KdV equation one can compute all the intersection numbers recursively by the knowledge of the "first" intersection number. We digress no longer. This is merely an example of the depth of the KdV equation. While there is no direct connection between shallow, solitary waves and the intersection numbers of the moduli space of curves, they are related by the KdV equation. This is of no surprise as the KdV equation is the first of countable number of equations that make up the KP hierarchy of equations. Thus, one expects to encounter the KdV equation often and many times when one considers integrable systems, as is the case for the shallow waves and the intersection numbers of the moduli space of curves.

2.2.2. KP Hierarchy. In the rest of the section, the KP hierarchy is introduced through the theory of isospectral deformations. We follow the ideas in [**Sat81**, **Mul84**]. First, let us consider the Lax equation with a further generalization where we have multiple parameters of deformation

$\{t_n\}_{n=0}^{\infty}$ corresponding to a family of evolution operators $\{B_n\}_{n=0}^{\infty}$ acting on an operator L. That is,

$$\frac{\partial L}{\partial t_n} = [B_n, L] \qquad n = 0, 1, 2, \ldots \tag{2.2.2}$$

We make a choice and let the operators $\{B_n\}_{n=0}^{\infty}$ and L be formal pseudo-differential operators. (See [**Tay08**] for more details on pseudo-differential operators.) In particular, we let

$$L := \sum_{n=0}^{\infty} u_n(x) \left(\frac{d}{dx}\right)^{1-n}, \qquad u_0(x) = 0, \tag{2.2.3}$$

and

$$B_n := [L^n]_+, \qquad n = 0, 1, 2, \ldots \tag{2.2.4}$$

where $[L^n]_+$ are the sum of the terms with non-negative exponents in the series expansion of L^n. For example, we have that

$$\begin{aligned}
B_0 &= 1\\
B_1 &= \frac{d}{dx}\\
B_2 &= \left(\frac{d}{dx}\right)^2 + 2u_1\frac{d}{dx} + u_{1,x} + 2u_2\\
B_3 &= \left(\frac{d}{dx}\right)^3 + 3u_1\left(\frac{d}{dx}\right)^2 + (3u_{1,x} + 3u_2 + 3u_1^2)\frac{d}{dx} + u_3 + 4u_1u_2 + 3u_{1,x} + u_{2,x},
\end{aligned}$$

where the second set of subscripts denote partial derivatives with respect to the variables indicated. We note that, for these computations, we find the commutation formula

$$\left[\left(\frac{d}{dx}\right)^n, u(x)\right] = \sum_{k=1}^{\infty} \binom{n}{k} \left(\frac{d}{dx}\right)^{n-k} \qquad n \in \mathbb{Z}$$

particularly useful, where $u^{(k)}(x)$ is the k^{th} derivative of $u(x)$ with respect to x. Note that the sum is finite for $n \geq 0$ since $\binom{n}{k} = 0$ for $k \geq n$ if n is positive. Moreover, note the resemblance to the generalized binomial theorem, and also that the formula holds for any $n \in \mathbb{Z}$. Indeed, one obtains the formula from repeated application of the product formula for derivatives, even for the negative case where the product formula is just integration by parts. Moreover, we may consider the conjugation $e^{-s(x)}Le^{s(x)}$ and by the previous formula we have the formal expansion

$$e^{-s(x)}Le^{s(x)} = \sum_{n=0}^{\infty} w_n(x)\left(\frac{d}{dx}\right)^{1-n}$$

where $w_n(x)$ are new functions that may be written in terms of $u_n(x)$'s and derivatives of $s(x)$. In particular, we have that $w_0(x) = u_0(x)$ and $w_1(x) = u_1(x) + \frac{ds(x)}{dt}$. So, if we let $s(x) = -\int u(x)dx$, we have that $w_1(x) = 0$. Therefore, we may assume from the beginning that L is normalized so that $u_1(x) = 0$, and the previous formulas become $B_0 = 1$, $B_1 = \frac{d}{dx}$, $B_2 = \left(\frac{d}{dx}\right)^2 + 2u_2$, and $B_3 = \left(\frac{d}{dx}\right)^3 + 3u_2\frac{d}{dx} + u_3 + u_{2,x}$. Then, by writing out the deformation equations for t_0 and t_1, we have that

$$u_{n,t_0} = 0 \qquad n = 0, 1, 2, \ldots$$

$$u_{n,t_1} = u_{n,x} \qquad n = 0, 1, 2, \ldots.$$

Therefore, we have that $u(x;t)$ is independent of t_0. Moreover, we have that $u(x;t)$ only depends on the sum $x + t_1$. Indeed, if we have a smooth function $f : \mathbb{C}^2 \to \mathbb{C}^2$ such that $f_x = f_y$, then there exist some smooth function $g : \mathbb{C} \to \mathbb{C}$ such that $f(x,y) = g(x+y)$. It suffices to take a coordinate change $(u,v) \mapsto (x,y)$ such that $u = x+y$ and $v = x-y$, and noting that we have $f_v = 0$. So, we have that $f(u,v) = f(u,0) = f(x+y,0)$. In particular, we have that $f(x,y) = f(x-a, y+a)$ for any constant $a \in \mathbb{C}$. So, we apply the previous discussion to the functions $u_n(x;t_1,t_2,\ldots)$ and we have that $u_n(x;t_1,t_2,\ldots) = u_n(0;t_1+x,t_2,\ldots)$ with $a = x$. Therefore, by a change of coordinates of t_1 with $t_1 + x$, we may identify the variable x_1 with t_1 and $\frac{d}{dx}$ with $\frac{\partial}{\partial t_1}$. Computing the next few terms in the deformation equations, we have that

$$u_{2,t_2} = 2u_{3,x} + u_{2,xx}$$

$$u_{3,t_2} = 2u_{4,x} + u_{3,xx} + 2u_2u_{2,x}$$

$$u_{2,t_3} = 3u_{4,x} + 3u_{3,xx} + u_{2,xxx} + 6u_2u_{2,x},$$

and manipulating these equations, we obtain the equation

$$3u_{2,t_2t_2} + \left(-4u_{2t_3} + u_{2,xxx} + 12u_2u_{2,x}\right)_x = 0.$$

This last equation is the *Kadomtsev - Petviashvili (KP) equation*, which is a generalization of the KdV equation (2.2.1) introduced in the previous sub-section. Indeed, the equation $-4u_{2t_3} + u_{2,xxx} + 12u_2u_{2,x} = 0$ is equivalent to the KdV equation under a different scaling. This leads us to the following definition:

DEFINITION 2.2.1. *The set of deformation equations given by the Lax equation* (2.2.2) *with the operators defined by* (2.2.3) *and* (2.2.4) *on a set of function* $\{u_n(x;t)\}$ *with deformation parameters* $t = (t_0, t_1, t_2, \dots)$ *is called the KP hierarchy.*

Moreover, one has the following technical result:

THEOREM 2.2.1 ([**Mul88**]). *The Lax equaition* (2.2.2) *of the KP hierarchy is equivalent to the linear total differential equation*

$$dU = \Omega U$$

for $U \in \mathcal{E}^x$*, and this equation is Frobenius integrable.*

REMARK 2.2.1. $\mathcal{E}^x$ *is a formal Lie algebra of ordinary pseudo-differential oprators over an infinite affine space* $\lim_{n\to\infty} \mathbb{C}^n$*. We invite the reader to see [**Mul84**] for more details.*

In the last subsection, we saw that solutions of the KdV equation have interesting properties. Thus, we expect that solutions to the KP hierarchy must also have interesting properties. In the next chapter, we will build up the theory to give an infinite family of solutions to the KP hierarchy. Moreover, we will show a close relation to interacting particle models.

2.3. Isomonodromic Deformations

In this section, we give a quick introduction to the theory "isomonodromic deformations" following the classical paper [**JMU81**]. Just as in the isospectral deformations we focuse on preserving certain quantitties of the evolution/deformation equation (2.1.1). In particular, we consider a deformation that preserve the "monodromy data", which we explain in detail in the following subsection. Similarly to the isospectral deformations, we will show that isospectral deformations will have to satisfy a Lax-type equation, which has been shown to be Frobenius integrable in [**JMU81**]. The major part of introducing this integrable system is setting up the "monodromy data", which is to preserved.

2.3.1. Monodromy: A simple example. In simple terms, the monodromy of a function describes how a function changes as one goes around a cycle around a singularity. Right away, we have to notice that we are working with multi-valued functions and that the monodromy must be path dependent. In our case, we only consider cases where the monodromy only depends on the homotopy type of the path taken, and in that case we may lift the multi-valued function to a single-valued function on the universal covering space of the initial domain.

One of the most simple and instructive examples is that of the $\log(x)$ function. Normally, we see that $\log(x)$ is defined on $\mathbb{C} - (-\infty, 0]$ as a single valued function, but we may define $\log(x)$ as a multi-valued function on $\mathbb{C} - \{0\}$, where we couldn't include 0 since there is a singularity there. That is, we have define

$$\log : re^{i\theta} \mapsto \log(r) + \theta i + 2\pi i n,$$

with the ambiguity of $n \in \mathbb{Z}$ giving the multi-value. Now, take any path $\gamma : [0,1] \to \mathbb{C} - \{0\}$ with $\gamma(0) = re^{i\theta}$ and $\gamma(1) = we^{i\phi}$. Then, if one analytically continues log along the path γ, we have that

$$\log(y) - \log(x) = \log(w/r) + (\phi - \theta)i + 2\pi i n_\gamma, \tag{2.3.1}$$

where $x = \gamma(0)$, $y = \gamma(1)$, and n_γ is the winding number of the path. Therefore, if we fix a base point $x_0 \in \mathbb{C} - \{0\}$, and consider paths $\gamma : [0,1] \to \mathbb{C} - \{0\}$ with fixed base points (i.e. $\gamma(0) = x_0$), then we have a single-valued function given by the above formula on the space of paths with fixed points, say $\mathcal{C}$, and recall that such a space is the universal covering space of $\mathbb{C} - \{0\}$ in algebraic topology [**Hat02**]. So, as we alluded, we defined a single-valued function on the covering space of $\mathbb{C} - \{0\}$ as promised. We note that the key feature that allowed us to have a well-defined function on the covering space $\mathcal{C}$ was the fact that equation (2.3.1) only depends on the homotopy type of the paths γ (i.e. depends only on the winding number n_γ). In fact, that is the monodromy. More precisely, the monodromy of log is the representation induced by log

$$\rho : \pi(\mathbb{C} - \{0\}, x_0) \to GL(1, \mathbb{C})$$

where

$$[\gamma] \mapsto e^{n_\gamma}.$$

Also, as we described earlier, one should note that this representation gives you the behavior of the multi valued function as you go around a closed cycle on the singularity. That is, given a closed path γ with base point x_0, we have that

$$\log(x_0) = \log(x_0) + 2\pi i \log(\rho([\gamma]))$$

Now, let us dig deeper to understand what are some of the important features of this example that will allow us to generalize this to other functions. That is, we want to know when we can apply this of type construction on a multi-valued function. As we noted earlier, we must have that the value of the function only depends on the homotopy type of the paths. In our example, we note that this condition is met since the log function always satisfies a differential equation (away from 0) regardless of the multi-value ambiguity. That is, log is a solution of the differential equation

$$\frac{dy}{dx} = \frac{1}{x},$$

and the solution is uniquely determined by given initial values. Thus, the uniqueness allows us to deform the paths γ homotopically without changing the values of the function at the end points. Of course, the differential equation is singular at $x = 0$, and therefore, we may not be allowed to deform the paths along $x = 0$, which gives us the non-trivial monodromy. In general, we are fortunate that this will be the case. That is, if we have a multi-valued function that only depends on the homotopy type of the paths taken, then such a function will be a solution of a certain differential equation (see Prop. 2.6 in [**JMU81**]).

2.3.2. Monodromy Data. In the previous subsection, we encountered a simple example where the monodromy is easily understood, but that is not all the information in a monodromy problem. In the general setting, with a multi-valued matrix function, there is still more information that is of interest. This is brought up by two further properties that arise in the general case. First, rather than just working with one singular point on the multi-valued function (or differential equation), we consider the case of multiple (finite) number of singular points. This will give rise to "connection matrices" which relate solution of the differential equations near one singular point to the solution near a different singular point, just as in the previous subsection where we related solutions around cycles of a singular point. Also, we will be working with holomorphic solutions near the singular points, which we requuire to have a specific asymptotic expansion, and thus we restrict the solutions to certain sectors around the singular points. Then, we will have matrices called "Stokes multipliers" that relate the holomorphic solutions on distinct sectors near a singular point. Thus, as the reader might see, there is more information to consider in the general case of the isomonodromy problem, and it is quite technical. So, in the following we will give precise definition of all this information, which we call the "monodromy data".

Take a multi-valued function

$$Y : M \to GL(m, \mathbb{C}),$$

where $M = \mathbb{C} - \{a_1, \ldots, a_n\}$ is the complement of n pair-wise distinct points in the complex plane $a_1, \ldots, a_n \in \mathbb{C}$. Let $\pi : \tilde{M} \to M$ be the universal covering space of M. Also, we require that $Y(x)$ is a solution to the differential equation

$$\frac{dY}{dx} = A(x)Y, \tag{2.3.2}$$

where $A(x)$ is a rational $m \times m$ matrix given by the following partial fraction decomposition at $n+1$ distinct points $a_1, \ldots, a_n$ and $a_\infty = \infty$

$$A(x) = A_\infty(x) + \sum_{\nu=1}^{n} A_\nu(x) \tag{2.3.3}$$

$$A_\nu(x) = \left\{ \begin{array}{ll} -\sum_{j=1}^{r_\infty} A_{\infty,-j} x^{j-1} & (\nu = \infty) \\ \sum_{j=0}^{r_\nu} A_{\nu,-j} \frac{1}{(x-a_\nu)^{j+1}} & (\nu \neq \infty) \end{array} \right\} \tag{2.3.4}$$

and the numbers $r_\nu \geq 0$ are the rank of each singular point a_ν. (If $r_\nu > 0$, then a_ν is an irregular singular point and otherwise a regular singular point.)

In the following, we will build a solution to the differential equation by first solving the equation using a formal series at each singular point. Then, at each singular point, we will find a sector where there is a holomorphic invertible solution that has an asymptotic expansion as the one given in the formal series. First, we normalize the rational matrix $A(x)$ by a gauge transformation as to diagonalize A_{ν,r_ν}. That is, we take a collection of invertible matrices $G^{(1)}, \ldots, G^{(n)}, G^{(\infty)}$ so that

$$T_{r_\nu}^{(\nu)} := \left(G^{(\nu)}\right)^{-1} A_{\nu,-r_\nu} G^{(\nu)}$$

is a diagonal matrix (with distinct entries for $r_\nu > 0$ and distinct entries modulo integers for $r_\nu = 0$). Also, we defined

$$A^{(\nu)} = \left(G^{(\nu)}\right)^{-1} A(x)G^{(\nu)}.$$

We note that there is some ambiguity in the choice of matrices $G^{(\nu)}$ for $\nu = 1, \dots, n, \infty$, but we chose to have $G^{(\infty)} = 1$ and assume that $A_{\infty,-r_\infty}$ is diagonal (with distinct entries if $r_\infty > 0$ and distinct entries modulo the integers if $r_\infty = 0$). Then, we call such a choice of matrices $G^{(1)}, \dots, G^{(n)}, G^{(\infty)}$, along with the partial fraction decomposition (2.3.3), a *singular data*. Moreover, the set of all possible singular data is a manifold $\mathcal{N}$ and we call it the singular data manifold.

PROPOSITION 2.3.1 (Prop. 2.1 [**JMU81**]). *There exist a unique formal series $Y^{(\nu)}(x)$ at $x = a_\nu$ of the following form*

$$Y^{(\nu)}(x) = \hat{Y}^{(\nu)}(x)e^{T^{(\nu)}(x)}$$

which is a solution to the differential equation

$$\frac{d}{dx}Y^{(\nu)} = A^{(\nu)}(x)Y^{(\nu)}.$$

Moreover, $T^{(\nu)}$ is a diagonal matrix of the form

$$T^{(\nu)}(x) = \left\{ \begin{array}{ll} \sum_{j=1}^{r_\infty} T_{-j}^{(\infty)} \frac{x^j}{(-j)} + T_0^{(\infty)} \log\left(\frac{1}{x}\right) & (\nu = \infty) \\ \sum_{j=1}^{r_\nu} T_{-j}^{(\nu)} \frac{(x-a_\nu)^j}{(-j)} + T_0^{(\infty)} \log\left(x - a_\nu\right) & (\nu \neq \infty) \end{array} \right\}, \tag{2.3.5}$$

and $\hat{Y}^{(\nu)}$ is a formal power series at $x = a_\nu$,

$$\hat{Y}^{(\nu)}(x) = \left\{ \begin{array}{ll} \sum_{j=0}^{\infty} \hat{Y}_j^{(\infty)} x^{-j} & (\nu = \infty) \\ \sum_{j=0}^{\infty} \hat{Y}_j^{(\nu)} (x - a_\nu)^j & (\nu \neq \infty) \end{array} \right\}, \tag{2.3.6}$$

with $Y_0^{(\nu)} = 1$.

The matrices $T_{-r_\nu}^{(\nu)}, \dots, T_0^{(\nu)}$ for $\nu = 1, \dots, n, \infty$ will be part of the "monodromy data". In particular, we have that the transformation of the solution given by a cycle around a singular point a_ν is directly dependent on the matrix $T_0^{(\nu)}$ (see (2.3.7)), which we call the *exponent matrix* of formal moodromy at a_ν.

Next, we consider holomorphic solutions of equation (2.3.2) on some sectors so that a_ν is in the sector and the solution has the asymptotic expansion given by the formal series solutions given.

Take a (small enough) neighborhood $V_\nu \subset M$ of a singular point a_ν, and define:

DEFINITION 2.3.1. *A set of sectors* $\mathcal{S}_1^{(\nu)}, \dots, \mathcal{S}_{k_\nu+1}^{(\nu)} \subset \pi^{-1}(V_\nu) \subset \tilde{M}$ *such that*

- $\mathcal{S}_\ell^{(\nu)} \cap \mathcal{S}_{\ell+1}^{(\nu)} \neq \varnothing$
- $\pi\left(\mathcal{S}_1^{(\nu)}\right) = \pi\left(\mathcal{S}_{k_\nu+1}^{(\nu)}\right)$
- $\pi\left(\bigcup_{\ell=1}^{k_\nu} \mathcal{S}_\ell^{(\nu)}\right) = V_\nu - \{a_\nu\}$

with $k_\nu = 2r_\nu + 1$

We have

PROPOSITION 2.3.2 (Prop. 2.4 [**JMU81**]). *There exist a choice (see. Prop. 2.4 [**JMU81**]) of a set of sectors* $\mathcal{S}_1^{(\nu)}, \dots, \mathcal{S}_{k_\nu+1}^{(\nu)}$ *such that there exists a unique holomorphic and invertible solution* $Y_\ell^{(\nu)}$ *having the asymptotic expansion*

$$Y_\ell^{(\nu)} \sim Y^{(\nu)}(x) \qquad in \;\; \mathcal{S}_\ell^{(\nu)},$$

where

$$Y_{2r_\nu+1}^{(\nu)} = Y_1^{(\nu)}(x^+) e^{2\pi i T_0^{(\nu)}},$$

where x^+ *is a point in* $\mathcal{S}_{2r_\nu+1}^{(\nu)}$ *satisfying* $\pi(x) = \pi(x^+)$.

Next, we make the observation that, by def. 2.3.1, we have $x \in \mathcal{S}_\ell^{(\nu)} \cap \mathcal{S}_{\ell+1}^{(\nu)} \neq \varnothing$, and so both the solutions $Y_{\ell+1}^{(\nu)}(x)$ and $Y_\ell^{(\nu)}(x)$ satisfy the same differential equation (2.3.2), which implies the existence of a constant $m \times m$ matrix $S_\ell^{(\nu)}$ such that

$$Y_{\ell+1}^{(\nu)}(x) = Y_\ell^{(\nu)}(x) S_\ell^{(\nu)}.$$

We call $S_\ell^{(\nu)}$ ($\ell = 1, \dots, 2r_\nu$) the *Stokes multiplies* at $x = a_\nu$, which will also be part of "monodromy data".

Lastly, we compare solutions of equation (2.3.2) at distinct singular points a_ν. Actually, since we can generate any permutation by transpositions of the type $(\nu \;\; \infty)$, it suffices to compare solutions for the pairs of singular point a_ν ($\nu = 1, \dots, n$) and a_∞. In particular, we have may choose a set of sectors (see Prop. 2.4 [**JMU81**]) so that $Y_1^{(\infty)}(x)$ may be analytically continued to $Y_1^{(\nu)}(x)$ along a path $\gamma_{(\nu\ \infty)}$ in $\mathcal{S}_1^{(\nu)}$. So, we have that $Y_1^{(\infty)}(x)$ and $G^{(\nu)}Y_1^{(\nu)}(x)$, by Prop. 2.3.1, satisfy the same differential equation (2.3.2), and just like in the case of Stokes multipliers, we have a collection of invertible $m \times m$ matrices $C^{(\nu)}$ for $\nu = 1, \dots, n, \infty$ so that

$$Y_1^{(\infty)}(x) = G^{(\nu)}Y_1^{(\nu)}C^{(\nu)},$$

and we call $C^{(\nu)}$ the connection matrix from a_ν to ∞. This brings us to the anticipated definition:

Definition 2.3.2. *For every singularity data (see paragraph before Prop. 2.3.1) we obtain solution matrices $Y_\ell^{(\nu)}$ for $\nu = 1, \dots, n, \infty$ and $\ell = 1, \dots, 2r_\nu + 1$ with the following monodromy data:*

$$\begin{array}{lllllllll}
\infty; & T^{(\infty)}_{-r_\infty}, & \dots, & T^{(1)}_0, & S^{(\infty)}_1, & \dots, & S^{(\infty)}_{k_\infty}, & C^{(\infty)} = 1, \\
a_1; & T^{(1)}_{-r_1}, & \dots, & T^{(1)}_0, & S^{(1)}_1, & \dots, & S^{(1)}_{k_1}, & C^{(1)}, \\
\vdots & & & & & & & \\
a_n; & T^{(n)}_{-r_n}, & \dots, & T^{(n)}_0, & S^{(n)}_1, & \dots, & S^{(n)}_{k_n}, & C^{(1)},
\end{array}$$

Finally, let us recapitulate the information we have from the monodromy data. There exist $m \times m$ constant matrices $C^{(\nu)}$ $(\nu = 1, \dots, n)$, $C^{(\infty)} = 1$, and $S_1^{(\nu)}, \dots, S_{k_\nu}^{(\nu)}$ $(\nu = 1, \dots, n, \infty)$ such that $Y(x)\left(C^{(\nu)}\right)^{-1}, Y(x)\left(C^{(\nu)}\right)^{-1}S_1^{(\nu)}, \dots, Y(x)\left(C^{(\nu)}\right)^{-1}S_1^{(\nu)} \cdots S_{k_\nu}^{(\nu)}$ have the same asymptotic expansion in $\mathcal{S}_1^{(\nu)}, \dots, \mathcal{S}_{k_\nu}^{(\nu)}$;

$$Y(x)\left(C^{(\nu)}\right)^{-1}S_1^{(\nu)} \cdots S_l^{(\nu)} \sim G^{(\nu)}\hat{Y}^{(\nu)}e^{T^{(\nu)}(x)}$$

for some $G^{(\nu)}$ $(\nu = 1, \dots, n)$ and $G^{(\infty)}$. Here $T^{(\nu)}(x)$ and $\hat{Y}^{(\nu)}(x)$ are of the form (2.3.6) and (2.3.5), respectively.

2.3.3. Isomonodromic Deformation. In the previous subsection, we spent a great deal of effort to define the *monodromy data* for a multi-valued matrix function $Y(x)$ on a manifold M given some singular data. Now, with this information we may define a "monodromy representation", which describes how the solution $Y(x)$ transforms through the analytic continuation along a cycle which contains a singular point a_ν (just as we did for the log example in Sec. 2.3.1). It is this monodromy representation that we wish to keep invariant as we deform equation (2.3.2) through some parameters t. In fact, in order to keep the "monodromy represetation" invariant as we deform equation (2.3.2), we must have that $A(x,t)$ satisfy a certain set of differential equations (or rather a Pfaffian differential system as described in Sec. 2.1.3), and this system of differential equations is Frobenius integrable. First, we have

Definition 2.3.3. *Give a differential equation* (2.3.2) *with a partial fraction decomposition* (2.3.3) *and singularity data* $G^{(1)},\dots,G^{(n)},G^{(\infty)}$ *and monodromy data as in Def 2.3.2, the monodromy representation is a homeomorphism*

$$Y_* : \pi_1(M,x_0) \to GL(m,\mathbb{C}), \qquad [\gamma] \mapsto R_\gamma.$$

This representation is generated by the image of counter clock-wise cycles γ_ν *that only encloses a singular point* a_ν $(\nu = 1,\dots,n,\infty)$ *whose image under this map is given by the matrices* R_ν*, where*

$$R_\nu = C^{(\nu)-1} e^{2\pi i T_0^{(\nu)}} S_{k_\nu}^{(\nu)-1} \cdots S_1^{(\nu)-1} C^{(\nu)}, \tag{2.3.7}$$

respectively.

Then, we have that $Y(x)$ is holomorphic and invertible in $\tilde{M}$. There exist $m \times m$ constant matrices $R_{(\nu)}$ $(\nu = 1,\dots,n,\infty)$ such that

$$Y(x^+) = Y(x) R_\nu.$$

Moreover, from a quick examination of the monodromy formula (2.3.7), we see that in the monodromy data if we keep the following fixed

$$\begin{aligned} &T_0^{(1)}, \quad S_1^{(\infty)}, \quad \dots, \quad S_{k_\infty}^{(\infty)}, \quad C^{(\infty)} = 1, \\ &T_0^{(1)}, \quad S_1^{(1)}, \quad \dots, \quad S_{k_1}^{(1)}, \quad C^{(1)}, \\ &\vdots \\ &T_0^{(n)}, \quad S_1^{(n)}, \quad \dots, \quad S_{k_n}^{(n)}, \quad C^{(1)}, \end{aligned} \tag{2.3.8}$$

then the monodromy representation will remain invariant. Thus, we will write the differential equations such that the data (2.3.8) remains fixed and so the monodromy representatiion stays invariant. So, we take an open set $V \subset \mathcal{N}$, so that $a_1(t),\dots,a_n(t)$ depend holomorphically on $t \in V$

and $a_\mu(t) \neq a_\nu(t)$ for $\mu \neq \nu$ and any $t \in V$. Moreover, we assume that the rest of the monodromy data depends holomorphically on t. Then,

THEOREM 2.3.1 (Thm. 3.3 [**JMU81**]). *The partial monodromy data* (2.3.8) *for equation* (2.3.2) *stays constant if and only if* $A^{(\nu)}(x)$ *and* $G^{(\mu)}$ *lie on a solution leaf of the following total differential equation on* $\mathcal{N}$*, the manifold of singular data:*

$$dA^{(\mu)} = \frac{\partial \Omega^{(\mu)}}{\partial x} + [\Omega^{(\mu)}, A^{(\mu)}] \qquad \mu = 1, \dots, n, \infty \tag{2.3.9}$$

$$dG^{(\nu)} = \Theta^{(\nu)} G^{(\nu)} \qquad \nu = 1, \dots, n. \tag{2.3.10}$$

Here $\Omega^{(\mu)} = \Omega^{(\mu)}(x,t)$ *and* $\Theta^{(\nu)} = \Theta^{(\nu)}(x,t)$ *are matrices of* 1*-forms given explicitly in Thm. 3.1 of [**JMU81**].*

We don't give all the formulas explicitly in the previous theorem as we are only interested in the result of the theorem. That is, we are only interested in knowing that the partial monodromy data (2.3.8) is invariant under deformations that satisfy some total differential equations (i.e. equations (2.3.9) and (2.3.10)). This way, we can determine if the system of isomonodromic deformations is Frobenius integrable by the Frobenius Theorem 2.1.3, which it is. Namely,

THEOREM 2.3.2 (Thm. 4.2 [**JMU81**]). *The ideal* $\mathcal{I}$*, generated by the total differential equations* (2.3.9) *and* (2.3.10)*, is d-closed:* $d\mathcal{I} \subset \mathcal{I}$*.*

CHAPTER 3

Random Walkers

We review [**HO07**] and [**BBT03**]. In the latter, the authors introduce the KP(Kadomtsev-Petviashvili) and TL (Toda Lattice) tau functions as the orbit of commuting flows on the Sato Grassmanian using the theory of infinite fermionic Fock spaces, and in the former the authors consider random walkers on possibly infinite graphs with hard core interactions using the theory of infinite fermionic Fock spaces and give the partition function of such processes as KP or TL tau functions.

3.1. Fermionic Fock Space

Let us first define fermionic Fock spaces. In physics, one takes a vector space with a specific choice of basis vector, and the basis vectors (properly normalized) are interpreted to be particles, either bosons or fermions. The distinguishing feature between these two types of particles is that multiple bosons may occupy the same "sate" at the same time while multiple fermions may not occupy the same "state" at the same time. Mathematically, this means that bosons correspond to symmetrization of tensor products of the original vector space and fermions corresponds to anti-symmetrization of tensor products of the original vector space (i.e. the wedge product). In the rest of the chapter, we will focus on developing fermionic Fock spaces and we develop the physical interpretation as needed.

3.1.1. Finite Fermionic Fock Space. Take a finite dimensional complex vector space V, say $\dim V = n$, with a specific choice of basis vectors $\{e_1, \dots, e_n\}$ which we use for convenience.

Definition 3.1.1. *The fermionic Fock space of V is the graded vector space*

$$\mathcal{F}(V) = \bigoplus_{i=0}^{n} \left(\bigwedge^{i} V \right)$$

with $deg\left(\bigwedge^i V\right) = i$. In the physics literature, the degree is referred as the "charge".

Elements in the vector space V represent a "state". That is, when one speaks of a particle being in a certain state, one is just specifying an element in the vector space. Thus, one calls V the "state space". Then, the vector space $\bigwedge^i V$ represents the "state space" of i fermions. We see that two particles may not be in the same state since $v \wedge v = 0$ for any $v \in V$, which is the defining characteristic of the wedge product.

Moreover, we may construct a *Clifford algebra* from the vector space V. (For more information on Clifford algebras see the appendix of [**JMMS80**].) We consider the vector space $W = V \oplus V^*$ with a natural symmetric bilinear form given by

$$\langle v_1 \oplus v_1^*, v_2 \oplus v_2^* \rangle = v_1^*(v_2) + v_2^*(v_1) \text{ for } v_1, v_2 \in V, \text{ and } v_1^*, v_2^* \in V^*.$$

Then, we may consider the Clifford algebra $C\ell(W)$. Given the dual basis of V^* as $\{e_1^*, \dots, e_n^*\}$, the algebra $C\ell(W)$ is generated by $\{e_1, \dots, e_n, e_1^*, \dots, e_n^*\}$ with the defining relations

$$e_i e_j^* + e_j^* e_i = \delta_{ij}, \quad e_i e_j + e_j e_i = 0, \quad e_i^* e_j^* + e_j^* e_i^* = 0, \quad i, j = 1, \dots, n. \tag{3.1.1}$$

We intend to define an action on the fermionic Fock space $\mathcal{F}(V)$ by the Clifford algebra $C\ell(W)$. To avoid confusion we will use an abstract basis for $C\ell(W)$, namely $\{\beta_1, \dots, \beta_n, \beta_1^*, \dots, \beta_n^*\}$, which obeys the same relations as (3.1.1). Then, we define the action of $C\ell(W)$ on $\mathcal{F}(V)$ by having β_i act by the exterior product

$$\beta_i \mapsto e_i \wedge \cdot \quad i = 1, \dots, n,$$

and β_i^* act by inner product

$$\beta_i^* \mapsto \langle e_i, \cdot \rangle \quad i = 1, \dots, n.$$

REMARK 3.1.1. *Notice that the fermionic Fock space $\mathcal{F}(V)$ is generated by the action of the Clifford algebra $C\ell(V \oplus V^*)$ on the degree zero identity element $1 \in \bigwedge^0 V = \mathbb{C}$. Also, one has that if v has homogeneous degree, then $deg(\beta_i v) = deg(v) + 1$ and $deg(\beta_i^* v) = deg(v) - 1$ for $i = 1, \dots, n$.*

Therefore, the elements β_i and β_i^* are referred to as fermionic operators, where the former are called creation operators and the latter are called annihilation operators.

3.1.2. Infinite Fermionic Fock Spaces. We have just defined fermionic Fock spaces on finite dimensional vector spaces, and now we wish to generalize the definition to infinite vector spaces (with countable basis). In the previous subsection, we defined the fermionic Fock space as a direct sum of wedge products of the vector space. That is fine for finite vector spaces since the N^{th} wedge product vanishes for N large enough, but for infinite dimensional vector spaces the wedge product doesn't vanish for large N. Thus, if one wishes to define the fermionic Fock space of an infinite dimensional vector space exactly as in the finite dimensional vector case, one would first have to make sense of the infinite wedge product. While this may be possible using direct limits, we instead use the observations in Remark 3.1.1 to define a fermionic Fock space for infinite dimensional vector space.

Take an infinite dimensional vector space V with a choice of a countable basis $\{e_r\}_{r\in\mathbb{Z}+\frac{1}{2}}$ which we use for convenience and where the index are half-integers as this labeling will be more natural in the following section 3.3. Now, note that, in spite of being infinite dimensional, the Clifford algebra $C\ell(V\oplus V^*)$ of V is still well-defined. Namely, we have an abstract basis $\{\beta_j,\beta_j^*\}_{j\in\mathbb{Z}+\frac{1}{2}}$ of $C\ell(V\oplus V^*)$ with the relations

$$\beta_i\beta_j^* + \beta_j^*\beta_i = \delta_{ij},\quad \beta_i\beta_j + \beta_j\beta_i = 0,\quad \beta_i^*\beta_j^* + \beta_j^*\beta_i^* = 0,\qquad i,j\in\mathbb{Z}+\frac{1}{2}. \tag{3.1.2}$$

Definition 3.1.2. *Let $|0\rangle$ be the "vacuum vector". Then, the infinite fermionic Fock space $\mathcal{F}_\infty(V)$ of an infinite dimensional vector space V (as above) is the vector space generated by the action of $C\ell(V\oplus V^*)$ on the "vacuum vector" $|0\rangle$. That is, the basis of $\mathcal{F}_\infty(V)$ is given by*

$$\beta_{i_1}\cdots\beta_{i_s}\beta_{j_1}^*\cdots\beta_{j_r}^*|0\rangle,\quad r,s\in\mathbb{N},\ \ i_1,\dots,i_s,j_1,\dots,j_r\in\mathbb{Z}+\frac{1}{2}$$

where $\beta_{-i}|0\rangle = \beta_i^*|0\rangle = 0$ for $i>0$.

REMARK 3.1.2. *Heuristically, one may think of the vacuum vector as a "infinite Dirac sea of fermions". That is,*

$$|0\rangle \text{``} = \text{''} e_{-1/2} \wedge e_{-3/2} \wedge e_{-5/2} \wedge \dots .$$

Also note that, if we were to define an infinite wedge product "$\bigwedge^\infty V$", there would be no way for the Clifford algebra to act on elements of "$\bigwedge^\infty V$" so that the result would be in a finite wedge product $\bigwedge^N V$. Thus, the infinite wedge products and finite wedge products would be disjoint through the action of the Clifford algebra (i.e. they would not be in the same orbit). So, in our definition of the infinite fermionic Fock space we essentially kept the part of the infinite wedge product that is of interest to us (i.e. making the action of the Clifford algebra $C\ell(V \oplus V^*)$ transitive).

In this construction, we can also define a grading by letting the vacuum vector have degree zero (i.e. $\deg(|0\rangle) = 0$) and having the creation operators increase the degree of an element by +1 and the annihilation operators decrease the degree of an element by −1. That is,

$$\deg\left(\beta_{i_1} \cdots \beta_{i_s} \beta^*_{j_1} \cdots \beta^*_{j_r} |0\rangle\right) = s - r.$$

Moreover, let us introduce the dual of the infinite fermionic Fock space $\mathcal{F}^*_\infty(V)$. In particular, we consider the dual of the vacuum vector, denoted by $\langle 0|$ and given the natural pairing

$$\mathcal{F}^*_\infty(V) \times \mathcal{F}_\infty(V) \to \mathbb{C}$$

we define the (adjoint/dual) action of $C\ell(V \oplus V^*)$ on $\mathcal{F}^*_\infty(V)$. As such, we define the *vacuum expectation value* as the number given by the natural pairing $\langle 0|\beta_{i_1} \cdots \beta_{i_s} \beta^*_{j_1} \cdots \beta^*_{j_r} |0\rangle$.

NOTATION 3.1.1. *In the rest of the chapter we drop the dependence of V when we write the infinite Fock space and we just write $\mathcal{F}_\infty$. Also, when we refer to the fermionic Fock space of a finite vector space, we will just write $\mathcal{F}_n$ where the subscript denotes the dimension of the finite vector space considered.*

3.2. Tau Function

A distinguishing feature of tau functions is that they satisfy the Hirota bilinear equation. In the following, we define the KP and TL tau functions as the "vacum expectation value" of an operator on the vacuum vector. We will make this precise in the following subsections. First, we introduce the action of $gl(n)$ and $GL(n)$ on the finite fermionic Fock space, and we generalize this action in the infinite case where one has to take care and not introduce any unwanted infinities. Also, we introduce the "fermionic field" which is a generating function of fermionic operators. Then, we have that the tau function is the projection onto the vacuum vector of the orbit of a generating function on an element of the infinite fermionic Fock space given by the action of some element in $GL(\infty)$.

3.2.1. $GL(n)$ Action on $\mathcal{F}_n$. The group action that we aim to consider for the infinite fermionic Fock space is rather special and interesting. Although, we may just describe the action for the infinite case, perhaps it is more instructive to consider the same type of action in the finite case, which will bring forth the proper interpretation of the action we define later. That is, let us consider the action of $GL(n)$ on $\mathcal{F}_n$. In particular, we show that the orbit of certain elements correspond to the embedding of Grassmanians into the projectivization of $\mathcal{F}_n$.

Recall that $\mathcal{F}_n$ is just the direct sum of wedge product of some finite vector space of dimension n. Then, the action of $GL(n)$ on $\mathcal{F}_n$ is just the component-wise action on each direct summand and the action on each summand is the action induced by the "canonical" action

$$GL(n) \times V \to V.$$

Then, the induced action is given by linearly extending the action

$$g \cdot (v_1 \wedge \cdots \wedge v_m) \longmapsto (g \cdot v_1) \wedge \cdots \wedge (g \cdot v_m)$$

for any $v_1, \ldots, v_m \in V$ and $g \in GL(n)$.

Now, recall that a set of vectors $v_1, \ldots, v_m \in V$ are linearly independent if and only if $v_1 \wedge \cdots \wedge v_m \neq 0$ if and only if the linear span of the vectors defines an m-hyperplane in V. Moreover, we have that the space of m-hyperplanes in an n-dimensional vector space V is a Grassmanian and it is denoted by $Gr(m, V)$. It is easy to check that the action of $GL(n)$ on m-hyperplanes induced by the action of $GL(n)$ on V is transitive. Therefore, we have the embedding into projective space

$$Gr(m,V) \hookrightarrow \{ge_1 \wedge \dots ge_m | g \in GL(n)\}/\mathbb{C} \subset \mathbb{P}\left(\bigwedge^m V\right)$$

for any set of independent vectors $e_1, \dots, e_m \in V$. This is the famous *Plücker embedding.* Since this is an algebraic map, the points of this embedding satisfy some algebraic conditions called *Plücker relations.* To see this, consider any other m-hyperplane $(x_1, \dots, x_m)$ (i.e. set of independent vectors) and expand it using a fixed basis $(e_1, \dots, e_n)$. That is, $x_j = \sum_{i=1}^n x_j^i e_i$ where we can then write the image of the m-hyperplane $(x_1, \dots, x_m)$ in the wedge product as

$$x_1 \wedge \dots \wedge x_m = \sum_{0<h_1<\dots<h_m<n+1}^{n} X^{h_1,\dots,h_m} e_{h_1} \wedge \dots \wedge e_{h_m}$$

where $X^{h_1,\dots,h_m} = \det\left(x_j^{h_i}\right)_{i,j=1}^m$, which are the *Plücker coordinates.* Then, one has that the coordinates satisfy the *Plücker bilinear relations*

$$\sum_{j=1}^{m+1} X^{k_1,\dots,h_j,\dots,k_{m-1}} X^{h_1,\dots,\hat{h}_j,\dots,h_{m+1}} = 0$$

for all $0 < k_1 < \dots < k_{m-1} < n+1$ and $0 < h_1 < \dots < h_{m+1} < n+1$ where $\hat{h}_j$ is omitted. The proof of these relations is well-known and can be found in many books (see [**BBT03**]).

3.2.2. $\hat{GL}(\infty)$ and $\hat{gl}(\infty)$ Action on $\mathcal{F}_\infty$. In the previous subsection, we saw that the action of $GL(n)$ on certain elements of the finite Fock space correspond to points on the Grassmanian, and the coordinates of these points satisfy the Plücker bilinear relations. In the rest of the section we generalize all of this for the infinite dimensional case. We begin by defining the action of the Lie algebra $\hat{gl}(\infty)$, which is a central extension of $gl(\infty)$, and its Lie group $\hat{GL}(\infty)$. (For a detailed definition of $GL(\infty)$ and $gl(\infty)$ see [**Sat81**].) Indeed, the action of the Lie group $GL(\infty)$ and its Lie algebra $gl(\infty)$ on $\mathcal{F}_\infty$ is not well-defined due to some infinities.

We have that $gl(\infty)$ is the set of infinite band matrices $(M)_{r,s\in\mathbb{Z}+\frac{1}{2}}$ (i.e. infinite matrices such that $M_{r,s} = 0$ for $|r-s| \gg 0$). Then, one would expect the action of an infinite band matrix to be

$$(M_{r,s})_{r,s\in\mathbb{Z}+\frac{1}{2}} \mapsto \sum_{r,s\in\mathbb{Z}+\frac{1}{2}} M_{r,s}\beta_r\beta_s^*$$

This action is not well-defined. Consider the simplest example the identity matrix , $M_{r,s} = \delta_{r,s}$, then by (3.1.2) and the definition of the vacuum vector, we have

$$\beta_r \beta_r^* |0\rangle = 0 \qquad r > 0$$

$$\beta_r \beta_r^* |0\rangle = (1 - \beta_r^* \beta_r)|0\rangle = |0\rangle \qquad r < 0.$$

Thus, we have that

$$(\delta_{r,s})_{\mathbb{Z}+\frac{1}{2}} \cdot |0\rangle = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \beta_r \beta_r^* |0\rangle = \sum_{r<0} |0\rangle \text{``}={}\text{''} \infty |0\rangle$$

and we see that the sum is not well-defined since it would infinite. We can remedy this situation by introducing *normal ordering* (see [**BBT03**] for more details). We denote the *normal order* of the product of two fermionic operators by $:\beta_r \beta_s^*:$ and define it by

$$\beta_r \beta_s^* =: \beta_r \beta_s^* : + \langle 0 | \beta_r \beta_s^* | 0 \rangle.$$

Then, reproducing the previous calculations with normal ordering instead, we have

$$:\beta_r \beta_r^*: |0\rangle = (\beta_r \beta_r^* - \langle 0|\beta_r \beta_r^* |0\rangle)|0\rangle = \beta_r \beta_r^* |0\rangle = 0 \qquad r > 0 \tag{3.2.1}$$

$$:\beta_r \beta_r^*: |0\rangle = (\beta_r \beta_r^* - \langle 0|\beta_r \beta_r^* |0\rangle)|0\rangle = -\beta_r^* \beta_r |0 > = 0 \qquad r < 0. \tag{3.2.2}$$

Thus, we resolved the problem of the infinite sum for $(\delta_{r,s})_{r,s \in \mathbb{Z}+\frac{1}{2}}$ by introducing normal ordering. In general, this is the only problem that arises (i.e. infinite non-zero entries on the diagonal with negative index), and so the action

$$(M_{r,s})_{r,s \in \mathbb{Z}+\frac{1}{2}} \mapsto \sum_{r,s \in \mathbb{Z}+\frac{1}{2}} M_{r,s} : \beta_r \beta_s^* :$$

is well-defined. Still, we are not done defining this action since the Lie bracket on the fermionic operators is not the same as the the Lie bracket on the infinte band matrices. That is,

(3.2.3)
$$\left[\sum_{r,s\in\mathbb{Z}+\frac{1}{2}} M_{r,s} :\beta_r\beta_s^*:, \sum_{n,m\in\mathbb{Z}+\frac{1}{2}} N_{n,m} :\beta_n\beta_m^*:\right] = \sum_{r,m\in\mathbb{Z}+\frac{1}{2}} [M,N]_{r,m} :\beta_r\beta_m^*: + \sum_{r,s\in\mathbb{Z}+\frac{1}{2}} M_{r,s}N_{s,r}(\theta(r)-\theta(s))$$

where the Lie bracket on the right-hand side is the usual Lie bracket on $gl(\infty)$ and the $\theta(m) = 1$ for $m > 0$ and zero otherwise. Therefore, we define

$$\hat{gl}(\infty) := gl(\infty) \oplus \mathbb{C},$$

the central extension of $gl(\infty)$, with the Lie bracket defined by (3.2.3). Therefore, we have defined an action of $\hat{gl}(\infty)$ on the infinite fermionic Fock space. Then, by taking the exponential map of $\hat{gl}(\infty)$, one may define a central extension $\hat{GL}(\infty)$ of $GL(\infty)$, and similarly one defines the action of $\hat{GL}(\infty)$ on $\mathcal{F}_\infty$. For our purposes, we consider group elements of the form

$$g = e^{M_1}e^{M_2}\cdots e^{M_k} \in \hat{GL}(\infty)$$

with $M_1, \ldots, M_k \in \hat{gl}(\infty)$, and the group action given by

$$e^M \longmapsto 1 + \sum_{r,s} M_{r,s} :\beta_r\beta_s^*: + \frac{1}{2!}\left(\sum_{r,s} M_{r,s} :\beta_r\beta_s^*:\right)^2 + \ldots$$

3.2.3. Fermionic Fields. We are almost to the point where we can define the KP and TL tau functions. Before we do that, we have to introduce the *fermionic fields*

$$\beta(z) := \sum_{r\in\mathbb{Z}+\frac{1}{2}} \beta_r z^{-r-1/2}$$
$$\beta^*(z) := \sum_{r\in\mathbb{Z}+\frac{1}{2}} \beta_{-r}^* z^{-r-1/2},$$

which are just generating functions of the fermionic operators. Moreover, we take the normal order of the product of the fermionic fields and we obtain

$$:\beta(z)\beta^*(z) := \sum_{n\in\mathbb{Z}} H_n z^{n-1}$$

where

$$H_n = \sum_{r\in\mathbb{Z}+\frac{1}{2}} :\beta_r\beta^*_{r+n}: . \tag{3.2.4}$$

Note that the operators H_n are just the image of infinite band matrices with only 1's on the n^{th} off-diagonal (i.e. $M = (\delta_{r,r+n})_{r\in\mathbb{Z}+\frac{1}{2}}$). From this point of view, one would expect that the H_n operator behave similar to shift operators, which they do in many applications, but there are some subtle differences that one should notice. In particular, we have that

PROPOSITION 3.2.1. *The operators $\beta(z)$, $\beta^*(z)$, and H_n for any $n \in \mathbb{Z}$ satisfy the following commutation relations:*

$$[H_n, \beta(z)] = z^{-n}\beta(z)$$

$$[H_n, \beta^*(z)] = -z^{-n}\beta^*(z)$$

$$[H_n, H_m] = -n\delta_{n+m,0}.$$

PROOF. We consider the first and last relation as the second relation is similar to the first relation. Also, we will disregard the cases $n = m = 0$, as they are more tedious than illuminating. For the first relation, it suffices to compare

$$:\beta_r\beta^*_{r+n}:\beta_s z^{-s-1/2} = \begin{cases} -\beta_r\beta_s\beta^*_{r+n}z^{-s-\frac{1}{2}} & s \neq r+n \\ \beta_r z^{-r-n-\frac{1}{2}} - \beta_r\beta_s\beta^*_{r+n}z^{-s-\frac{1}{2}} & s = r+n \end{cases}$$

with

$$\beta_s z^{-s-1/2} :\beta_r\beta^*_{r+n}:= -\beta_r\beta_s\beta^*_{r+n}z^{-s-\frac{1}{2}},$$

which gives us the desired result.

Now, the last relation is a bit more subtle, but it still just boils down to comparing a few terms. We have

$$:\beta_r\beta^*_{r+m}::\beta_s\beta^*_{s+n}:=\begin{cases}-\beta_r\beta_s\beta^*_{r+m}\beta^*_{s+m} & s\neq r+m\\ :\beta_r\beta^*_{r+m+n}:+\langle 0|\beta_r\beta^*_{r+m+n}|0\rangle-\beta_r\beta_s\beta^*_{r+m}\beta^*_{s+m} & s=r+m\end{cases}$$

and

$$:\beta_s\beta^*_{s+n}::\beta_r\beta^*_{r+m}:=\begin{cases}-\beta_r\beta_s\beta^*_{r+m}\beta^*_{s+m} & r\neq s+n\\ :\beta_{r-n}\beta^*_{r+m}:+\langle 0|\beta_{r-n}\beta^*_{r+m}|0\rangle-\beta_r\beta_s\beta^*_{r+m}\beta^*_{s+m} & r=s+n.\end{cases}$$

Then, making sure not to interchange the infinite sums and keeping track of the labels, we have that

$$\begin{aligned}[H_n,H_m]&=\sum_{r\in\mathbb{Z}+\frac{1}{2}}\left(\langle 0|\beta_r\beta^*_{r+m+n}|0\rangle-\langle 0|\beta_{r-n}\beta^*_{r+m}|0\rangle\right)\\&=-n\delta_{n+m,0},\end{aligned}$$

giving us the desired result. □

Remark 3.2.1. *One should note that the commutation relations just proved are very similar to the commutation relations for* $(\delta_{r,r+n})_{r\in\mathbb{Z}+\frac{1}{2}}$. *We leave the exact comparison to the interested reader. Instead, we mention that the differences are not due to the introduction of the normal order of the product of fermionic operator, but instead from the fact that the fermionic operators are different objects from regular vector and carry a richer structure. In the physics literature, these fermions are referred to as "spinors" or particles with* 1/2*-spin. One often sees the description of these objects as "square roots of vectors", but they are more accurately described as elements of a "spin representation". For a curious reader, we suggest [**Fre80**] to get a complete idea of the distinction.*

3.2.4. The Tau-Function.

Definition 3.2.1. *The KP tau-function is the generating function*

$$\tau_{KP}(\vec{t};g):=\langle 0|e^{H(\vec{t})}g|0\rangle,\qquad g\in\hat{GL}(\infty)$$

where $H(\vec{t}) = \sum_{n>0} t_n H_{-n}$ *and* $\vec{t} = (t_1, t_2, \dots)$ *are formal parameters. Similarly, one defines the TL tau function*

$$\tau_{TL}(\vec{t}, \vec{t}'; g) := \langle 0| e^{H(\vec{t})} g e^{\bar{H}(\vec{t}')} |0\rangle, \qquad g \in \hat{GL}(\infty)$$

where $\bar{H}(\vec{t}) = \sum_{n>0} \bar{t}_n H_n$ *and* $\vec{t}' = (t'_1, t'_2, \dots)$ *are formal parameters.*

Note that the tau-functions are completely determined by the the element $g \in GL(\infty)$, or rather the element $g|0\rangle$ which oddly enough has a distinct interpretation.

Definition 3.2.2. *Let*

$$\tilde{GM}(n,\infty) := \begin{cases} \left\{ g\beta_{n-\frac{1}{2}} \cdots \beta_{\frac{1}{2}} |0\rangle | g \in GL(\infty) \right\} \Big/ \mathbb{C} & n \geq 0 \\ \left\{ g\beta^*_{-n+\frac{1}{2}} \cdots \beta^*_{-\frac{1}{2}} |0\rangle | g \in GL(\infty) \right\} \Big/ \mathbb{C} & n < 0 \end{cases}$$

be the infinite Sato Grassmanian of relative dimension n.

There are different definitions of the infinite Sato Grassmanian in the literature. In particular, we have the original definitions in [**Sat81**,**SW85**] where the authors define the infinite Grassmanian as a projective limit of finite Grassmanians or as linear subspace of a polarized separable Hilbert space which satisfy certain conditions under a specific projection operator, respectively. Here, we take an unusual and more pragmatic approach and define the Sato Grassmanian as a subspace of the infinite Fock space. The important feature in all of these generalizations of finite Grassmanians is that there are relations analogous to the bilinear Plücker relations. In our case, these are as follows:

Proposition 3.2.2 ([**BBT03**]). *The KP tau-function satisfies the Hirota bilinear identities*

$$\oint \frac{dz}{2\pi i} \exp\left(\sum_{n>0} 2z^n y_n \right) \exp\left(-\sum_{n>0} \frac{z^{-n}}{n} \frac{\partial}{\partial y_n} \right) \tau_{KP}(t+y; g) \tau_{KP}(t-y; g) = 0.$$

It turns out that this system of bilinear equations are equivalent to the KP hierarchy. More precisely, we have that

Proposition 3.2.3 ([**BBT03**]). *Define the Baker-Akhiezer function as follows*

$$\Psi(t,z) := \frac{\tau(\vec{t} - [z^{-1}])}{\tau(\vec{t})} e^{\xi(\vec{t};z)}$$

where $[z^{-1}] = \left(\frac{1}{z}, \frac{1}{2z^2}, \frac{1}{3z^3}, \dots\right)$ *and* $\xi(\vec{t}; z) = \sum_{i=1}^{\infty} t_i z^i$. *Then, the Baker-Akhiezer is an eigenfunction of the KP-hierarchy equations (def. 2.2.1). Namely, we have that*

$$(L - z)\,\Psi = 0$$
$$\left(\frac{\partial}{\partial t_n} - B_n\right)\Psi = 0,$$

where L *and* B_n *for* $n = 0, 1, \dots$ *are defined in equations* (2.2.3) *and* (2.2.4).

The proof of the previous two statements depend on properly manipulating the Plücker relations for the Sato Grassmanian. We choose to omit the proofs since they are long, tedious, and can be found in the book [**BBT03**].

3.3. Random Walkers on a Graph

In this section, we use the tools developed in the previous sections and apply them to random processes from probability theory. More specifically, we systematically represent elements of an infinite Fock space as particles on an integer lattice and, equivalently, as Young diagrams. Then, we establish the "dynamics" of a random process, be it of random walkers or, equivalently, of random growing interfaces. Also, we see that the tau-function is the partition function of such random processes. Among the different random processes described, we obtain the Plancherel growth process as a specific specialization of our construction. Most of the work here is inspired by [**HO07**].

3.3.1. Particles on an Integer Lattice. Recall our chosen basis of the infinite fermionic Fock space. That is,

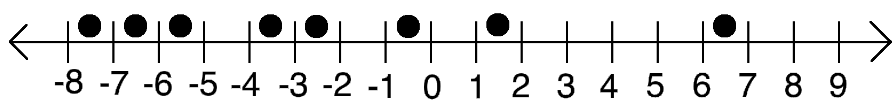
$$\beta_{i_1} \cdots \beta_{i_s} \beta^*_{j_1} \cdots \beta^*_{j_r} |0\rangle$$

such that $j_1 < \cdots < j_r < 0$ and $i_1 > \cdots > i_s > 0$. Then, to each basis vector we may assign a particle configuration on the lattice $\mathbb{Z} + \frac{1}{2}$ where the position of the n^{th} particle is given by

$$x_n := \begin{cases} i_n & 1 \le n \le s \\ -n + s + \frac{1}{2} - \sum_{m=1}^{r} \theta(j_m + n - s) & n > s \end{cases}$$

where $\theta(m) = 1$ for $m < 0$ and zero otherwise.

We thus have a list $(x_n)_{n=1}^{\infty}$ of strictly deceasing numbers $x_1 > x_2 > x_3 > \ldots$, meaning that no two particles may be at the same position. Although this might seem like a simple consequence from the anti-commutation relation of the fermionic operators, it does imply that all the "processes" that we consider on the infinite fermionic Fock space have an *exclusion property* where no two particles may occupy the same location at any given time. The exclusion property is a characteristic property of "fermion" (as opposed to "boson") particles in physics, and it is what we are most familiar with in nature as we all know that we can't walk through walls! Also, let us note that it is often useful and convenient to represent the particle configuration $(x_n)_{n=1}^{\infty}$ with black dots on the real line at the position of the particles. For example, the following diagram



represents the particle configuration $x_1 = 13/2 > x_2 = 3/2 > x_3 = -1/2 > x_4 = -5/2 > x_5 = -7/2 > x_6 = -11/2 > \cdots > x_n = -n + 1/2$, which also represents the infinite Fock space element $\beta_{13/2}\beta_{1/2}\beta^*_{-3/2}\beta^*_{-9/2}|0\rangle$.

Given our interpretation of the basis vectors of the infinite fermionic Fock space $\mathcal{F}_\infty$, let us consider an interpretation for any vector in $\mathcal{F}_\infty$. For this, we go back to the probabilistic interpretation given in "Quantum Mechanics" (see [**Dir30**]). That is, for any vector $|v\rangle \in \mathcal{F}_\infty$, the coefficients of its basis expansion

$$|v\rangle = \sum_{\vec{i},\vec{j}} C_{\vec{i},\vec{j}} \left(\beta_{i_1}\cdots\beta_{i_s}\beta^*_{j_1}\cdots\beta^*_{j_r}|0\rangle\right)$$

give the "relative" probability of being at any given particle configuration as described by the basis vectors. That is, the probability of being in the particle configuration given by the basis vector $\beta_{i_1}\cdots\beta_{i_s}\beta^*_{j_1}\cdots\beta^*_{j_r}|0\rangle$ is

$$P_{|v\rangle}(\beta_{i_1}\cdots\beta_{i_s}\beta^*_{j_1}\cdots\beta^*_{j_r}|0\rangle) := \frac{|C_{\vec{i},\vec{j}}|^2}{\sum_{\vec{i},\vec{j}}|C_{\vec{i},\vec{j}}|^2} = \frac{\langle v|\beta_{i_1}\cdots\beta_{i_s}\beta^*_{j_1}\cdots\beta^*_{j_r}|0\rangle\langle 0|\beta^*_{i_1}\cdots\beta^*_{i_s}\beta_{j_1}\cdots\beta_{j_r}|v\rangle}{\langle v|v\rangle}.$$

More precisely, let

$$\mathcal{B} := \left\{\beta_{i_1}\cdots\beta_{i_s}\beta^*_{j_1}\cdots\beta^*_{j_r}|0\rangle\right\} \tag{3.3.1}$$

be the set of basis vectors given in the definition of $\mathcal{F}_\infty$, then we have that for any non-zero $|v\rangle \in \mathcal{F}_\infty$ the triplet $(\mathcal{B}, 2^{\mathcal{B}}, P_{|v\rangle})$ defines a discrete probability space where $\mathcal{B}$ is the sample space, $2^{\mathcal{B}}$ is the event space, $P_{|v\rangle}$ is the probability measure. This is an important point of view that allows us to define precisely what we mean by a "random process" on the infinite fermionic Fock space $\mathcal{F}_\infty$. In particular, given the probability spaces, we define our "random processes" to be stochastic processes where the Markov matrix of such a process is given by the operator acting on the infinite fermionic Fock space. (See [**Spi70**] for more details on Markov processes.) In any case, we keep this probability interpretation throughout the rest of our discussion.

3.3.2. Young Diagrams and Random 2D Interface. There are other interpretation of the basis vectors of the infinite fermionic Fock space $\mathcal{F}_\infty$. Of course, when we say interpretation, we mean bijection. In the previous subsection, we saw that the basis vectors are in one-to-one correspondence with a strictly decreasing sequence of numbers $(x_n)_{n=1}^{\infty}$ such that $x_i \in \mathbb{Z} + \frac{1}{2}$ for every $i \in \mathbb{N}$ and there is only finite number of terms such that $x_i - x_{i+1} > 1$. From this sequence of numbers, we may also define a sequence of non-increasing integers $(\tilde{\lambda}_n)_{n=1}^{\infty}$ by

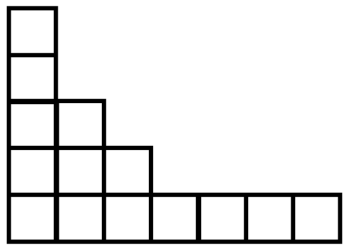
$$\tilde{\lambda}_i := x_i + i - \frac{1}{2} \qquad i \geq 0,$$

where we have that $\tilde{\lambda}_i = \tilde{\lambda}_{i+1}$ for $i \gg 0$. In particular, for a given basis vector $\beta_{i_1} \cdots \beta_{i_s} \beta^*_{j_1} \cdots \beta^*_{j_r} |0\rangle$, $\tilde{\lambda}_i = s - r$ for $i \gg 0$ (i.e. the value of the sequence stabilizes at the degree of the basis vector). Therefore, we have that basis vector of degree zero are in one-to-one correspondence with integer partitions. In fact, if we know the degree of our basis vector, say deg $= n$, we have an integer partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0 \geq 0 \geq \dots)$ where $\lambda_i := \tilde{\lambda}_i - n$ for $i \geq 1$. Thus, from now on, we write our basis vectors as follows

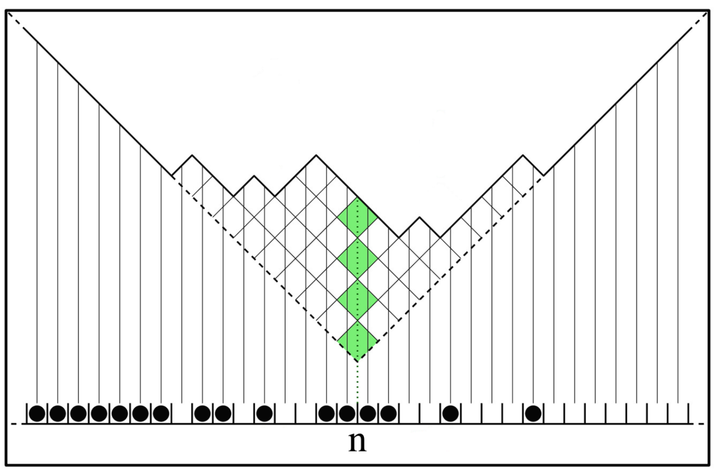
$$|\lambda, n\rangle := \beta_{i_1} \cdots \beta_{i_s} \beta^*_{j_1} \cdots \beta^*_{j_r} |0\rangle$$

where n is the degree of our basis vector (i.e $n = s - r$). Moreover, we may construct a *Young diagram* from a integer partition. Given a non-increasing sequence of integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0 \geq 0 \geq 0 \geq \dots)$, the corresponding Young diagram is a stack of boxes where the i^{th} row has λ_i boxes.

For example, given the basis vector $\beta_{13/2}\beta_{1/2}\beta^*_{-3/2}\beta^*_{-9/2}|0\rangle$, we saw in the previos subsection that it has a particle configuration given by $x_1 = 13/2 > x_2 = 3/2 > x_3 = -1/2 > x_4 = -5/2 > x_5 = -7/2 > x_6 = -11/2 > \cdots > x_n = -n + 1/2$. We note that the degree of the basis element is 0, and thus it has an integer partition $\lambda = (7, 3, 2, 1, 1, 0, 0, \dots)$ with a Young diagram as follows:



In fact, if we rotate our Young diagram 45^o, we can explicitly see the bijection to the particle configuration diagram, seen in the previous section. For example,



In the diagram, we have that the slope of the solid lines is +1 over empty sites and the slope is −1 over the occupied sites. Also, we have that the dashed line corresponds to the degree n vacuum vector(i.e. $|\varnothing, n\rangle$). We say that the rotated Young diagram is a *2D Interface* on an infinite wedge. Applying specific operators to elements in $\mathcal{F}_\infty$, along with this interpretation, yields a "random growth interface", which lead to many interesting physical and mathematical results. (See [**KPZ86**, **Cor12**] for a review and results on "growing interface models".)

Lastly, we note that the introduction of new notation to represent the basis vectors of the infinite Fock space is not only more compact and succinct, but also it is more convenient in our computations. Indeed, we have the following formulas [**HO07**]

$$\langle n, \varnothing | e^{H(\vec{t})} = \sum_{\lambda \in P} \langle \lambda, n | s_\lambda(\vec{t}), \qquad e^{\bar{H}(\vec{t})} | \varnothing, n \rangle = \sum_{\lambda \in P} | \lambda, n \rangle s_\lambda(\vec{t}) \tag{3.3.2}$$

where P is the set of all integer partitions and $s_\lambda(t)$ is the Schur polynomial corresponding to the plane partition λ. Schur functions can be defined as character of irreducible representations of general linear groups (i.e. $GL(n,\mathbb{C})$) where Young tableaux (i.e. Young diagrams with a certain labeling) are used to label all the irreducible representations. (See [**FH91**] for more details on this.) Thus, it is of no surprise that such formulas appear since we are working with $GL(\infty)$. (For a precise definition of Schur polynomial see [**Mac95**])

3.3.3. Partition Function and Generator of Random Processes. As we now interpret the basis vectors of the infinite Fock space $\mathcal{F}_\infty$ as a particle configuration, let us consider the action of the operators $:\beta_r \beta^*_{r+n}:$ on the particle configurations. Take any particle configuration $\vec{x} = (x_i)_{i=1}^\infty$ as discussed in section 3.3.1 (i.e. with $x_i - x_{i+1} = 1$ for $i \gg 0$), then for the new particle configuration,

$$:\beta_r \beta^*_{r+n}: \vec{x} \mapsto \vec{x}',$$

with $n \neq 0$, we have two possibilities for the new particle configuration $\vec{x}'$ depending on the initial particle configuration $\vec{x}$. If there is no particle at position $r+n$ or if there is a particle at position r, then $\vec{x}' = 0$ due to the defining relations of the fermionic operators. Otherwise, if there is a particle at position $r+n$ and no particle at position r, then $\vec{x}'$ will be the particle configuration same as that of $\vec{x}$ except for that there will now be a particle at position r and no particle at position $r+n$. That is, the operator $:\beta_r \beta^*_{r+n}:$ made the particle at position $r+n$ "jump" to the position r, and if there is no particle at position $r+n$ or the position r is occupied, there will be no "jump". Note how the exclusion rule is to be enforced by only having jumps to vacant sites. Therefore, these operator will be the building block that we will use to define random walks on graphs.

We start with some definitions for a quick reference of probability. First, Definition 3.3.1. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space,* $(\Sigma, \mathcal{S})$ *a measurable space, and* T *a totally ordered set. Then, a family of* Σ*-valued random variables,*

$$\{X_t : \Omega \to \Sigma | t \in T\},$$

is a stochastic process.

We invite the reader to reference [**Lig10**] for further conditions on continuity for the stochastic processes. Then, a family of σ-algebras, $\{\mathcal{F}_t\}_{t\in T}$, defined by $\mathcal{F}_t = \sigma\left(\bigcup_{s\leq t} X_s\right)$, is called an *adapted filtration* to the stochastic process $\{X_t\}_{t\in T}$. So that,

Definition 3.3.2. *A stochastic process (as defined in def. 3.3.1) has the Markov property if for every* $t > s$*,*

$$\mathbb{P}(X_t \in A|\mathcal{F}_s) = \mathbb{P}(X_t \in A|X_s)$$

where $\{\mathcal{F}_t\}_{t\in T}$ *is an adapted filtration of the stochastic process* $\{X_t\}_{t\in T}$*.*

We note that the previous definition is non-trivial since in general $\mathcal{F}_t = \sigma\left(\bigcup_{s\leq t} X_s\right) \neq \sigma(X_t)$. In fact, this is a strict condition, which gives processes with this property a very rich structure. Thus,

Definition 3.3.3. *A Markov process is a Stochastic process with the Markov property.*

These are enough definitions so that we can define a Markov process by using the $\hat{gl}(\infty)$-action on the infinite fermionic Fock space $\mathcal{F}_\infty$.

Given a graph $\mathcal{G}$ on the vertex set $\mathbb{Z} + \frac{1}{2}$ without self edges, let $\tilde{A}_{\mathcal{G}} = (a_{i,j})_{i,j\in\mathbb{Z}+\frac{1}{2}}$ be a weighted incident matrix and we require that $a_{i,j} = 0$ for $|i - j| \gg 0$ so that $\tilde{A}_{\mathcal{G}} \in gl(\infty)$. Then, given $\tilde{A}_{\mathcal{G}}$, we define the fermionic operator

$$A_{\mathcal{G}} = \sum_{i,j\in\mathbb{Z}+\frac{1}{2}} a_{i,j} :\beta_i\beta_j^* : \in \hat{gl}(\infty), \tag{3.3.3}$$

which we call the *generator* of our Markov process, and we define a stochastic process.

Definition 3.3.4. *Given a graph* $\mathcal{G}$ *with an incidence Markov "matrix"* $A_{\mathcal{G}} \in \hat{gl}(\infty)$*. Let:*

- *a sample space be* $\Sigma = \mathcal{B}$*, the set of all basis vector of the infinite fermionic Fock space* $\mathcal{F}_\infty$*,*
- *the sigma algebra of* Σ *be* $\mathcal{S} = 2^\Sigma$*, the power set of* Σ*,*

- *a sample space* $\Omega = \{f : T \to \mathcal{S} | f(0) = |\varnothing, 0\rangle\}$ *,* $T = \mathbb{R}_{\geq 0}$.
- *the sigma algebra of* Ω *be* $\mathcal{Q}$ *defined by the "infinite product topology" (i.e. generated by the cylinders* $\{f : T \to \mathcal{B} | f(t) = |\lambda, n\rangle\}$ *for any* $t \in T$*,* $n \in \mathbb{Z}$*, and* λ *a partition),*
- *the probability measure on* $(\Omega, \mathcal{Q})$ *be* $\mathbb{P}\Big(f(t) = |\lambda, n\rangle\Big) = \frac{|\langle n,\lambda|e^{A_{\mathcal{G}}t}|\varnothing,0\rangle|^2}{\langle 0,\varnothing|e^{(A_{\mathcal{G}}^{\dagger}+A_{\mathcal{G}})t}|\varnothing,0\rangle}$*, and*
- *define a measurable function* $X_t : (\Omega, \mathcal{Q}) \to (\Sigma, \mathcal{S})$ *so that* $f \mapsto f(t)$*,*

where $A_{\mathcal{G}}^{\dagger}$ *is the Hermitian conjugate of* $A_{\mathcal{G}}$*. Then,* $\{X_t\}_{t \in T}$ *is a stochastic process that we call "Random Walker on* $\mathcal{G}$*".*

In the following subsection, we will choose a specific generator $A_{\mathcal{G}}$ so that the stochastic process just defined will be a Markov process. For that, we must show that the sum of infinitesimal transition probabilities from a fixed state to any state is constant regardless of the original state. In the literature, this condition is often stated that the constant is 0, but in our case we only require it to be any constant since we normalize our probability measure by the factor $\langle 0, \varnothing|e^{(A_{\mathcal{G}}^{\dagger}+A_{\mathcal{G}})t}|\varnothing, 0\rangle$.

Lastly, let us recall that $\langle\varnothing, n|e^{H(\vec{t})} = \sum_{\lambda\in P}\langle\lambda, n|s_\lambda(\vec{t})$ (from equation (3.3.2)), and take $A_{\mathcal{G}}$, a generator of random walkers. Then,

$$\begin{aligned}\langle\varnothing, 0|e^{H(\vec{t})}e^{A_{\mathcal{G}}t}|0, \varnothing\rangle &= \sum_{\lambda\in P}\langle\lambda, 0|s_\lambda(\vec{t})e^{A_{\mathcal{G}}t}|0, \varnothing\rangle \\ &= \sum_{\lambda\in P} s_\lambda(\vec{t})\langle 0, \lambda|e^{A_{\mathcal{G}}t}|\varnothing, 0\rangle \\ &= \sum_{\lambda\in P} e^{i\phi t}\left|\mathbb{P}\Big(f(t) = |\lambda, 0\rangle\Big)\right|^{1/2} s_\lambda(\vec{t})\end{aligned}$$

is the partition function of the random walkers on a graph $\mathcal{G}$. Note that there is a phase factor $e^{i\phi t}$, which will become clear in the example in next subsection. Therefore,

Proposition 3.3.1. *The partition function of the stochastic process, defined in 3.3.4, corresponding to random walkers on a graph* $\mathcal{G}$ *with an incident matrix* $\tilde{A}_{\mathcal{G}} \in gl(\infty)$ *and generator* $A_{\mathcal{G}} \in \hat{gl}(\infty)$ *(eq. (3.3.3)),*

$$\langle 0|e^{H(\vec{t})}e^{A_{\mathcal{G}}t}|0\rangle,$$

is KP-tau function. Moreover, it satisfies the KP-hierarchy.

PROOF. As we saw in the previous argument, $\langle 0|e^{H(\bar{t})}e^{A_{\mathcal{G}}t}|0\rangle$ is the partition function of the Markov process and by definition is a KP τ-function, and as we know, KP-tau functions satisfy the KP-hierarchy. So, with the background established in this chapter, the proof follows. □

3.3.4. Integrability of Random Walkers on the Line. In the previous subsection, we defined, given an incident matrix $A_{\mathcal{G}} \in gl(\infty)$ without self loops, a stochastic processes on the fermionic Fock space $\mathcal{F}_\infty$, which we call random walkers on $\mathcal{G}$, and we showed that the generating function of these processes are KP τ-function. Now, we choose a specific $A_{\mathcal{G}} \in gl(\infty)$ so that the stochastic process is also a Markov process. For this, our process must satisfy the Markov property (see def. 3.3.3). In turn, this property can be written in terms of (infinitesimal) transitional/conditional probabilities. Let, $p\left(|\lambda, n\rangle \middle| |\mu, m\rangle\right)$ be the (infinitesimal) transitional/conditional probability of the stochastic process, then the Markov property is equivalent to

$$\sum_{|\lambda,n\rangle \in \mathcal{B}} p(|\lambda, n\rangle \| \mu, m\rangle) = constant,$$

independent of $|\mu, m\rangle$. This is a typical condition of continuous Markov matrices (see [**Lig10**]) which ensures that the probability measure is conserved throughout the process (i.e. that the total measure of $\mathbb{P}$ is 1 at each t). We note that the previous condition might be easily confused with the similar condition that $\sum_{i \in \mathbb{Z}+\frac{1}{2}} a_{i,j} = 0$, where $a_{i,j}$ are the coefficients of the incident matrix $A_{\mathcal{G}}$, as this property of a continuous Markov matrix is often stated in terms of the coefficients of the Markov matrix. Indeed, we note that the incident matrix $\tilde{A}_{\mathcal{G}} \in gl(\infty)$ is not the Markov matrix of our Markov process. Indeed, our Markov process is generated by the operator $A_{\mathcal{G}}$ on $\mathcal{F}_\infty$ with a basis $\mathcal{B}$. Therefore, if we write the operator $A_{\mathcal{G}}$ as a matrix with basis $\mathcal{B}$, we don't obtain the matrix $\tilde{A}_{\mathcal{G}}$, necessarily. So, one must take care when checking this second requirement of the continuous Markov process. In the following, we consider an example where an incident matrix gives a continuous Markov process: the Plancherel growth process.

The Plancherel growth process is characterized by being totally asymmetric, simple, exclusion stochastic process with the transition probability rates depending on the configuration of the process, meaning that it has a totally asymmetric jump rate (depending on the configuration of the walkers), only jumping to the neighboring site on the right, and with only one particle per site, respectively. Also, one has to specify the geometry (rather the graph $\mathcal{G}$) of the available sites and the initial configuration of the particles (i.e. initial conditions). In our case, we work with the infinite lattice line with the sites left of the origin occupied (with the rest vacant) and a hoping rate of 1 to the right. That is, the initial configuration is given by $|\varnothing, 0\rangle$ and the incident matrix $\tilde{A}_{\mathbb{Z}} = (a_{i,j})_{i,j \in \mathbb{Z}+\frac{1}{2}}$ is given by

$$a_{i,j} = \begin{cases} 1 & \text{if } i - j = 1 \\ 0 & \text{otherwise} \end{cases} .$$

Then, we have that the operator $A_{\mathbb{Z}} \in \hat{gl}(\infty)$ is the same as $H_1 \in \hat{gl}(\infty)$ (see eqn. (3.2.4)). Moreover, we have that $\mathcal{S} = \{|\lambda, 0\rangle \in \mathcal{B}\}$, where the only requirement that we have is that the charge of the basis vectors is 0. Now, in order to check the Markov property on the operator $A_{\mathbb{Z}}$, we will use the particle configuration introduced in subsection 3.3.1, and we will denote the particle configuration of a basis vector $|\lambda, n\rangle$ by $X(|\lambda, n\rangle) = (x_k(|\lambda, n\rangle))_{k=1}^{\infty}$. Then,

$$\langle m, \mu | A_{\mathbb{Z}} | \lambda, n\rangle = \begin{cases} 1 & \text{if } X(|\mu, m\rangle) - X(|\lambda, n\rangle) = (\delta_{j,k})_{k=1}^{\infty} \\ 0 & \text{otherwise} \end{cases} ,$$

Moreover, we note that

$$A_{\mathbb{Z}}^n |\varnothing, 0\rangle = \sum_{\lambda \vdash n} (\dim \lambda) |\lambda, 0\rangle,$$

where $(\dim \lambda)$ is the dimension of the irreducible representation of the symmetric group S_n (see [**FH91**, **KC03**]), and recall that we have that

$$\sum_{\lambda \vdash n} (\dim \lambda)^2 = n!,$$

by Burnside's lemma. Alternatively, we have that

$$\dim \mu = \sum_{|\lambda, n\rangle \in \Delta(|\mu, m\rangle)} \dim \lambda,$$

with $\dim \varnothing = 1$ and $\Delta(|\mu, m\rangle) \coloneqq \left\{|\lambda, n\rangle | X(|\mu, m\rangle) - X(|\lambda, n\rangle) = (\delta_{j,k})_{k=1}^{\infty} \text{ for some } j \in \mathbb{N}\right\}$. Then,

$$
\begin{aligned}
\mathbb{P}\Big(f(t) = |\lambda, n\rangle\Big) &= \frac{\left|\langle 0, \lambda| e^{A_{\mathbb{Z}}}\rangle\right|^2}{\langle 0, \varnothing| e^{(A_{\mathbb{Z}}^\dagger + A_{\mathbb{Z}})t}|\varnothing, 0\rangle} \\
&= \frac{\left|\langle 0, \lambda| \sum_{n=0}^\infty \frac{t^n}{n!} A_{\mathbb{Z}}^n |\varnothing, 0\rangle\right|^2}{\langle 0, \varnothing| e^{(A_{\mathbb{Z}}^\dagger + A_{\mathbb{Z}})t}|\varnothing, 0\rangle} \\
&= \frac{\left|\sum_{n=0}^\infty \frac{t^n}{n!} \langle 0, \lambda| A_{\mathbb{Z}}^n |\varnothing, 0\rangle\right|^2}{\Big(\sum_{n=0}^\infty \frac{t^n}{n!}\langle \varnothing, 0| (A_{\mathbb{Z}}^\dagger)^n\Big)\Big(\sum_{m=0}^\infty \frac{t^m}{m!} A_{\mathbb{Z}}^n |\varnothing, 0\rangle\Big)} \\
&= \frac{\left|\frac{t^{|\lambda|}}{|\lambda|!}(\dim\ \lambda)\right|^2}{\Big(\sum_{n=0}^\infty \sum_{\mu\vdash n} \frac{t^n}{n!}\langle \mu, 0|(\dim\ \mu)\Big)\Big(\sum_{m=0}^\infty \sum_{\nu\vdash m} \frac{t^m}{m!}(\dim\ \nu)|\nu, 0\rangle\Big)} \\
&= \frac{\left|\frac{t^{|\lambda|}}{|\lambda|!}(\dim\ \lambda)\right|^2}{\sum_{n=0}^\infty \sum_{\mu\vdash n} \frac{t^{2n}}{(n!)^2}(\dim\ \mu)^2} \\
&= \frac{\left|\frac{t^{|\lambda|}}{|\lambda|!}(\dim\ \lambda)\right|^2}{\sum_{n=0}^\infty \frac{t^{2n}}{n!}} \\
&= t^{2|\lambda|} e^{t^2} \frac{(\dim\ \lambda)^2}{(|\lambda|!)^2},
\end{aligned}
$$

where $|\lambda| = \sum_{i=1}^\infty \lambda_i$ is the sum of the partition (i.e. $\lambda \vdash |\lambda|$). That is, in a more compact form, we have that the probability of being at state $|\lambda, 0\rangle$ at time t is given by

$$
\mathbb{P}\Big(f(t) = \lambda\Big) = t^{2|\lambda|} e^{t^2} \frac{(\dim\ \lambda)^2}{(|\lambda|!)^2}, \tag{3.3.4}
$$

which is indeed a probability measure for each t. Also, we can compute the infinitesimal transition probabilities. Take two partitions, $\lambda \vdash n$ and $\mu \vdash n-1$, and we compute first the infinitesimal probability that a path goes through μ then through λ, which corresponds to adding a single block to μ. Thus, we must have that $X(|\lambda, n\rangle) - X(|\mu, n-1\rangle) = (\delta_{j,k})_{k=1}^\infty$ for some j, and we will denote this condition by $\mu \nearrow \lambda$. Then, the infinitesimal probability that a path goes through μ then λ is given by

$$
\frac{\langle 0, \lambda| A_{\mathbb{Z}}^n |\varnothing, 0\rangle\langle 0, \varnothing|(A_{\mathbb{Z}}^\dagger)^{n-1}|\mu, 0\rangle}{\langle 0, \varnothing|(A_{\mathbb{Z}}^\dagger)^n A_{\mathbb{Z}}^n|\varnothing, 0\rangle} = \frac{(\dim\ \lambda)(\dim\ \mu)}{n!},
$$

and with the probability of being in the state $|\mu, 0\rangle$

$$\frac{\left|\langle 0,\lambda|A_{\mathbb{Z}}^{n}|\varnothing,0\rangle\right|^2}{\langle 0,\varnothing|(A_{\mathbb{Z}}^{\dagger})^n A_{\mathbb{Z}}^{n-1}|\varnothing,0\rangle} = \frac{(\dim\,\mu)^2}{(n-1)!},$$

we have that the infinitesimal conditional probability of being in state λ given that we were in state μ is given by

$$p(\lambda|\mu) := \frac{\dim\,\lambda}{n\dim\,\mu},$$

where $\lambda \vdash n$, $\mu \vdash n-1$, and $\mu \nearrow \lambda$. Note that if we don't have that $\mu \nearrow \lambda$, then the infinitesimal conditional probability is 0. Now, we can show the second requirement that the transitional probabilities add up to 1. That is, fix $\mu \vdash n-1$, then

$$\begin{aligned}\sum_{\lambda\in\mathcal{S}} p(\lambda|\mu) &= \sum_{\mu\nearrow\lambda}\frac{\dim\,\lambda}{n\dim\,\mu}\\ &= 1.\end{aligned}$$

The last equality follows from a formula about induced representations in representation theory

$$\mathrm{Ind}_{S_{n-1}}^{S_n}\pi_\mu = \bigoplus_{\lambda:\mu\nearrow\lambda}\pi_\lambda,$$

where S_n (S_{n-1}) is the symmetric group on n (resp. $n-1$) and π_λ (resp. π_μ) is the irreducible representations labeled by the partition λ (resp. μ) of S_n (resp. S_{n-1}). (See the book [**KC03**] for more details.). Therefore, now that we have checked the Markov properties for the generator $A_{\mathbb{Z}}$ (of Plancherel growth model), we have

THEOREM 3.3.1. *The continuous stochastic process defined by 3.3.4 is a Markov process and the generating function of the process*

$$\langle 0|e^{H(\vec{t})}e^{A_{\mathbb{Z}}t}|0\rangle = \sum_{\lambda} e^{t^2}\left(\mathbb{P}\Big(f(t) = |\lambda,0\rangle\Big)\right)^{1/2} s_\lambda(\vec{t})$$

is a KP τ-function.

REMARK 3.3.1. *It was already been known that the Plancherel growth model is a continuous Markov process in ([**Ker99**, **BOO00**]). Instead, the main focus of our statement is that the generating function of such process is a KP τ-function and thus it is KP integrable. Also, we* *like to point out that we have chosen a specific choice of generator $A_{\mathbb{Z}}$ to prove our result, but it would be interesting to know if there are other generators $A_{\mathcal{G}}$ where our result also holds. We also considered applying our methods to the models with jumps to both the right and left neighboring sites, but checking the Markov property in that case is more difficult. Moreover, we like to point out that we based our construction of the infinite fermionic Fock space on the Lie algebra $gl(\infty)$, and this construction can be generalized for any affine Kac-Moody algebra [**BBT03**].*

CHAPTER 4

Periodic ASEP Model

4.1. Introduction

4.1.1. Background. The asymmetric simple exclusion process (ASEP) is a continuous-time Markov process that arose in the study of biopolymers [**MGP68**] and was first analyzed mathematically by Spitzer [**Spi70**]. It has since become a significant object of study in the fields of non-equilibrium statistical mechanics and interacting particle systems [**Mal11**]. The process involves a set of particles located at distinct integral points on the real number line. At various times, a particle may attempt to move one unit to the right with a probability p — known as the *hopping rate* — or one unit to the left with probability $q = 1 - p$, with the caveat that two particles cannot occupy the same site. The term *asymmetric* refers to the fact that p and q are not necessarily equal; the term *simple* refers to the fact that particles hop one unit; and the term *exclusion* refers to the fact that particles cannot occupy the same site. In this paper, we focus exclusively on the periodic ASEP, in which case the sites lie on a ring. The simple rules governing the ASEP belie profoundly complex dynamics that make it a particularly interesting and useful model to consider.

The Bethe ansatz is a technique that dates back to the seminal paper of Bethe [**Bet31**] on the Heisenberg XXX spin chain and has since been applied to a wide variety of models in quantum and statistical mechanics. For the reader looking for the general history and applications of the Bethe ansatz, there are various well-written surveys in the literature [**Bax07**, **GC14**, **Sut04**]. The version of the ansatz that we study in this paper is often referred to as the coordinate Bethe ansatz, which has been applied to variations of the ASEP model since the work of Gwa and Spohn [**GS92**]. More recently, Tracy and Widom [**TW08**] used formulas inspired by the coordinate Bethe ansatz to show that ASEP on $\mathbb{Z}$ with step initial condition belongs to the KPZ universality class [**Cor14**].

The issue of the completeness of the Bethe ansatz — in other words, whether it yields all possible eigenstates — has been addressed rigorously in only a handful of cases. Determining completeness is not only appealing from the standpoint of mathematical rigour, but is also desirable for various other reasons, such as the asymptotic analysis of ASEP starting from finite models. In 1994, Dorlas [**Dor93**] proved completeness for the nonlinear Schrödinger model but stated: "In the Heisenberg chain the existence of bound states presents a problem in finite volume." Indeed, for the ASEP and Heisenberg XXZ models on the 1-dimensional lattice $\mathbb{Z}$, completeness was only recently proven by Borodin, Corwin, Petrov and Sasamoto [**BCPS15**] as a corollary to powerful new algebraic insights into the structure of integrable systems. However, the finite periodic case has remained immune to these methods.

In the 1990s, Langlands and Saint-Aubin [**LSA95**, **LSA97**] outlined an argument based upon classical results in algebraic geometry and topology that reduces the problem of completeness of the Bethe ansatz for the XXZ model to a certain enumeration of spanning forests. In this paper, we expand on the argument of Langlands and Saint-Aubin and apply it to the case of the periodic ASEP. We hope to make these powerful techniques more well-known and accessible to a wider audience. Our main result is the following.

Theorem 4.1.1. *For generic values of the hopping rate p, the Bethe ansatz for the periodic ASEP is complete.*

In this theorem, we interpret p as a point in the Riemann sphere, rather than a real number in the interval $[0,1]$, in order to employ tools from algebraic geometry and topology. Furthermore, we use the term *generic* to mean that p lies in a Zariski open subset of $\mathbb{CP}^1$. In other words, the theorem is true for all but finitely many values of p. At such non-generic values of the hopping rate p, the Bethe ansatz does not yield a complete basis of eigenstates. However, a key observation is that the generic completeness guaranteed by Theorem 4.1.1 allows one to use a limiting procedure to obtain a complete eigenspace decomposition for any value of p. In general, the basis given by this limiting procedure does not diagonalize the operator, but decomposes it into Jordan canonical form.

4.1.2. ASEP and the Bethe ansatz. The ASEP model on the integer lattice $\mathbb{Z}$ is a continuous-time Markov process η_t, where we set $\eta_t(x) = 1$ if the site $x \in \mathbb{Z}$ is occupied by a particle at time t and $\eta_t(x) = 0$ if the site $x \in \mathbb{Z}$ is vacant at time t [**Lig12**]. Each particle is controlled by an exponential timer with parameter 1 such that, when the timer activates, the particle attempts to jump from its current site at x to an adjacent site $y = x \pm 1$ with a certain probability $p(x,y)$. If $y = x \pm 1$ is occupied, then $p(x,y) = 0$; if $y = x+1$ is unoccupied, then $p(x,y) = p$; and if $y = x-1$ is unoccupied, then $p(x,y) = q = 1-p$. In this paper, we focus exclusively on the ASEP model with periodic boundary conditions, in which each particle occupies one of L sites on a ring. In other words, we work on $\mathbb{Z}_L$ rather than $\mathbb{Z}$.

For a positive integer n, we introduce the notation

$$[n] = \{1,2,\dots,n\} \qquad \text{and} \qquad [n]^{(2)} = \{(i,j) \in \mathbb{Z}^2 \mid 1 \le i < j \le n\}.$$

Let $\mathcal{X} = \bigotimes_{i=1}^{L} \mathbb{C}_i^2$ and consider the basis $\{u_S \mid S \subseteq [L]\}$, where $\mathbb{C}_i^2 = \langle u_i^+, u_i^- \rangle$ and

$$u_S = \left(\bigotimes_{i \in S} u_i^+ \right) \otimes \left(\bigotimes_{i \in [L] \setminus S} u_i^- \right).$$

The basis element u_S naturally represents the state in which the sites in S are occupied while the remaining sites are empty. Define the operator

$$H(t;p) : \mathcal{X} \to \mathcal{X},$$

where

$$\sum_{S \subseteq [L]} a^S u_S \mapsto \sum_{S \subseteq [L]} b^S u_S$$

and

$$b^S = \sum_{S'} p(a^S - a^{S'}) + \sum_{S''} q(a^S - a^{S''}).$$

Here, the summations range over all states S' that differ from S by a single particle that moved one position to the right and states S'' that differ from S by a single particle that moved one position to the left. Note that $|S| = |S'| = |S''|$, so that the number of particles is conserved. Thus, the operator $H(t;p)$ decomposes into a direct sum of operators

$$H_N(t;p) : \mathcal{X}_N \to \mathcal{X}_N,$$

where $\mathcal{X}_N = \text{span}\{u_S \mid |S| = N\}$ is the subspace with dimension $\binom{L}{N}$ whose basis elements correspond to states with N particles.

One can solve the periodic ASEP model by diagonalizing the operator $H_N(t;p)$ using the Bethe ansatz, as was carried out by Gwa and Spohn [**GS92**]. Let $x_1, x_2, \dots, x_N$ denote the sites occupied by the N particles. The probability $u(X;t)$ of being in state $X = (x_1, \dots, x_N)$ at time t satisfies the so-called *master equation*

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{N} [pu(x_i - 1)\delta_{i,i-1} + qu(x_i + 1)\delta_{i+1,i} - pu(x_i)\delta_{i+1,i} - qu(x_i)\delta_{i,i-1}]. \tag{4.1.1}$$

Here, we use the notation $u(x_i \pm 1)$ as a shorthand for $u(x_1, \dots, x_{i-1}, x_i \pm 1, x_{i+1}, \dots x_N; t)$ and we define $\delta_{i,j} = 0$ if $x_i = x_j + 1$ and $\delta_{i,j} = 1$ otherwise. The periodic boundary conditions impose the constraint $u(x_1, \dots, x_N; t) = u(x_2, \dots, x_N, x_1 + L; t)$ and the initial state $Y = (y_1, \dots, y_N)$ imposes the constraint $u(x_1, \dots, x_N; 0) = \delta_{X,Y}$. We are not concerned here with the full computation of $u(X;t)$, so we shall not mention initial conditions again.

The coordinate Bethe ansatz is derived from the observation that the master equation (4.1.1) greatly simplifies if one assumes that the particles are sufficiently far apart from each other to be considered non-interacting. In this case, one obtains the so-called *free equation*

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{N} [pu(x_i - 1) + qu(x_i + 1) - u(x_i)]. \tag{4.1.2}$$

This free equation becomes equivalent to the master equation (4.1.1) once we impose the boundary conditions

$$pu(x_i, x_i) + qu(x_i + 1, x_i + 1) - u(x_i, x_i + 1) = 0. \tag{4.1.3}$$

The Bethe ansatz proposes solutions of the form

$$u_{\bar{z}}(x_1, \dots, x_N) = \sum_{\sigma \in S_N} A_\sigma \prod_{i=1}^{N} z_{\sigma(i)}^{x_i}, \tag{4.1.4}$$

where $\vec{z} = (z_1, \dots, z_N)$ is a tuple of complex parameters and, for each σ, A_σ is a function of $z_1, \dots, z_N$ chosen to satisfy the boundary conditions of equation (4.1.3). As a result, one obtains

$$A_\sigma(z_1, \dots, z_N) = \prod_{\substack{\text{inversions}\\(i,j)}} -\frac{p + qz_iz_j - z_i}{p + qz_iz_j - z_j}, \tag{4.1.5}$$

where we recall that an *inversion* of a permutation σ is a pair (i, j) such that $i < j$ and $\sigma(i) > \sigma(j)$.

Now imposing the periodicity constraint $u(x_1, \dots, x_N; t) = u(x_2, \dots, x_N, x_1 + L; t)$ on the ansatz of equation (4.1.4) leads to the *Bethe ansatz equations*

$$z_j^L = (-1)^{N-1} \prod_{i=1}^{N} \frac{p + qz_jz_i - z_j}{p + qz_jz_i - z_i} \qquad \text{for } j = 1, 2, \dots, N. \tag{4.1.6}$$

Note that these equations imply the condition $\prod_{i=1}^N z_i^L = 1$.

In this paper, we count the number of solutions to these equations and thereby determine the number of eigenstates produced by the Bethe ansatz. The structure of the chapter is as follows.

- In Section 2, we analyze the case of $N = 2$ particles in detail. We introduce the algebro-geometric perspective and topological tools — in particular, the Lefschetz theorem — required to approach the general case.
- In Section 3, we consider the geometric set-up of the general case of N particles. Inadmissibility conditions are introduced to classify those solutions of the Bethe ansatz equations that do not actually lead to eigenstates. We show that solutions of the Bethe ansatz equations correspond to coincidences between certain algebraic functions ψ and ϕ. Such coincidences can then be counted by the Lefschetz theorem.
- In Section 4, we compute the traces appearing in the Lefschetz theorem and show how they relate to the enumeration of trees. By a careful combinatorial argument, we deduce that the number of admissible solutions to the Bethe ansatz equations is $L(L-1)\cdots(L-N+1)$, as required.
- In Section 5, we consider non-generic values of the hopping rate p, at which the Bethe ansatz fails to describe all possible eigenstates. We argue that a limiting procedure can always be used in conjunction with the Bethe ansatz to express the operator in Jordan canonical form.

- In Appendix A, we discuss the algebro-geometric notion of blow-ups in the context of the maps ψ and ϕ defined earlier. Whereas these are rational maps and hence, only defined on a Zariski open subset of their domains, the Lefschetz theorem requires a continuous maps between compact spaces. We therefore blow up the domains appropriately and prove the existence of smooth resolutions of the maps ψ and ϕ.

4.2. The two-particle case

In this section, we examine the case of two particles on a ring with L sites. This simple example illustrates how algebraic geometry, topology and combinatorics come into play. In later sections, these ideas are combined to prove completeness of the Bethe ansatz for the general case of N particles on a ring with L sites.

4.2.1. Naive approach. In the case $N = 2$, the Bethe ansatz equations (4.1.6) reduce to the following, where $p + q = 1$.

$$z_1^L = -\frac{p + qz_1z_2 - z_1}{p + qz_1z_2 - z_2} \qquad\qquad z_2^L = -\frac{p + qz_1z_2 - z_2}{p + qz_1z_2 - z_1} \tag{4.2.1}$$

The Bethe ansatz then asserts that, for z_1 and z_2 satisfying these equations, one should seek eigenstates of the operator H_2 given by

$$u(x_1, x_2) = A_{12}z_1^{x_1}z_2^{x_2} + A_{21}z_1^{x_2}z_2^{x_1},$$

where the amplitudes A_{12} and A_{21} are thus related by equation (4.1.5).

$$A_{12} = -A_{21}\frac{p + qz_1z_2 - z_1}{p + qz_1z_2 - z_2}$$

However, note that simultaneous solutions to equations (4.2.1) do not necessarily yield non-trivial eigenstates. First, if $z_1 = 0$ or $z_2 = 0$, then it is clear that $u(x_1, x_2) = 0$ for all x_1 and x_2. Second, if $z_1 = z_2$, then we have $A_{12} = -A_{21}$ and $u(x_1, x_2) = 0$ for all x_1 and x_2. Therefore, we declare solutions to equations (4.2.1) with $z_1 = 0$, $z_2 = 0$ or $z_1 = z_2$ as *inadmissible*. On the other hand, one can check that *admissible* solutions — in other words, those with distinct $z_1, z_2 \in \mathbb{C}^*$ — do indeed yield non-trivial eigenstates. Furthermore, two admissible solutions (z_1, z_2) and (z_1', z_2') yield the same eigenstate if and only if they are permutations of each other. Therefore, we aim to show that there are $L(L-1)$ admissible solutions to equations (4.2.1), from which we may deduce that there are $\frac{1}{2}L(L-1) = \binom{L}{2}$ distinct eigenstates.

Let us initially consider a naive approach to the problem, in order to better understand the nature of the inadmissible solutions. Rewrite the system of equations (4.2.1) as

$$z_2 = \epsilon z_1^{-1} \text{ with } \epsilon^L = 1 \qquad \text{and} \qquad z_1^L = -\frac{p + q\epsilon - z_1}{p + q\epsilon - \epsilon z_1^{-1}}. \tag{4.2.2}$$

At first glance, it appears as though there are L^2 solutions, since there are L choices for ϵ and each one yields L solutions for z_1. However, it is necessary to exclude the inadmissible solutions with $z_1 = z_2$. In the case $z_1 = z_2$, equations (4.2.2) reduce to

$$\epsilon z_1^{-1} = z_1 \text{ with } \epsilon^L = 1 \qquad \text{and} \qquad z_1^L = -1.$$

Therefore, for each of the L choices for ϵ, there is exactly one solution $z_1 = \pm\epsilon^{1/2}$ that is compatible with the equation $z_1^L = -1$. So we obtain L inadmissible solutions with $z_1 = z_2$. Note that inadmissible solutions with $z_1 = 0$ or $z_2 = 0$ only arise when $p = 0$, which we presently exclude from our consideration.

Note that, clearing the denominator of (4.2.2) and obtaining a polynomial in z_1 and p ($q = 1-p$), the inadmissible solution corresponds to factoring the polynomial

$$(p + q\epsilon)z_1^L - \epsilon z_1^{L-1} - z_1 + (p + q\epsilon) = (z_1 \pm \epsilon^{1/2}) f(z_1, p). \tag{4.2.3}$$

Thus, we have that for each ϵ, and for every p, there is is an inadmissible solution, and the rest of the solutions, which number $L(L-1)$ as we claimed, should be admissible solutions.

4.2.2. Algebro-geometric perspective. We now approach the problem of determining the number of admissible solutions using a more sophisticated perspective. Solutions to equations (4.2.1) correspond to zeros of the ideal

$$I = \big((p + qz_1z_2 - z_2)z_1^L + (p + qz_1z_2 - z_1), (p + qz_1z_2 - z_1)z_2^L + (p + qz_1z_2 - z_2)\big),$$

whose generators are obtained by clearing the denominators in equations (4.2.1). There is a small subtlety in clearing the denominators, since one may inadvertently introduce new solutions in the case that $p + qz_1z_2 - z_1 = p + qz_1z_2 - z_2 = 0$. However, these equations imply that $z_1 = z_2$, which yields an inadmissible solution that will eventually be excluded. Since these generators are polynomials in z_1, z_2 and p, one might expect to be able to bring the tools of algebraic geometry to bear. Thus, we consider the following commutative diagram of natural ring homomorphisms.

$$\begin{array}{ccc} \mathbb{C}[z_1, z_2, p]/I & \longleftarrow & \mathbb{C}[z_1, z_2, p] \\ & \nwarrow & \uparrow \\ & & \mathbb{C}[p] \end{array}$$

Algebro-geometrically, this corresponds to the following morphisms between varieties.

$$\begin{array}{ccccc} \Sigma := \operatorname{Spec} \mathbb{C}[z_1, z_2, p]/I & \longrightarrow & \operatorname{Spec} \mathbb{C}[z_1, z_2, p] = \mathbb{C}^3 & \hookrightarrow & \left(\mathbb{CP}^1\right)^3 \\ & \overset{\pi}{\searrow} & \downarrow & & \downarrow \\ & & \operatorname{Spec} \mathbb{C}[p] = \mathbb{C} & \hookrightarrow & \mathbb{CP}^1 \end{array}$$

Note that we have extended the commutative diagram to include the natural embeddings $\mathbb{C} \to \mathbb{CP}^1$ and $\mathbb{C}^3 \to (\mathbb{CP}^1)^3$. This will play a role later on, when we introduce topological tools that require us to work with compact spaces.

Moreover, it follows easily from basic results in algebraic geometry (cf. [**Har77**] Prop. 1.13 p.7) that the space Σ is a 1-dimensional complex manifold (i.e. an orientable real surface with a complex structure). Thus, in the case of two particles, the map $\pi : \Sigma \to \mathbb{C}$ of surfaces is generically L^2-to-1, and by factoring the inadmissible solutions, we have shown that Σ decomposes into a union of algebraically independent components

$$\Sigma = \Sigma_{\text{ad}} \cup \Sigma_{\text{in}}$$

such that π factors into maps

$$\Sigma_{\text{ad}} \xrightarrow{L(L-1):1} \mathbb{C} \qquad \text{and} \qquad \Sigma_{\text{in}} \xrightarrow{L:1} \mathbb{C},$$

where the first map is the map of admissible solutions and the second map is the map of inadmissible solutions. These observations follow from the factorization (4.2.3) where the roots found correspond to the inadmissible solutions.

In the general case of N particles, we wish to define a surface Σ and a map $\pi : \Sigma \to \mathbb{C}$ via the Bethe ansatz equations. Then we decompose Σ into the space of admissible solutions, Σ_{ad}, and the space of inadmissible solutions, Σ_{in}, and show that the induced map

$$\Sigma_{\text{ad}} \to \mathbb{C}$$

is generically $N!\binom{L}{N}$ to 1.

This will be accomplished by counting fixed points (correspondences) via the Lefschetz theorem [**Lef42**] and using the combinatorics of rooted trees to get the proper count. Still, even though the maps $\Sigma_{\text{ad}} \xrightarrow{\pi} \mathbb{C}$ are generically (i.e. for all but finite p) $N!\binom{L}{N}$ to 1, there will be special points $p_r \in \mathbb{C}$, called ramification points, such that $|\pi^{-1}(p_r)| < N!\binom{L}{N}$. For these points, the Bethe ansatz might not be complete, but knowing that Σ_{ad} is independent from Σ_{in}, we can perform a limiting procedure that will transform $H_N(t; p_r)$ into Jordan canonical form. We develop this further in Section 4.5.

We have a map of surfaces $\pi : \Sigma \to \mathbb{C}$, where the fiber $\pi^{-1}(p)$ for each $p \in \mathbb{C}$ consists of solutions to the Bethe ansatz equations, both admissible and otherwise. In the previous section, we find L inadmissible solutions. We now perform the count of admissible solutions using topological tools that will allow us to generalize to the case of N particles.

Start by fixing a generic value of $p \in \mathbb{C}$ and consider the rational map $\psi : (\mathbb{CP}^1)^3 \dashrightarrow (\mathbb{CP}^1)^3$ defined by

$$([z_0^1 : z_1^1], [z_0^2 : z_1^2], [\omega_0 : \omega_1]) \mapsto ([\omega_0 : \omega_1], [\omega_1 : \omega_0], [pz_0^1 z_0^2 + qz_1^1 z_1^2 - z_1^1 z_0^2 : -(pz_0^1 z_0^2 + qz_1^1 z_1^2 - z_1^1 z_0^2)]).$$

It is well-defined away from the codimension two subvariety

$$Z := \bigcup_{i=0,1} \left\{([2q : 1 + (-1)^i \sqrt{1 - 4pq}], [2q : 1 + (-1)^i \sqrt{1 - 4pq}])\right\} \times \mathbb{CP}^1 \subseteq (\mathbb{CP}^1)^3.$$

Furthermore, consider the map $\phi : (\mathbb{CP}^1)^3 \to (\mathbb{CP}^1)^3$ defined by

$$([z_0^1 : z_1^1], [z_0^2 : z_1^2], [\omega_0 : \omega_1]) \mapsto ([(z_0^1)^L : (z_1^1)^L], [(z_0^2)^L : (z_1^2)^L], [\omega_0 : \omega_1]).$$

Then, we choose a chart $U = (z_0^1 \neq 0, z_0^2 \neq 0, \omega_0 \neq 0)$ and $f : U \to \mathbb{C}^3$ given by

$$([z_0^1 : z_1^1], [z_0^2 : z_1^2], [\omega_0 : \omega_1]) \mapsto (z_1^1/z_0^1, z_1^2/z_0^2, \omega_1/\omega_0)$$

Moreover, on this chart and away from Z we have that the maps $f \circ \psi \circ f^{-1}, f \circ \phi \circ f^{-1} : \mathbb{C}^3 \dashrightarrow \mathbb{C}^3$ are given by

$$f \circ \phi \circ f^{-1}(z_1, z_2, w_{12}) = (z_1^L, z_2^L, w_{12})$$
$$f \circ \psi \circ f^{-1}(z_1, z_2, w_{12}) = \left(w_{12}, w_{12}^{-1}, -\frac{p + qz_1z_2 - z_1}{p + qz_1z_2 - z_2}\right).$$

Now, in this chart, we note that the system of equations (4.2.1) is equivalent to the equation

$$f \circ \psi \circ f^{-1}(z_1, z_2, w_{12}) = f \circ \phi \circ f^{-1}(z_1, z_2, w_{12}). \tag{4.2.4}$$

We wish to count the number of solutions to (4.2.4) by applying the Lefschetz theorem 4.2.1. However, this result applies only to continuous maps between compact spaces. Therefore, it is necessary to resolve $\psi : (\mathbb{CP}^1)^3 \dashrightarrow (\mathbb{CP}^1)^3$ into a smooth map. We do this by blowing up the domain along the subvariety Z and compactifying the codomain. We explain the details of this construction for the general case in Appendix A.1. Then, we define

$$C := \mathrm{Blow}_Z((\mathbb{CP}^1)^3)$$
$$X := (\mathbb{CP}^1)^3$$

There exist smooth maps $\tilde{\psi}, \tilde{\phi} : C \to X$ that extend the maps $\psi, \phi : X \dashrightarrow X$ on the subvariety Z. Now, with these smooth resolutions, we can apply the Lefschetz theorem 4.2.1. Actually, we will use the Lefschetz theorem to count the number of solutions of

$$\tilde{\psi}(pt) = \tilde{\phi}(pt) \tag{4.2.5}$$

on the whole domain C rather than just the chart U. Of course, this way we will inadvertently introduce extra solutions outside the chart U that don't correspond to eigenstate, but we will classify and discard those solutions along the way.

4.2.3. Topological tools. We now present the topological technique that we use to compute the number of points in X that satisfy (4.2.5), denoted by $\lambda(\tilde{\psi}, \tilde{\phi})$.

THEOREM 4.2.1 (Strong Lefschetz theorem). *Given two differentiable maps $\phi, \psi : X \to Y$ of compact spaces the number of solutions (with multiplicity) of the equation $\phi(x) = \psi(x)$, the coincidence number of ϕ and ψ, is given by*

$$\lambda(\psi, \phi) \coloneqq \sum_{i=0}^{\dim Y} (-1)^i \operatorname{Tr}(\psi_i \phi^i), \tag{4.2.6}$$

where $\psi_i : H_i(X) \to H_i(Y)$ is the pushforward in homology and $\phi^i : H_i(Y) \to H_i(X)$ is the Poincaré dual of the pullback in cohomology. We also call $\lambda(\psi, \phi)$ the Lefschetz number.

This theorem, in this generality, is not found in many textbooks, but it is discussed in old textbooks by Lefschetz himself, such as in [**Lef42**]. It is a quick check to see that our formulas satisfy the hypothesis of the theorem. As a matter of fact, this is the reason that we need to apply blow-ups to our original domain. Otherwise, we would have a singular domain and the theorem would not apply. Thus, we can use this formula to find the number of solutions of equation (4.3.6).

REMARK 4.2.1. *In Theorem 4.2.1, we use homology with $\mathbb{C}$-coefficients. This a choice made by the authors for convenience in the proofs to follow. In any case, we will use $\mathbb{C}$ throughout the paper. So, we will denote $H_*(X;\mathbb{C})$ and $H_*(C;\mathbb{C})$ by $H_*(X)$ and $H_*(C)$, accordingly.*

We compute the trace of the induced maps on the cohomology vector spaces

$$\psi_i : H_i(C) \to H_i(X)$$
$$\phi^i : H_i(X) \to H_i(C)$$

where the last map is defined via Poincaré duality.

Since $X = (\mathbb{CP}^1)^3$ and $C = \mathrm{Blow}_Z(X)$, we have

$$\begin{aligned} H_*(X) &= H_0(X) \oplus H_2(X) \oplus H_4(X) \oplus H_6(X) \\ &= \langle 1 \rangle \oplus \langle e_1, e_2, e_{12} \rangle \oplus \langle e_1 \otimes e_2, e_1 \otimes e_{12}, e_2 \otimes e_{12} \rangle \oplus \langle e_1 \otimes e_2 \otimes e_{12} \rangle \\ H_*(C) &= H_*(X) \oplus H. \end{aligned}$$

Here, the second summand comes from the blow-up (cf. Prop. 4.3.1). One can show that $\phi^i(H_*(X)) \subseteq H_*(X)$, making the direct summand H of $H_*(C)$ irrelevant to our computations, and so we ignore it for the remainder. This is explained in more detail and more generality in Appendix A.1 and Proposition 4.3.1.

So the induced maps are

$$\left(\bigotimes_{i\in S} e_i\right) \otimes \left(\bigotimes_{t\in T} e_t\right) \overset{\phi^*}{\mapsto} L^{|\{1,2\}\setminus S|}\left(\bigotimes_{i\in S} e_i\right) \otimes \left(\bigotimes_{t\in T} e_t\right)$$

where $S \subseteq \{1,2\}$ and $T \subseteq \{(1,2)\}$.

The ψ_* is determined in the appendix for the general case. For two particles, we can simply compute directly and find

$$1 \overset{\psi_*}{\mapsto} 1 \qquad e_1 \otimes e_2 \overset{\psi_*}{\mapsto} 0$$

$$e_1, e_2 \overset{\psi_*}{\mapsto} e_{12} \qquad e_1 \otimes e_{12}, e_2 \otimes e_{12} \overset{\psi_*}{\mapsto} e_1 \otimes e_{12} + e_2 \otimes e_{12}$$

$$e_{12} \overset{\psi_*}{\mapsto} e_1 + e_2 \qquad e_1 \otimes e_2 \otimes e_{12} \overset{\psi_*}{\mapsto} 0.$$

So, we compute the trace and obtain

$$\lambda(\psi, \phi) = L^2 + 2L,$$

which counts all of the solutions, admissible or otherwise. Subtracting the L inadmissible solutions as we did in the elementary algebraic approach, it still seems as though we have $2L$ too many. These extra $2L$ solutions are the solutions at infinity that we have inadvertently added by compactifying our spaces. So, we now have another class of inadmissible solution: $z_1 = z_2$ or $z_i = 0, \infty$. The second set of inadmissible solutions really corresponds to the zeros and poles of z_i^L, and having the poles and zeros at infinity with non-trivial multiplicity complicates the problem unnecessarily. Thus, following Langlands and Saint-Aubin [**LSA95**], we replace the polynomial map $z_i \mapsto z_i^L$ with the rational map

$$z_i \mapsto R(z_i) = \prod_{i=1}^{L} \frac{z - \alpha_i}{z - \beta_i},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_L$ are distinct and

$$\beta_i = \frac{1}{q} - \frac{p}{q\alpha_i} \qquad \text{for } i = 1, 2, \ldots, L.$$

This new function is homotopy equivalent to z_i^L and it will thus give us the same results in the computation of the trace of the Lefschetz numbers, and the new inadmissibility conditions are: $z_1 = z_2$, $z_i = \alpha_j$, and $z_i = \beta_j$.

For each inadmissibility condition, we consider the set of solutions corresponding to it by defining a closed subvariety that corresponds to such a condition and counting the solutions via the Lefschetz Theorem 4.2.1 applied to this subvariety. For example, for the condition $z_1 = z_2$, we define

$$X\langle z_1 = z_2\rangle := \{(z_1, z_2, w_{12}) \in X \mid z_1 = z_2\} \subseteq X$$

$$C\langle z_1 = z_2\rangle := \{(z_1, z_2, w_{12}) \in C \mid z_1 = z_2\} \subseteq C.$$

Then, we have

$$\lambda(X\langle z_1 = z_2\rangle) = L$$

where we have abused notation since ψ and ϕ are fixed and the subvarites are the main diference. Similarly, we define closed subspaces for other inadmissibility conditions and compute their Lefschetz numbers.

$$\lambda(X\langle z_1 = \alpha_i\rangle) = 1 \qquad \lambda(X\langle z_1 = \alpha_i, z_2 = \alpha_j\rangle) = 0$$

$$\lambda(X\langle z_1 = \beta_i\rangle) = 1 \qquad \lambda(X\langle z_1 = \alpha_i, z_2 = \beta_j\rangle) = \delta_{i,j}$$

$$\lambda(X\langle z_2 = \alpha_i\rangle) = 1 \qquad \lambda(X\langle z_1 = \beta_i, z_2 = \alpha_j\rangle) = \delta_{i,j}$$

$$\lambda(X\langle z_2 = \beta_i\rangle) = 1 \qquad \lambda(X\langle z_1 = \beta_i, z_2 = \beta_j\rangle) = 0.$$

Then, the number of inadmissible solutions, via the Inclusion-Exclusion principle of set theory, is

$$\lambda(X\langle z_1 = z_2\rangle) + \sum_i \left[\lambda(X\langle z_1 = \alpha_i\rangle) + \lambda(X\langle z_1 = \beta_i\rangle) + \lambda(X\langle z_2 = \alpha_i\rangle) + \lambda(X\langle z_2 = \beta_i\rangle)\right]$$

$$- \sum_{i,j} \left[\lambda(X\langle z_1 = \alpha_i, z_2 = \alpha_j\rangle) + \lambda(X\langle z_1 = \alpha_i, z_2 = \beta_j\rangle) + \lambda(X\langle z_1 = \beta_i, z_2 = \alpha_j\rangle) + \lambda(X\langle z_1 = \beta_i, z_2 = \beta_j\rangle)\right],$$

which computes to $3L$. That is, the number of admissible solutions is $L^2 + 2L - 3L = L^2 - L$ just as we showed in the previous section. This process generalizes completely, and it becomes a combinatorial problem to compute the traces of the homology maps on the closed subspaces of the inadmissible solutions.

4.3. The geometric set-up

In this section, we discuss the geometric set-up for the general case of the Bethe ansatz for the periodic ASEP with N particles.

4.3.1. Inadmissibility conditions. Recall that the Bethe ansatz equations (4.1.6) state that

$$z_j^L = (-1)^{N-1} \prod_{i=1}^{N} \frac{p + qz_jz_i - z_j}{p + qz_jz_i - z_i}, \qquad \text{for } j = 1, 2, \dots, N. \tag{4.3.1}$$

Clearing the denominators, we define the polynomials

$$f_j(\vec{z}, p) = z_j^L \prod_{i=1}^{N} (p + qz_jz_i - z_i) + (-1)^N \prod_{i=1}^{N} (p + qz_jz_i - z_j), \qquad \text{for } j = 1, 2, \dots, N.$$

Define the ideal $I = (f_1, f_2, \dots, f_N) \subseteq \mathbb{C}[\vec{z}, p]$ and consider the inclusion and projection maps

$$\mathbb{C}[p] \to \mathbb{C}[\vec{z}, p] \to \mathbb{C}[\vec{z}, p]/I.$$

By applying the contravariant functor Spec, one obtains a morphism of varieties

$$\Sigma' \to \mathbb{C}^{N+1} \to \mathbb{C},$$

with Σ the closure of Σ' in $(\mathbb{CP}^1)^{N+1}$. By considering $\Sigma' \subseteq \mathbb{C}^{N+1} \subseteq (\mathbb{CP}^1)^{N+1}$ and $\mathbb{C} \subseteq \mathbb{CP}^1$, we may extend this composition to a map between compact spaces

$$\Sigma \xrightarrow{\pi} \mathbb{CP}^1.$$

Also, we wish to describe the inadmissible solutions, and we make this notion precise.

Definition 4.3.1. *We say that a solution $\vec{z}$ to the Bethe ansatz equations is* inadmissible *if the resulting eigenstate satisfies $u_{\vec{z}}(\vec{x}) = 0$ for all $\vec{x}$.*

Inadmissible solutions admit a simple description via the following result, which generalises the observations of Section 4.2 in the case of two particles.

Lemma 4.3.1. *Suppose that $\vec{z}$ is a solution to the Bethe ansatz equations such that $z_i \in \mathbb{C}^*$ for $i = 1, 2, \ldots, N$. Then $\vec{z}$ is inadmissible if and only if $z_i = z_j$ for two distinct indices i and j.*

Remark 4.3.1. *As the Bethe ansatz equations stand, we cannot have a solution to* (4.3.1) *such that $z_k = 0$ or ∞. However, in the next section, we will reformulate these equations with auxilary maps where* (4.3.1) *is not necessarily satisfied. Indeed, in the reformulation in the next section, if we have that all the $z_k \neq 0$ or ∞, then* (4.3.1) *will be satisfied. Thus, in hindsight of the following section, we add the condition that $z_k \neq 0$ or ∞.*

Proof. Recall the ansatz (4.1.4)

$$u_{\vec{z}}(\vec{x}) = \sum_{\sigma \in S_N} A_\sigma \prod_{i=1}^{N} z_{\sigma(i)}^{x_i}.$$

Suppose that $z_i = z_j$ for two distinct indices i and j and let $\tau \in S_N$ be the transposition that swaps i and j. Then we have $A_\sigma = -A_{\sigma\tau}$ and for all $\vec{x}$ we have

$$u_{\vec{z}}(\vec{x}) = \sum_{\sigma \in S_N} A_\sigma \prod_{j=1}^{N} z_{\sigma(j)}^{x_j} = \sum_{\sigma \in S_N} A_{\sigma\tau^2} \prod_{j=1}^{N} z_{\sigma\tau^2(j)}^{x_j} = \sum_{\sigma' \in S_N} A_{\sigma'\tau} \prod_{j=1}^{N} z_{\sigma'\tau(j)}^{x_{\tau(j)}} = \sum_{\sigma' \in S_N} -A_{\sigma'} \prod_{j=1}^{N} z_{\sigma'(j)}^{x_j} = -u_{\vec{z}}(\vec{x}).$$

It follows that $\vec{z}$ is inadmissible, as claimed.

Conversely, suppose that $\vec{z}$ is inadmissible so that $u_{\vec{z}}(\vec{x}) = 0$ for all $\vec{x}$. Since we have that $z_i \in \mathbb{C}^*$, we can write $z_j = \epsilon_j z_1$ for $j = 1, 2, \ldots, N$ and we obtain

$$0 = u_{\vec{z}}(\vec{x}) = \sum_{\sigma \in S_N} A_\sigma \prod_{j=1}^{N} z_{\sigma(j)}^{x_j} = \sum_{\sigma \in S_N} A_\sigma \prod_{j=1}^{N} \epsilon_{\sigma(j)}^{x_j} z_1^{x_j} = z_1^{\sum x_j} \sum_{\sigma \in S_N} A_\sigma \prod_{j=1}^{N} \epsilon_{\sigma(j)}^{x_j}.$$

Thus, we have that

$$\sum_{\sigma \in S_N} A_\sigma \prod_{i=1}^{N} \epsilon_{\sigma(i)}^{x_i} = 0.$$

Although this equation looks rather similar to the original equation. Now, we have that $\epsilon_1 = 1$. Moreover, we have

$$\sigma(j) = 1 \Rightarrow \epsilon_{\sigma(j)}^{x_j} = 1.$$

With such an observation we have the decomposition of the sum

$$\sum_{\sigma \in S_N} A_\sigma \prod_{i=1}^{N} \epsilon_{\sigma(i)}^{x_i} = \sum_{j=1}^{N} \sum_{\substack{\sigma \in S_N \\ \sigma(j)=1}} A_\sigma \prod_{\substack{i=1 \\ i \neq j}}^{N} \epsilon_{\sigma(i)}^{x_i} = 0.$$

Note that each term

$$\sum_{\substack{\sigma \in S_N \\ \sigma(j)=1}} A_\sigma \prod_{\substack{i=1 \\ i \neq j}}^{N} \epsilon_{\sigma(i)}^{x_i}$$

is independent of the variable x_j. Then, one can show that each term must be constant. In particular, we have that

$$\sum_{\substack{\sigma \in S_N \\ \sigma(1)=1}} A_\sigma \prod_{\substack{i=1 \\ i \neq j}}^{N} \epsilon_{\sigma(i)}^{x_i} = const. \tag{4.3.2}$$

Since we have that $\{\sigma \in S_N \mid \sigma(1) = 1\} \cong S_{N-1}$, we can prove our result by induction on N and the weaker assumption 4.3.2. Thus, it suffices to show this result for the case $N = 2$. That is, it only remains to show that if

$$A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_1^{x_2} z_2^{x_1} = const$$

then $z_1 = z_2$. Recall that we showed this in the two-particle case when $const = 0$. Now, we show this for any constant and $N = 2$. Again, we write $z_2 = \epsilon z_1$, and we have

$$\begin{aligned} const &= z_1^{x_1} z_2^{x_2} + A_{21} z_1^{x_2} z_2^{x_1} \\ &= z_1^{x_1+x_2} \left(\epsilon^{x_2} + A_{21} \epsilon^{x_1} \right). \end{aligned}$$

Then, rearranging the terms we have

$$\epsilon^{x_2} = -A_{21}\epsilon^{x_1} + \frac{const}{z_1^{x_1+x_2}}.$$

Note that the left-hand-side of the equation is independent of x_1, so we obtain

$$\epsilon^{x_2} = const_2$$

for all x_2. Therefore, we must have $\epsilon = 1$ and $z_1 = z_2$, as we wished to show. □

4.3.2. Maps ψ and ϕ. Now that we know how to classify the inadmissible solutions, we'll count the number of admissible solutions. For example, we count all the solutions and subtract the solutions such that any two z_i are equal, but in doing so we over count the number of inadmissible solutions as in the case when three z_i are equal. We must then take care to do the proper count. In fact, what we want to count is the cardinality of the set $\pi^{-1}(p)$ for a generic p, and we can use the Lefschetz theorem 4.2.1. Following Langlands and Saint-Aubin [**LSA95**], we introduce $\binom{N}{2}$ variables and decompose the Bethe ansatz equations (4.3.1) into

$$\omega_{(k,l)} = -\frac{p + qz_k z_l - z_k}{p + qz_k z_l - z_l} \text{ for } 1 \le k < l \le N \tag{4.3.3}$$

$$R(z_i) = \left(\prod_{1 \le j < i} \omega_{(j,i)} \right) \left(\prod_{i < j \le N} \omega_{(i,j)}^{-1} \right) \qquad \text{for } i = 1, \dots, N \tag{4.3.4}$$

where $R : \mathbb{CP}^1 \to \mathbb{CP}^1$ has L zeros at $\alpha_i \in \mathbb{CP}^1$ and L poles at $\beta_i \in \mathbb{CP}^1$ with $\beta_i = \frac{1}{q} - \frac{p}{q\alpha_i}$ which is a generalization of the function $z \mapsto z^L$. Now, note that our equations (4.3.3) and (4.3.4) are over the space $\mathbb{C}^N \times \mathbb{C}^{\binom{N}{2}}$ except for the divisor where the numerator and denominator of the right hand side of (4.3.3) vanish. In order to use the Lefschetz theorem 4.2.1 we need to compactify the spaces we are working with and make the right-hand of (4.3.3) well-defined.

As in the $N = 2$ case, we will count the set of solutions to the system of equations (4.3.3) and (4.3.4) by counting the number of coincidences of two smooth maps with compact domain and range. Again, we start with the domain and range of our maps to be $X := (\mathbb{CP}^1)^{\frac{N(N+1)}{2}}$ and denote the projection maps into its $\frac{N(N+1)}{2}$ factors by

$$\begin{aligned} \pi_i : X \to \mathbb{CP}^1 \qquad &\text{for} \qquad i = 1, \dots, N \\ \pi_{(k,l)} : X \to \mathbb{CP}^1 \qquad &\text{for} \qquad 1 \le k < l \le N \end{aligned}$$

Define the rational map $\psi_N : X \dashrightarrow X$ by

$$\psi_N : (z, \omega) \mapsto (z', \omega')$$

with

$$\begin{aligned} z &= ([z_0^i : z_1^i])_{i=1}^N \\ z' &= ([z'^i_0 : z'^i_1])_{i=1}^N \\ \omega &= ([\omega_0^{(k,l)} : \omega_1^{(k,l)}])_{1 \le k < l \le N} \\ \omega' &= ([\omega'^{(k,l)}_0 : \omega'^{(k,l)}_1])_{1 \le k < l \le N} \end{aligned}$$

and

$$\begin{aligned} [z'^i_0 : z'^i_1] &= \left[\prod_{1 \le j < i} \omega_0^{(j,i)} \prod_{i < j \le N} \omega_1^{(i,j)} : \prod_{1 \le j < i} \omega_1^{(j,i)} \prod_{i < j \le N} \omega_0^{(i,j)} \right] \\ [\omega'^{(k,l)}_0 : \omega'^{(k,l)}_1] &= [p z_0^k z_0^l + q z_1^k z_1^l - z_0^k z_1^l : -(p z_0^k z_0^l + q z_1^k z_1^l - z_0^l z_1^k)] \end{aligned}$$

Note that $\psi_N : X \dashrightarrow X$ is well-defined away from the codimension 2 subvariety

$$Z := \bigcup_{1 \le k < l \le N} \left(Z^0_{(k,l)} \cup Z^1_{(k,l)} \right),$$

where $Z_{(k,l)} := \{pt \in X | \pi_k(pt) = \pi_l(pt) = [2q : 1 + (-1)^i \sqrt{1-4pq}]\}$. Also, we define the smooth map $\phi_N : X \to X$ by

$$\phi_N : (z, \omega) \mapsto (z'', \omega'')$$

with

$$z = ([z_0^i : z_1^i])_{i=1}^N$$
$$z'' = ([z''^i_0 : z''^i_1])_{i=1}^N$$
$$\omega = ([\omega_0^{(k,l)} : \omega_1^{(k,l)}])_{1 \le k < l \le N}$$
$$\omega'' = ([\omega''^{(k,l)}_0 : \omega''^{(k,l)}_1])_{1 \le k < l \le N}$$

and

$$[z'^i_0 : z'^i_1] = R\left([z_0^i : z_1^i]\right)$$
$$[\omega'^{(k,l)}_0 : \omega'^{(k,l)}_1] = [\omega_0^{(k,l)} : \omega_1^{(k,l)}]$$

where $R : \mathbb{CP}^1 \to \mathbb{CP}^1$ is the function defined by equation (4.3.4).

Now define a chart $U := (z_0^i \neq 0, \omega_0^{(k,l)} \neq 0 \mid i = 1, \ldots, N \text{ and } 1 \le k < l \le N)$ and $f : U \to \mathbb{C}^{\frac{N(N+1)}{2}}$ with $(z, \omega) \mapsto (\vec{z}, \vec{\omega})$ with

$$z = ([z_0^i : z_1^i])_{i=1}^N$$
$$\vec{z} = (z_i)_{i=1}^N$$
$$\omega = ([\omega_0^{(k,l)} : \omega_1^{(k,l)}])_{1 \le k < l \le N}$$
$$\vec{\omega} = (\omega_{(k,l)})_{1 \le k < l \le N}$$

and

$$z_i = z_1^i / z_0^i$$
$$\omega = \omega_1^{(k,l)} / \omega_0^{(k,l)}$$

Then, note that the system of equations (4.3.3) and (4.3.4) on $\mathbb{C}^{\frac{N(N+1)}{2}} - f(U \cap Z)$ is equivalent to

$$f \circ \psi_N \circ f^{-1}(\vec{z}, \vec{\omega}) = f \circ \phi_N \circ f^{-1}(\vec{z}, \vec{\omega}). \tag{4.3.5}$$

Thus, our strategy is to count the admissible solutions of $\psi_N(p) = \phi_N(p)$ and discard the solutions that are not on U. We formalize this count in the next section. But as we have mentioned in the $N = 2$ case, we first need to obtain a smooth resolution of $\psi_N : (\mathbb{P}^1)^{\frac{N(N+1)}{2}} \dashrightarrow (\mathbb{P}^1)^{\frac{N(N+1)}{2}}$ to apply Theorem 4.2.1. We do this resolution by applying a sequence of blow-ups to our domain which results in a smooth space C with a projection map $\pi : C \to X$ such that $C - \pi^{-1}(Z) \cong X - Z$. Then, we define smooth maps $\tilde{\psi}_N, \tilde{\phi}_N : C \to X$ such that these maps agree with our maps ψ_N and ϕ_N on $C - \pi^{-1}$. We give the details of this construction in Appendix A.1.

Therefore, we have that the solutions of the Bethe ansatz equations (admissible and inadmissible) are the same as the solutions to the equation

$$\tilde{\phi}_N(pt) = \tilde{\psi}_N(pt) \qquad \text{for } pt \in C$$

Proposition 4.3.1. *Let* $\tilde{C} = \mathrm{Blow}_Z(C)$. *There exist smooth maps* $\tilde{\psi}, \tilde{\phi} : \tilde{C} \to X$ *that agree with* ψ, ϕ *away from the exceptional divisor (i.e. away from the subvariety where we have performed the blow-up). Furthermore, if we have* $H_*(\tilde{C}) = H_*(C) \oplus H$, *then* $\tilde{\phi}_*(H) = 0$.

Proof. See Appendix A.1 for the existence of the maps. For the second statement, it follows from the general properties of the blow-up that

$$H_*(C) = \mathrm{Im}\ \pi^* \oplus H$$

where $\mathrm{Im}\pi^* \cong H_*(X)$ [**GH78**]. Moreover, since $\tilde{\phi} = \phi \circ \pi$, we have

$$\tilde{\phi}^*(H_*(X)) \cap H = \varnothing.$$

□

4.3.3. Lefschetz theorem. We reduced the problem of finding eigenstates of the Bethe ansatz to finding solutions of the equation

$$\phi(pt) = \psi(pt). \tag{4.3.6}$$

In algebraic topology, one can answer how many such solutions exist by the Lefschetz theorem.

Our cohomology spaces have dimensions

$$\dim H_{2i}(X) = \binom{N + \binom{N}{2}}{i}$$
$$\dim H_{2i+1}(X) = 0.$$

To compute the trace of the map $\psi_i \phi^i$, we choose a basis of $H_*(X)$. In this setting, we can choose a basis of $H_*(X)$ as

$$\{e_S \otimes e_T \mid S \subseteq [N] \text{ and } T \subseteq [N]^{(2)}\}$$

where $e_S \otimes e_T$ is represented by the submanifold

$$\left(\prod_{i \in S} \mathbb{CP}^1_i\right) \times \left(\prod_{(k,l) \in T} \mathbb{CP}^1_{k,l}\right).$$

We would like to compute the induced maps ψ_i and ϕ^i. Still, we have yet to determine a basis for $H_*(C)$, but we have, from the Appendix A.1 and Proposition 4.3.1 on blow-ups, that

$$H_*(C) = H_*(X) \oplus H.$$

Moreover, we have the commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\tilde{\phi}} & X \\ \big\downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \phi} & \\ X & & \end{array}$$

where $\pi : C \to X$ is the projection from the blow-up and $\phi : (\vec{z}, \vec{w}) \mapsto (\vec{z}^L, \vec{w})$ on an affine chart. So, we have that $\tilde{\phi}^* = \pi^* \circ \phi^*$ with $\pi^*(H_*(X)) \cap H = \varnothing$. Thus, we can factor the maps

$$\begin{array}{ccccc} H_*(X) & \xrightarrow{\tilde{\phi}^*} & H_*(C) & \xrightarrow{\tilde{\psi}_*} & H_*(X) \\ & \searrow{\scriptstyle \phi^*} & \big\uparrow{\scriptstyle \pi^*} & \nearrow{\scriptstyle \psi_*} & \\ & & H_*(X) & & \end{array}$$

This means that, when we are computing $\tilde{\psi}_*$, we only need to consider the image of the elements in $H_*(X) \subseteq H_*(C)$ where we have already established a basis. Now, let's consider some examples. For instance,

$$1 \overset{\phi^*}{\mapsto} L^N \cdot 1.$$

We have that the element $1 \in H_0(X)$ is represented by a point (i.e. $1 = [pt]$). Then,

$$\phi^*[pt] = [\phi^{-1}(pt)] = [L^N pts] = L^N[pt] = L^N \cdot 1.$$

Also, we have that

$$e_i \overset{\phi^*}{\mapsto} L^{N-1} \cdot e_i.$$

Indeed, we can represent $e_i = [(1, \dots, 1, z_i, 1, \dots, 1)]$ and we have that

$$\phi^{-1}((1, \dots, 1, z_i, 1, \dots, 1)) = \bigcup_{k_j=1,\dots,L} (e^{2\pi i k_1/L}, \dots, z_i, \dots, e^{2\pi i k_N/L}).$$

In general, we have the following result.

Lemma 4.3.2.

$$e_S \otimes e_T \overset{\phi^*}{\mapsto} L^{N-|S|} e_S \otimes e_T$$

Now, let's consider the ψ_* map. For example, we have

$$1 \overset{\psi_*}{\mapsto} 1$$

since $\psi_*[pt] = [\psi(pt)] = [pt] = 1$. Also, we have that

$$e_i \overset{\psi_*}{\mapsto} \sum_{i<j} e_{ij} + \sum_{i>j} e_{ji}$$

since we have that the image of $\mathbb{CP}^1_i$ is

$$\left(\bigcup_{i>j} \mathbb{CP}^1_{ij}\right) \cup \left(\bigcup_{i<j} \mathbb{CP}^1_{ji}\right).$$

Similarly, we have that

$$e_{ij} \overset{\psi_*}{\mapsto} e_i + e_j$$

since the image of $\mathbb{CP}^1_{ij}$ is $\mathbb{CP}^1_i \cup \mathbb{CP}^1_j$. Also, we consider one more instructive example where $N = 3$, where we have that

$$e_1 \otimes e_2 \otimes e_3 \otimes e_{1,2} \otimes e_{1,3} \otimes e_{23} \overset{\psi_*}{\mapsto} 0.$$

This follows from the fact that we may represent $e_1 \otimes e_2 \otimes e_3 \otimes e_{1,2} \otimes e_{1,3} \otimes e_{23} = [X]$, we have that $\dim_{\mathbb{R}} X = 12$, and $\dim_{\mathbb{R}} \psi(X) < 12$. The latter can easily be seen since the coordinates of the image are not independent. Note that

$$(w_{12}w_{13})(w_{12}^{-1}w_{2,3})(w_{13}^{-1}w_{23}^{-1}) = 1.$$

Therefore, the map ψ_* is not as straightforward as ϕ^*, but in general, this is the only situation that we will encounter that will give us a zero in our computations.

Remark 4.3.2. *In the rest of the paper, we will fix the maps $\phi, \psi : C \to X$ and we will compute the Lefschetz number for different restrictions of the maps ψ and ϕ to certain subvarieties $C(\mathcal{A}, \mathcal{B}) \subseteq C$ and $X(\mathcal{A}, \mathcal{B}) \subseteq X$, introduced in the next section. For convenience, we will avoid the cumbersome notation $\lambda(\psi|_{C(\mathcal{A},\mathcal{B})}, \phi_{C(\mathcal{A},\mathcal{B})})$ and simply write*

$$\lambda(\mathcal{A}, \mathcal{B}).$$

4.4. Proof of Completeness

4.4.1. Computing traces by counting trees. In this section, we show how to compute the Lefschetz numbers of the subvarieties containing inadmissible solutions by enumerating rooted forests.

Proposition 4.4.1. *Given $\psi, \phi : C \to X$ as above, with the induced homology maps $\psi_i : H_i(C) \to H_i(X)$ and $\phi^i : H_i(X) \to H_i(C)$, we have*

$$\sum_{i=0}^{\dim X} (-1)^i \operatorname{Tr}(\psi_i \phi^i) = \sum_{f \in \mathcal{F}_N} L^{n(f)},$$

where $\mathcal{F}_N$ denotes the set of planted forests on the vertex set $[N]$ and $n(f)$ denotes the number of components of a forest f.

Proof. First, note that from the commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\phi} & X \\ \Big\downarrow \pi & \nearrow \phi' & \\ X & & \end{array}$$

where π is the projection from the blow-up and by definition $\phi = \phi' \circ \pi$. We obtain the following commutative diagram in homology

$$\begin{array}{ccccc} H_*(X) & \xrightarrow{\phi^*} & H_*(C) & \xrightarrow{\psi_*} & H_*(X) \\ & \searrow (\phi')^* & \Big\uparrow \pi^* & \nearrow (\psi_*)|_{H_*(X)} & \\ & & H_*(X) & & \end{array}$$

In Appendix A.1 and Proposition 4.3.1, we explain that indeed π_* is indeed an orthogonal projection. From this, we have that

$$\mathrm{Tr}(\psi_* \phi^*) = \mathrm{Tr}((\psi_*)|_{H_*(X)}(\phi')^*).$$

Thus, we only consider computations on the direct summand $H_*(X)$ of $H_*(C)$, and we will abuse notation and use $(\phi')^*$, $(\psi_*)|_{H_*(X)}$ and ϕ^*, ψ_* interchangeably, respectively.

Next, we introduce a basis for $H_*(X)$ via Kunneth's theorem. That is, given the inclusion maps of the different factors of $\mathbb{CP}^1$ in X

$$j_i : \mathbb{CP}^1 \hookrightarrow X \qquad \text{for } i = 1, \ldots, N$$

$$j_{(k,l)} : \mathbb{CP}^1 \hookrightarrow X \qquad \text{for } 1 \leq i < j \leq N$$

we define elements in $H_2(X)$

$$e_i := (j_i)_*([\mathbb{CP}^1]) \text{ for } i = 1, \ldots, N$$

$$e_{(k,l)} := (j_{(k,l)})_*([\mathbb{CP}^1]) \qquad \text{for } 1 \leq i < j \leq N$$

and given $S \subseteq [N]$ and $T \subseteq [N]^{(2)}$ we define elements in $H_{2|S|}(X)$ and $H_{2|T|}(X)$, respectively

$$e_S := \bigotimes_{i \in S} e_i$$

$$e_T := \bigotimes_{(k,l) \in T} e_{(k,l)}.$$

Then, by Kunneth's theorem we have a basis for $H(X)$

$$\mathbb{B} := \{e_S \otimes e_T \mid S \subseteq [N] \text{ and } T \subseteq [N]^{(2)}\}$$

and note that $\deg(e_S \otimes e_T) = 2|S| + 2|T|$.

Moreover, for convenience, we define an inner product

$$\langle \cdot, \cdot \rangle : H(X)^{\otimes 2} \to \mathbb{C}$$

such that $\mathbb{B}$ is an orthonormal basis. Then, we have that

$$\begin{aligned}\sum_{i=0}^{\dim_{\mathbb{R}} X} (-1)^i Tr(\psi_i \phi^i) &= \sum_{b \in \mathbb{B}} (-1)^{\deg(b)} \langle \psi_* \phi^*(b), b \rangle \\ &= \sum_{b \in \mathbb{B}} (-1)^{\deg(b)} \langle \psi_*(b), \phi^*(b) \rangle\end{aligned}$$

where the last equality comes from the fact that ϕ^* is represented by a diagonal matrix under the basis $\mathbb{B}$. Also, it is clear that for every basis element we have

$$\phi^*(e_S \otimes e_T) = L^{N-|S|} e_S \otimes e_T.$$

Thus, we concentrate on computing

$$\langle \psi_*(e_S \otimes e_T), e_S \otimes e_T \rangle.$$

First, we define $X_1 = (\mathbb{CP}^1)^N$ and $X_2 = (\mathbb{CP}^1)^{\binom{N}{2}}$ and we write $X = X_1 \times X_2$. Now, given inclusions

$$j_1 : X_1 \hookrightarrow X$$

$$j_2 : X_2 \hookrightarrow X$$

and projections

$$\pi_1 : X \to X_1$$

$$\pi_2 : X \to X_2$$

we consider

$$\psi_{11} : X_1 \to X_1$$

$$\psi_{12} : X_1 \to X_2$$

$$\psi_{21} : X_2 \to X_1$$

$$\psi_{22} : X_2 \to X_2$$

and we note that ψ_{11} and ψ_{22} are constant. Thus, we have that

$$\langle \psi_*(e_S \otimes e_T), e_S \otimes e_T \rangle = \langle (\psi_{12})_*(e_S), e_T \rangle \langle (\psi_{21})_*(e_T), e_S \rangle$$

One easy consequence of this is that $\deg e_S = \deg e_T$ for the above quantity to be non-zero, which means that $|S| = |T|$.

Now, we work on another decomposition of X by considering a map from the basis elements $\mathbb{B}$ and the set of graphs on N labeled vertices $\mathcal{G}_N$ given by

$$\gamma : e_S \otimes e_T \mapsto ([N], T)$$

where $[N]$ is the vertex set and T is the edge set. Then, fixing T we have a decomposition of $([N], T)$ into

$$([N], T) = \left[\bigcup_{x=1}^{r} (V_x, T_x) \right] \cup (V_{r+1}, \varnothing)$$

where (V_x, T_x), for $1 \le x \le r$, are connected components with at least one edge and V_{r+1} is the set of vertices without edges (i.e. $V_{r+1} = [N] \smallsetminus \bigcup_{x=1}^{r} V_x$). Also, we define $T_{r+1} \coloneqq [N]^{(2)} \smallsetminus T$ and the spaces

$$G_x \coloneqq (\mathbb{CP}^1)^{|V_x|} \times (\mathbb{CP}^1)^{|T_x|}$$

with the projections of X onto the i^{th} $\mathbb{CP}^1$ factors and $(k,l)^{th}$ $\mathbb{CP}^1$ factors such that $i \in V_x$ and $(k,l) \in T_x$

$$p_x : X \to G_x \qquad \text{for } 1 \le x \le r+1.$$

Thus, we obtain the decomposition

$$X = \prod_{x=1}^{r+1} G_x.$$

Next, given the inclusions

$$q_x : G_x \hookrightarrow X \qquad \text{for } 1 \le x \le r+1,$$

we define the maps

$$\psi_{x,y} := p_y \circ \psi \circ q_x : G_x \to G_y,$$

and we note that $\psi_{x,y}$ is constant unless $x = y$. Then, given a basis element $e_S \otimes e_T$, we can write via Kunneth's formula

$$e_S \otimes e_T = \left[\bigotimes_{x=1}^{r} (e_{S_x} \otimes e_{T_x})\right] \otimes e_{S_{r+1}}$$

where $S_x := V_x \cap S$ for $1 \le x \le r+1$. Thus, from this decompostion, we obtain

$$\langle \psi_*(e_S \otimes e_T), e_S \otimes e_T \rangle =$$

$$\langle (\psi_{r+1,r+1})_*(e_{S_{r+1}}), e_{S_{r+1}} \rangle \prod_{x=1}^{r} \langle (\psi_{x,x})_*(e_{S_x} \otimes e_{T_x}), e_{S_x} \otimes e_{T_x} \rangle$$

Thus, we may assume that $([N],T)$ is connected and that

$$S(T) := \bigcup_{(i,j)\in T} \{i,j\} = [N]$$

Also, we have that $S \subseteq [N]$ and from before we have that $|S| = |T|$. Then, $|T| \le N$. Since $([N],T)$ is a connected graph, we must have that $|T| = N$, which means that $([N],T)$ is a connected graph with one cycle, or $|T| = N-1$, which means that $([N],T)$ is a tree. Otherwise, if $|T| < N-1$, then $([N],T)$ is not connected.

Then, given the following lemma

LEMMA 4.4.1. *Given $T \subseteq [N]^{(2)}$ such that the graph $([N],T)$ is connected,*

$$\langle(\psi_{21})_*(e_T), e_S\rangle = \delta_{|S|,N-1} = \langle(\psi_{12})_*(e_S), e_T\rangle.$$

We have that, if $([N],T)$ is connected, then $\langle\psi_*(e_S \otimes e_T), e_S \otimes e_T\rangle = 1$ if and only if $|T| = |S| = N-1$. This means that $([N],T)$ is a tree and that $[N]-S = \{\rho\}$. Therefore, we have that $([N],T,[N]-S)$ is a rooted tree. So we obtain

$$\begin{aligned}\sum_{i=0}^{\dim_{\mathbb{R}} X}(-1)^i Tr(\psi_i\phi^i) &= \sum_{e_S\otimes e_T\in\mathbb{B}}(-1)^{\deg(e_S\otimes e_T)}\langle\psi_*(e_S\otimes e_T),\phi^*(e_S\otimes e_T)\rangle\\ &= \sum_{e_S\otimes e_T\in\mathbb{B}}(-1)^{2|S|+2|T|}L^{N-|S|}\langle\psi_*(e_S\otimes e_T),e_S\otimes e_T\rangle\\ &= \sum_{f\in\mathcal{F}}L^{\#roots}\\ &= \sum_{f\in\mathcal{F}}L^{n(f)}. \qquad\square\end{aligned}$$

Now, with this Proposition 4.4.1 we have reduced the problem of counting "fixed points" to a combinatorial problem of counting planted forests. Still we need to further refine the combinatorics to account for the inadmissible solutions. Indeed, the count from 4.4.1 includes inadmissible solutions. Fortunately, we can classify inadmissible solutions using 4.3.1 and "special" partitions of $[N]$, which we introduce.

DEFINITION 4.4.1. *Let L and N be positive integers. A* labeled enhanced partition of N *is a pair*

$$(\mathcal{A},\mathcal{B}) = (\{A_1,\ldots,A_s\},\{(B_1,\overline{B}_1,d_1),\ldots,(B_b,\overline{B}_b,d_b)\}),$$

where $A_1,A_2,\ldots,A_s,B_1,B_2,\ldots,B_b,\overline{B}_1,\overline{B}_2,\ldots,\overline{B}_b$ are pairwise disjoint non-empty sets whose union is $[N]$ and $d_1,d_2,\ldots,d_b \in [L]$. We use the notation $\mathbb{EP}^l(N)$ to denote the set of all labeled enhanced partitions of N.

Similarly, an enhanced partition of N *is a pair*

$$(\mathcal{A},\mathcal{B}) = (\{A_1,\ldots,A_s\},\{(B_1,\overline{B}_1),\ldots,(B_b,\overline{B}_b)\}),$$

where $A_1, A_2, \ldots, A_s, B_1, B_2, \ldots, B_b, \overline{B}_1, \overline{B}_2, \ldots, \overline{B}_b$ are pairwise disjoint non-empty sets whose union is $[N]$. We use the notation $\mathbb{EP}(N)$ to denote the set of all enhanced partitions of N.

From this, we have a "forgetful map" $\sigma : \mathbb{EP}^l(N) \to \mathbb{EP}(N)$ defined by

$$(\{A_1, \ldots, A_s\}, \{(B_1, \overline{B}_1; d_1), \ldots, (B_b, \overline{B}_b; d_b)\}) \mapsto (\{A_1, \ldots, A_s\}, \{(B_1, \overline{B}_1), \ldots, (B_b, \overline{B}_b)\})$$

which "forgets" the label of a partition. Then, given an enhanced partition we can define subvarieties of X and C as follows:

$$C(\mathcal{A}, \mathcal{B}) := \left\{ p \in C \;\middle|\; \begin{array}{l} \tilde{\pi}_i(p) = \tilde{\pi}_j(p) \text{ if } i, j \in A \text{ for some } A \in \mathcal{A}, \\ \tilde{\pi}_k(p) = \alpha_r \text{ if } k \in B \text{ for some } (B, \overline{B}, r) \in \mathcal{B} \\ \tilde{\pi}_l(p) = \beta_r \text{ if } k \in \overline{B} \text{ for some } (B, \overline{B}, r) \in \mathcal{B} \end{array} \right\}$$

and

$$X(\mathcal{A}, \mathcal{B}) := \left\{ p \in X \;\middle|\; \begin{array}{l} \pi_i(p) = \pi_j(p) \text{ if } i, j \in A \text{ for some } A \in \mathcal{A}, \\ \pi_k(p) = [1:0] \text{ if } k \in B \text{ for some } (B, \overline{B}, r) \in \mathcal{B} \\ \pi_l(p) = [0:1] \text{ if } k \in \overline{B} \text{ for some } (B, \overline{B}, r) \in \mathcal{B} \end{array} \right\}$$

One should now note that if $pt \in C(\mathcal{A}, \mathcal{B})$ for any non-trivial labeled enhanced partition $(\mathcal{A}, \mathcal{B})$ and

$$\tilde{\phi}(pt) = \tilde{\psi}(pt)$$

then pt is an inadmissible solution. Thus, we will use (labeled) enhanced partitions $(\mathcal{A}, \mathcal{B})$ and the corresponding subvarieties $X(\mathcal{A}, \mathcal{B})$ and $C(\mathcal{A}, \mathcal{B})$ to count inadmissible solutions. That is, we will count the number of solutions of

$$\tilde{\phi}|_{C(\mathcal{A}, \mathcal{B})}(pt) = \tilde{\psi}|_{C(\mathcal{A}, \mathcal{B})}(pt)$$

to count the number of inadmissible solutions satisfying "conditions" $(\mathcal{A}, \mathcal{B})$.

First, we define the analogous set of planted forests $\mathcal{F}_N$. Given a tuple of conditions $(\mathcal{A}, \mathcal{B})$, we define $\mathcal{F}(\mathcal{A}, \mathcal{B})$ to be the set of planted forests with $\mathcal{A} \cup \mathcal{B}$ as vertices where vertices from $\mathcal{B}$ may only be roots. Moreover, we define a multiplicity for each $f \in \mathcal{F}(\mathcal{A}, \mathcal{B})$. For this, we denote as $E(f)$ the set of edges of a planted forest. We have that the root on each component of f induces an orientation on each component where the root can be thought of as a source and the edges are oriented away from the root. Then, we write the edges of f as tuples (a,b) where the edge connects vertices a and b, and it is oriented from a to b. Recall, that a is an $\mathcal{A}$-set or a $\mathcal{B}$-set, and denote by $\#(a)$ the cardinality of the set a if $a \in A$ or the cardinality of $a(1) \cup a(2)$ if $a \in \mathcal{B}$. Now, we can define the multiplicity of a forest $f \in \mathcal{F}(\mathcal{A},\mathcal{B})$ as

$$m(f) := \prod_{(a,b)\in E(f)} \#(a).$$

We have the following result generalizing Proposition 4.4.1.

Lemma 4.4.2. *Given a labeled enhanced partition* $(\mathcal{A},\mathcal{B})$*, we have that the number solutions of* $\phi(pt) = \psi(pt)$ *satisfying the inadmissibility conditions* $(\mathcal{A},\mathcal{B})$ *is given by*

$$\sum_{f\in\mathcal{F}(\mathcal{A},\mathcal{B})} m(f)\, L^{n(f)-|\mathcal{B}|}.$$

Proof. We obtain this result from the Lefschetz Theorem 4.2.1 and Proposition 4.4.1. The difference is that we are looking for solutions to the Lefschetz problem in subvarieties of C and X that are determined by the conditions $(\mathcal{A},\mathcal{B})$. Namely, consider the projection maps onto the i^{th} $\mathbb{CP}^1$ factors of X and C, respectively

$$\pi_i : X \to \mathbb{CP}^1$$

$$\tilde{\pi}_i := \pi_i \circ \pi : C \to \mathbb{CP}^1$$

where $\pi : C \to X$ is the projection from the blow-up.

Note that we have inclusion maps

$$j_C : C(\mathcal{A},\mathcal{B}) \hookrightarrow C$$

$$j_X : X(\mathcal{A},\mathcal{B}) \hookrightarrow X$$

and we define maps $\phi_{\mathcal{A},\mathcal{B}}, \psi_{\mathcal{A},\mathcal{B}} : C(\mathcal{A},\mathcal{B}) \to X(\mathcal{A},\mathcal{B})$ such that the following diagram commutes

$$
\begin{array}{ccc}
C & \xrightarrow{\phi,\psi} & X \\
{\scriptstyle j_C}\big\uparrow & & {\scriptstyle j_X}\big\uparrow \\
C(\mathcal{A},\mathcal{B}) & \xrightarrow{\phi_{\mathcal{A},\mathcal{B}},\psi_{\mathcal{A},\mathcal{B}}} & X(\mathcal{A},\mathcal{B})
\end{array}
\tag{4.4.1}
$$

Then, by the Lefschetz Theorem 4.2.1, we have that the number of solutions (with multiplicity) that satisfy the inadmissibility conditions $(\mathcal{A},\mathcal{B})$ is given by

$$\lambda(\mathcal{A},\mathcal{B}) := \sum_{i=0}^{\dim_{\mathbb{R}} X(\mathcal{A},\mathcal{B})} (-1)^i Tr((\psi_{\mathcal{A},\mathcal{B}})_i(\phi_{\mathcal{A},\mathcal{B}})^i).$$

Again, we compute this number by introducing a basis of $H(X(\mathcal{A},\mathcal{B}))$ and an inner product that makes this basis orthonormal. Then, along with functorial properties of the commutative diagram (4.4.1), we can compute $\lambda(\mathcal{A},\mathcal{B})$ using Proposition 4.4.1. So, we introduce elements in $e_{A_i} \in H_2(X(\mathcal{A},\mathcal{B}))$ for $1 \leq i \leq s$ where

$$(j_C)^*(e_k) = (j_X)^*(e_k) = e_{A_i} \qquad \text{for any } k \in A_i \tag{4.4.2}$$

$$(j_C)_*(e_{A_i}) = (j_X)_*(e_{A_i}) = \sum_{k\in A_i} e_k \tag{4.4.3}$$

and for any subset $S \subseteq A$, we define

$$e_S = \bigotimes_{A_i \in S} e_{A_i}.$$

Also, note that the elements $e_{(k,l)} \in H_2(X)$ remain unaffected by the inclusion map. That is, we have $(j_X)^*(j_X)_*(e_{(k,l)}) = e_{(k,l)}$, and by abuse of notation we denote the image of $e_{(k,l)} \in H_2(X)$ under the push-forward of the inclusion map by $e_{(k,l)} \in H_2(X(\mathcal{A},\mathcal{B}))$. For $T \in [N]^{(2)}$, we let $e_T \in H(X(\mathcal{A},\mathcal{B}))$ be the push-forward of $e_T \in H(X)$ under the inclusion map j_X. Then, we have a basis for $H(X(\mathcal{A},\mathcal{B}))$ defined as follows

$$\mathbb{B}_{\mathcal{A},\mathcal{B}} := \{e_S \otimes e_T \mid S \subseteq A \text{ and } T \subseteq [N]^{(2)}\}$$

and an inner product that makes this basis orthonormal

$$\langle \cdot, \cdot \rangle_{\mathcal{A},\mathcal{B}} : H(X(\mathcal{A},\mathcal{B}))^{\otimes 2} \to \mathbb{C}.$$

First, one can see that

$$\langle (\phi_{\mathcal{A},\mathcal{B}})^*(e_S \otimes e_T), e_S \otimes e_T \rangle_{\mathcal{A},\mathcal{B}} = L^{|A|-|A\cap S|}. \tag{4.4.4}$$

This follows from the fact that

$$\langle (j_X)^*(e_S \otimes e_T), (j_C)^*(e_S \otimes e_T) \rangle_{\mathcal{A},\mathcal{B}} = \langle e_S \otimes e_T, e_S \otimes e_T \rangle$$

and from the commutative diagram (4.4.1)

$$\langle (\phi_{\mathcal{A},\mathcal{B}})^*(j_X)^*(e_S \otimes e_T), (j_C)^*(e_S \otimes e_T) \rangle_{\mathcal{A},\mathcal{B}} = \langle \phi^*(e_S \otimes e_T), e_S \otimes e_T \rangle.$$

We note that the number of components is the same as the number of roots and we will show shortly that elements in B and elements in the complement of S in A must be roots. We have that $n(f) - |\mathcal{B}| = |A| - |A \cap S|$ as we wished.

Now, we wish to compute

$$\langle (\psi_{\mathcal{A},\mathcal{B}})_*(e_S \otimes e_T), e_S \otimes e_T \rangle_{\mathcal{A},\mathcal{B}}.$$

First, note that, from (4.4.3), we have

$$\frac{\langle (j_C)_*(e_S \otimes e_T), (j_X)_*(e_S \otimes e_T) \rangle}{\prod_{i=i}^{s} |A_i|} = \langle e_S \otimes e_T, e_S \otimes e_T \rangle_{\mathcal{A},\mathcal{B}}$$

and so, by the commutative diagram and the functoriality of the push-forward, we have that

$$\langle (\psi_{\mathcal{A},\mathcal{B}})_*(e_S \otimes e_T), e_S \otimes e_T \rangle_{\mathcal{A},\mathcal{B}} = \frac{\langle \psi_*(j_C)_*(e_S \otimes e_T), (j_X)_*(e_S \otimes e_T) \rangle}{\prod_{i=1}^{s} |A_i|}.$$

Again, since we know $(j_C)_*$ and $(j_X)_*$ from (4.4.2), (4.4.3), and by Proposition 4.4.1, we have that

$$\langle \psi_*(j_C)_*(e_S \otimes e_T), (j_X)_*(e_S \otimes e_T) \rangle = 1$$

if and only if there exist a choice $a_i \in A_i$ for $i = 1, \dots s$ such that $([N], T, [N] - \{a_1, \dots, a_s\})$ is a planted forest. Note that there is no element of B in $\{a_1, \dots, a_s\}$ since the elements in B are not in the image of $(j_C)_*$, which makes elements of B roots in $([N], T, [N] - \{a_1, \dots, a_s\})$. Thus, we consider the subset $\tilde{\mathcal{F}}(\mathcal{A}, \mathcal{B}) \subseteq \mathcal{F}$ of planted forest on N labeled vertices where $\tilde{f} \in \tilde{\mathcal{F}}(\mathcal{A}, \mathcal{B})$ if and only if there are no edges within the elements of A_i for $1 \le i \le s$ and vertices from B_j for $1 \le j \le r$ may only be roots. Then, the above becomes

$$\langle \psi_*(j_C)_*(e_S \otimes e_T), (j_X)_*(e_S \otimes e_T) \rangle = 1$$

if and only if there exists a choice $a_i \in A_i$ for $i = 1, \dots s$ such that $([N], T, [N] - \{a_1, \dots, a_s\})$ is in $\tilde{\mathcal{F}}(\mathcal{A}, \mathcal{B})$. Moreover, note that given a planted forest $\tilde{f} \in \tilde{\mathcal{F}}(\mathcal{A}, \mathcal{B})$ we may define a planted forest $f \in \mathcal{F}(\mathcal{A}, \mathcal{B})$, in the set of planted forests with $A \cup B$ as the vertex set. Indeed, for every directed edge (k, l) in $\tilde{f}$, where the direction is induced from the roots, we define a directed edge (v, w) such that $k \in v$ and $l \in w$ and this directed edge makes f into a planted forest in $\mathcal{F}(A, B)$. We denote this map by

$$F : \tilde{\mathcal{F}}(\mathcal{A}, \mathcal{B}) \to \mathcal{F}(\mathcal{A}, \mathcal{B}).$$

Therefore, we have

$$\begin{aligned}
&\sum_{e_S \otimes e_T \in \mathbb{B}(\mathcal{A}, \mathcal{B})} \langle (\psi_{\mathcal{A},\mathcal{B}})_*(e_S \otimes e_T), e_S \otimes e_T \rangle_{\mathcal{A},\mathcal{B}} \\
&= \sum_{e_S \otimes e_T \in \mathbb{B}(\mathcal{A}, \mathcal{B})} \frac{\langle \psi_*(j_C)_*(e_S \otimes e_T), (j_X)_*(e_S \otimes e_T) \rangle}{\prod_{i=1}^s |A_i|} \\
&= \sum_{\tilde{f} \in \tilde{\mathcal{F}}(\mathcal{A}, \mathcal{B})} \frac{1}{\prod_{i=1}^s |A_i|} \\
&= \sum_{f \in \mathcal{F}(\mathcal{A}, \mathcal{B})} \sum_{\tilde{f} \in F^{-1}(f)} \frac{1}{\prod_{i=1}^s |A_i|}
\end{aligned}$$

Now, one has that $|F^{-1}(f)| = \prod_{(v,w) \in E(f)} \#(v)\#(w)$ since we have $\#(v)\#(w)$ choices of edges (k, l) that map to (v, w) under F. Then,

$$\sum_{f \in \mathcal{F}(\mathcal{A}, \mathcal{B})} \sum_{\tilde{f} \in F^{-1}(f)} \frac{1}{\prod_{i=1}^s |A_i|} = \prod_{(v,w) \in E(f)} \#(v) = m(f),$$

since each vertex only has only one incoming edge at most and otherwise its a root. So,

$$\sum_{e_S \otimes e_T \in \mathbb{B}(\mathcal{A},\mathcal{B})} \langle (\psi_{\mathcal{A},\mathcal{B}})_*(e_S \otimes e_T), e_S \otimes e_T \rangle_{\mathcal{A},\mathcal{B}} = \sum_{f \in \mathcal{F}(\mathcal{A},\mathcal{B})} m(f),$$

and from (4.4.1), we have the desired result

$$\sum_{e_S \otimes e_T \in \mathbb{B}(\mathcal{A},\mathcal{B})} \langle (\psi_{\mathcal{A},\mathcal{B}})_*(\phi_{\mathcal{A},\mathcal{B}})^*(e_S \otimes e_T), e_S \otimes e_T \rangle_{\mathcal{A},\mathcal{B}\mathcal{A},\mathcal{B}} = \sum_{f \in \mathcal{F}(\mathcal{A},\mathcal{B})} L^{n(f)-|\mathcal{B}|} m(f). \qquad \square$$

Now, we have combinatorial equations counting the inadmissible solutions using the labeled enhanced partitions. Although this refinement is enough to obtain the proper count of admissible solutions, we refine our equations still slightly more to remove the label in the labeled enhanced partitions. Thus, for an enhanced partition $(\mathcal{A},\mathcal{B})$, we define subvarieties of C and X by taking the union of the subvarieties defined by the labeled enhanced partitions which match the unlabeled enhanced partition $(\mathcal{A},\mathcal{B})$. That is,

$$X(\mathcal{A},\mathcal{B}) = \bigcup_{(\tilde{\mathcal{A}},\tilde{\mathcal{B}}) \in \sigma^{-1}(\mathcal{A},\mathcal{B})} X(\tilde{\mathcal{A}},\tilde{\mathcal{B}})$$

$$C(\mathcal{A},\mathcal{B}) = \bigcup_{(\tilde{\mathcal{A}},\tilde{\mathcal{B}}) \in \sigma^{-1}(\mathcal{A},\mathcal{B})} C(\tilde{\mathcal{A}},\tilde{\mathcal{B}})$$

This union washes out the dependence on the labels, and we have a similar result to 4.4.2.

LEMMA 4.4.3. *The number of solutions to $\phi(p) = \psi(p)$ satisfying the inadmissibility conditions prescribed by the enhanced partition $(\mathcal{A},\mathcal{B})$ is*

$$\sum_{f \in \mathcal{F}(\mathcal{A},\mathcal{B})} m(f)\, L^{n(f)}.$$

PROOF. This follows directly from the fact that $C(\mathcal{A},\mathcal{B})$ is the disjoint union of the subvarieties $C(\tilde{\mathcal{A}},\tilde{\mathcal{B}})$ with $(\tilde{\mathcal{A}},\tilde{\mathcal{B}}) \in \sigma^{-1}(\mathcal{A},\mathcal{B})$. Also, we have that $|\sigma^{-1}(\mathcal{A},\mathcal{B})| = L^{|\mathcal{B}|}$. $\square$

4.4.2. The enumeration of forests. With this, we are almost ready to perform the full count of admissible solutions to the Bethe ansatz. The general strategy is to count all solutions and then subtract the inadmissible solutions using equations from Lemma 4.4.3. One should note that an inadmissible solution satisfies many inadmissibility conditions $(\mathcal{A},\mathcal{B})$. That is, there are multiple enhanced partitions $(\mathcal{A},\mathcal{B})$ such that $pt \in C(\mathcal{A},\mathcal{B})$. For example, if $(\mathcal{A},\mathcal{B}) = (\{1\},\{2\},\ldots,\{N\};\varnothing)$, we have $C(\mathcal{A},\mathcal{B}) = C$ and so all inadmissible solutions satisfy these trivial inadmissibility conditions. Thus, we need to be careful in subtracting the number of inadmissibility conditions, making sure that we only subtract each inadmissibility condition exactly once. For this, we will introduce a combinatorial factor on enhanced partitions that will perform the count properly, but first we define a partial order on $\mathbb{EP}(N)$ that will aid in the combinatorics. For any two enhanced partitions, we say that

$$(\mathcal{A},\mathcal{B}) \leq (\mathcal{A}',\mathcal{B}') \qquad \text{if and only if} \qquad X(\mathcal{A},\mathcal{B}) \subseteq X(\mathcal{A}',\mathcal{B}').$$

Now, we introduce the combinatorial factor on the enhanced partitions needed for the computations. For this, we introduce the following weights on enhanced partitions.

Definition 4.4.2. *The* weight *of a set S is*

$$\omega(S) = (-1)^{|S|-1}(|S|-1)!.$$

Let $(\mathcal{A},\mathcal{B})$ be an enhanced partition. Then the weight *of the tuple $(\mathcal{A},\mathcal{B})$ is*

$$\omega(\mathcal{A},\mathcal{B}) = \prod_{A\in\mathcal{A}} \omega(A) \prod_{B\in\mathcal{B}} \omega(B\cup\overline{B}).$$

Note that the weight of the tuple corresponding to X is

$$\omega((\{1\},\{2\},\dots,\{N\};\varnothing)) = 1.$$

Lemma 4.4.4. *Let $(\mathcal{A}(pt),\mathcal{B}(pt))$ denote the intersection of all inadmissible varieties $(\mathcal{A},\mathcal{B})$ that contain pt. Then*

$$\sum_{(\mathcal{A},\mathcal{B})\geq(\mathcal{A}(pt),\mathcal{B}(pt))} \omega(\mathcal{A},\mathcal{B}) = 0$$

unless pt is admissible, in which case the sum is simply unity as noted above.

Proof. First, we will show that we only need to prove this for the case where $(\mathcal{A}(pt),\mathcal{B}(pt)) = (A_1)$, i.e. when $z_1 = \cdots = z_N$. In this case, we are summing over all partitions of $[N]$, refinements of $\{\{1,2,\dots,N\}\}$, in which case we will see that the definition of the weights is natural and gives us our lemma.

Given a tuple $(\mathcal{A}, \mathcal{B})$, consider it as a partition

$$P_{\mathcal{A},\mathcal{B}} := \{A_1, \dots, A_s, B_1 \cup \overline{B}_1, \dots, B_b \cup \overline{B}_b\}$$

of $[N]$ in which we have listed separately the numbers that belong to $B_1, \dots, B_b$ in a set we call

$$B' = B_1 \cup \cdots \cup B_b.$$

Now, if we take any refinement of $P_{\mathcal{A},\mathcal{B}}$, we can map it onto an inadmissible variety by comparing each set in the partition with B':

1. If the set is disjoint with B' or contained in B', then it is an A set.

2. If the set has nonempty intersection with B' but is not contained in B', then it is a B set with those elements from B' distributed accordingly. Note that the weight is preserved under this map.

Example: Let $(\mathcal{A}, \mathcal{B}) = (\{\{1,2\}\}, \{\{3,4\}, \{5\}\})$. Then $P_{\mathcal{A},\mathcal{B}} = \{\{1,2\}, \{3,4,5\}\}$ and $B' = \{3,4\}$. Consider a refinement such as $P' = \{\{1\}, \{2\}, \{3\}, \{4,5\}\}$. This corresponds to $(\{\{1\}, \{2\}, \{3\}\}, \{\{4\}, \{5\}\})$.

Since every refinement corresponds to a unique inadmissible variety containing $(\mathcal{A}(pt), \mathcal{B}(pt))$, it only remains to prove the lemma for inadmissible varieties without roots and poles, i.e. with A sets only. However, the multiplicativity of ω reduces this to the case of a single A set, the case $(\mathcal{A}(pt), \mathcal{B}(pt)) = (A_1)$, $z_1 = \cdots = z_N$.

We will proceed by induction. The base case is obvious. To get a sense of how the cancellation occurs, check $N = 2$:

$$\omega(\{\{1,2\}\}) + \omega(\{\{1\},\{2\}\}) = -1 + 1 = 0.$$

Let $A = \{1, 2, \dots, N\}$. Consider every partition in which N is in a set by itself, e.g. $\{\{1, 2, \dots, N-1\}, \{N\}\}$. By the induction hypothesis, the sum of the weights of these vanishes since the weight of the singleton N can be factored out. Now consider every partition in which N is in a pair, such as $\{\{1, 2, \dots, N-2\}, \{N-1, N\}\}$. Again, the sum of the weights is zero by induction since the weight of the pair including N can be factored out of every term. Continuing in this way, we are left with the partitions in which N is in a set of cardinality $N-1$ and the original partition. The sum of these weights is

$$\omega(A) = \omega(\{1,2,\dots,N\}) + (N-1)\omega(\{\{1\},\{2,\dots,N\}\})$$
$$= (-1)^{N-1}(N-1)! + (N-1)(-1)^{N-2}(N-2)! = 0.$$

□

Corollary 4.4.1. *The number of admissible solutions to the Bethe ansatz equations is*

$$\sum_{(\mathcal{A},\mathcal{B})} \omega(\mathcal{A},\mathcal{B})\,\lambda(\mathcal{A},\mathcal{B}).$$

4.4.3. The final count. Now that we need to compute the final sum in Corollary 4.4.1, it is actually more natural to relabel the forests with permutation cycles. As in our proof of Lemma 4.4.4, we note that an enhanced partition can be considered as the usual sort of set partition with certain elements marked. Now, the weight of that partition gives the number of permutations with that cycle structure with the sign encoding the sign of the permutation. So, instead of labeling our vertices with sets from the partition, we label them with cycles. Next, we consider the family of forests with vertices labeled by cycles and edges labeled by an element of the cycle labeling the vertex from which it issues. This clearly gives us a larger collection of forests, but there is no need for the weights m and ω any more.

Consider a permutation $\pi \in S_N$ and the forests whose vertices are labelled by the cycles of π. Mark some of the elements of the cycles as roots, but not all elements in a given cycle. Now, mark each edge with an element of the cycle in the vertex closer to the root. To each of these objects, we assign the weight $(-1)^{s(\pi)}L^{c(\pi)}$, where s denotes the sign of the permutation and c denotes the number of disjoint cycles in the permutation. Note that $(-1)^{s(\pi)}L^{c(\pi)} = (-1)^N(-L)^{c(\pi)}$. What we'd like to show is that adding up all of these weights gives $L(L-1)(L-2)\cdots(L-N+1)$. Recall that

$$L(L-1)(L-2)\cdots(L-N+1) = \sum_{k=1}^{N} s(N,k)L^k,$$

where $s(N,k)$ is the Stirling number of the first kind with $|s(N,k)|$ giving the number of permutations in S_N with exactly k cycles. This implies that the forests which contribute to this sum are only those with marked elements. It remains to show that the unmarked forests cancel out.

To this end, we construct a simple sign-reversing involution. Take the smallest unmarked element s. If s is part of a cycle of length 1, then contract the edge adjacent to this vertex closer to the root, and slot the label s in the cycle, just after the label of the contracted edge. If s is part of a cycle of length greater than 1, then extend a new edge, labelled with the element that was just before s in the cycle. At the other end of this new edge, label the vertex with the 1-cycle s, and carry the tree of s-labelled edges (and its ancestors) with it. This is clearly an involution and the sign changes because the number of vertices has changed by 1. Thus, when we sum the terms $(-1)^N(-L)^{c(\pi)}$ for all of the forests, the unmarked ones cancel out to zero and those that remain give us exactly

$$\sum_{k=1}^{N} s(N,k)L^k$$

by construction and hence the number of admissible solutions to the Bethe ansatz equations is

$$L(L-1)(L-2)\cdots(L-N+1) = N!\binom{L}{N}$$

as desired for completeness.

4.5. Non-generic values of the hopping rate

We noted that the Bethe ansatz doesn't work for all $p_o \in \mathbb{CP}^1$. Indeed for fixed N and L, by the Riemann-Hurwitz formula [**Har77**], we have that there are at most a finite number of points in $\mathbb{CP}^1$ where the ansatz doesn't work, and we call these points *ramification points*. The problem is not with our method of counting but rather with the fact that, for the ramification points, certain solutions will have higher multiplicity. So despite obtaining the correct count, not all the solutions are distinct.

Let's make this more explicit and take a p_o that is not a ramification point. Then, for this p_o, we obtain $M := \binom{L}{N}$ distinct vectors $\vec{z}_1(p_o), \ldots, \vec{z}_{oM}(p)$ such that $\{u_{\vec{z}_i(p_o)}(\vec{x}) \mid i = 1, \ldots, M\}$ forms a complete basis of the ASEP system with eigenvalues $\{E_{\vec{z}_i(p_o)}(\vec{x}) \mid i = 1, \ldots, M\}$, respectively. Note that we make the dependence of the $\vec{z}_i$ on p_o explicit, as these vectors change as we change p_o. In fact, when we take p_o to be a ramification point p_r, two or more of these vectors are the same, and the basis can no longer be complete. That is, for some $i \neq j$ we have $\vec{z}_i(p_r) = \vec{z}_j(p_r)$ and $\vec{z}_i(p) \neq \vec{z}_j(p)$ for all $p \neq p_r$ in a neighborhood of p_r. Thus, the ansatz is not complete at p_r, but all is not lost. We can use the fact that the ansatz is complete at every point in a neighborhood of p_r to perform a limiting process that will complete the ansatz at p_r. We will show our reasoning on a toy model then apply it to our particular case.

Take a 2×2 matrix H_t on $\mathbb{C}^2$, which depends on a parameter t, with eigenvectors $\vec{v}_t$ and $\vec{w}_t$ and eigenvalues λ_t and μ_t, respectively. That is,

$$H_t \vec{v}_t = \lambda_t \vec{v}_t$$

$$H_t \vec{w}_t = \mu_t \vec{w}_t$$

If $\vec{v}_t \neq \vec{w}_t$, then we have $\{\vec{v}_t, \vec{w}_t\}$ is a basis of $\mathbb{C}^2$ and it diagonalizes the matrix H_t. Now, assume that $\vec{v}_t \neq \vec{w}_t$ for all $t \neq t_r$ in a neighborhood of t_r, and $\vec{v}_{t_r} = \vec{w}_{t_r}$. We wish to find a basis of $\mathbb{C}^2$ for t_r since $\{\vec{v}_{t_r}, \vec{w}_{t_r}\}$ is no longer a basis. For this, we note that $\{(\vec{v}_t + \vec{w}_t), (\vec{v}_t - \vec{w}_t)\}$ is also a basis but it no longer diagonalizes the matrix H_t. We have

$$H_t \left[\frac{1}{2} (\vec{v}_t + \vec{w}_t) \right] = \frac{1}{2} (\lambda_t \vec{v}_t + \mu_t \vec{w}_t)$$
$$H_t \left[\vec{v}_t - \vec{w}_t \right] = \lambda_t \vec{v}_t - \mu_t \vec{w}_t$$

In the limit $t \to t_r$, we get that the top equation becomes an eigenvalue equation, but the second equation becomes $0 = 0$. Still, we can manipulate the second equation to obtain something interesting, such as

$$\begin{aligned} H_t \left[\vec{v}_t - \vec{w}_t \right] &= \lambda_t \vec{v}_t - \mu_t \vec{w}_t \\ &= (\lambda_t - \mu_t) \vec{v}_t + \mu_t (\vec{v}_t - \vec{w}_t) \\ \Rightarrow H_t \left[\frac{\vec{v}_t - \vec{w}_t}{\lambda_t - \mu_t} \right] &= \left(\vec{v}_t + \mu_t \frac{\vec{v}_t - \vec{w}_t}{\lambda_t - \mu_t} \right) \end{aligned}$$

Then, if we take the limit $t \to t_r$, we have

$$\left\{ \vec{v}_{t_r}, \frac{d\vec{v}_t/dt - d\vec{w}_t/dt}{d\lambda_t/dt - d\mu_t/dt} \Big|_{t=t_r} \right\}$$

is a basis in which the matrix becomes

$$H_{t_r} = \begin{bmatrix} \lambda_{t_r} & 1 \\ 0 & \lambda_{t_r} \end{bmatrix}.$$

We complete the Bethe ansatz the same way at the ramification points. Let p_r be a ramification point where

$$\begin{gathered} \vec{z}^{1}_{j_1} = \cdots = \vec{z}^{1}_{j_{k_1}} \\ \vdots \\ \vec{z}^{s}_{j_1} = \cdots = \vec{z}^{s}_{j_{k_s}} \end{gathered}$$

and $\vec{z}^{a}_{m} \neq \vec{z}^{b}_{n}$ if $a \neq b$. Then, the complete basis for p_r becomes

$$\begin{aligned} &\bigcup \left\{ \frac{du_{\vec{z}^1_{j_2}}/dp_o - du_{\vec{z}^1_{j_1}}/dp_o}{dE_{\vec{z}^1_{j_2}}/dp_o - dE_{\vec{z}^1_{j_1}}/dp_o}, \ldots, \frac{du_{\vec{z}^1_{j_{k_1}}}/dp_o - du_{\vec{z}^1_{j_1}}/dp_o}{dE_{\vec{z}^1_{j_{k_1}}}/dp_o - dE_{\vec{z}^1_{j_1}}/dp_o} \right\} \\ &\bigcup \left\{ \frac{du_{\vec{z}^2_{j_2}}/dp_o - du_{\vec{z}^2_{j_1}}/dp_o}{dE_{\vec{z}^2_{j_2}}/dp_o - dE_{\vec{z}^2_{j_1}}/dp_o}, \ldots, \frac{du_{\vec{z}^2_{j_{k_1}}}/dp_o - du_{\vec{z}^2_{j_1}}/dp_o}{dE_{\vec{z}^2_{j_{k_1}}}/dp_o - dE_{\vec{z}^2_{j_1}}/dp_o} \right\} \\ &\vdots \\ &\bigcup \left\{ \frac{du_{\vec{z}^s_{j_2}}/dp_o - du_{\vec{z}^s_{j_1}}/dp_o}{dE_{\vec{z}^s_{j_2}}/dp_o - dE_{\vec{z}^s_{j_1}}/dp_o}, \ldots, \frac{du_{\vec{z}^s_{j_{k_1}}}/dp_o - du_{\vec{z}^s_{j_1}}/dp_o}{dE_{\vec{z}^s_{j_{k_1}}}/dp_o - dE_{\vec{z}^s_{j_1}}/dp_o} \right\}. \end{aligned}$$

Actually, this new basis for p_r may not be complete as two or more of the new basis vectors may coincide. One should note that this is the problem we are trying to fix for the point p_r, but this does not mean all is lost. In fact, one should note that if we end up in the case when the new basis vectors coincide, we can keep applying the same limiting process, and this procedure terminates. Indeed, we have that the $\vec{z}_i(p_o)$'s are distinct holomorphic functions depending on p_o, and if this limiting procedure doesn't terminate, it would mean that two distinct $\vec{z}_i(p_o)$'s have the same Taylor expansion at p_r making the two $\vec{z}_i(p_o)$'s equal, which is a contradiction.

CHAPTER 5

Painlevé 1 and the Quantum Curve

5.1. Introduction

Painlevé transcendents are remarkable special functions which appear in many areas of mathematics and physics (e.g., [**AFN06**]). These are solutions of certain nonlinear ordinary differential equations known as *Painlevé equations.* These equations were discovered by Painlevé and Gambier more than 100 years ago ([**Pai02**]), and solutions have the so-called *Painlevé property*; i.e., any movable singularity must be a pole. One particular property of the Painlevé equations is existence of the *Lax pair*; that is, each Painlevé equation describes an *isomonodromic deformation* of a certain meromorphic linear ordinary differential equation ([**JM81**, **JMU81**]). The monodromy data of the linear ODEs gives a conserved quantity of the Painlevé transcendents. The *Riemann-Hilbert method*, as well as *exact WKB analysis* are applied to analyze the properties of Painlevé transcendents ([**TAT98**, **AFN06**, **Kap88**, **KT96**, **Tak99**]).

On the other hand, *quantum curves* attract both mathematicians and physicists since they are expected to encode the information of many quantum topological invariants, such as Gromov-Witten invariants, quantum knot invariants etc. These are concieved in physics literature including [**MAV12**, **MAV06**, **RDM11**, **?**]. A quantum curve is an ordinary differential (or difference) equation containing a formal parameter $\hbar$ (which plays the role of the Planck constant), like a Schrödinger equation. The quantum invariants appear in the coefficients of the *WKB (Wentzel-Kramers-Brillouin) solution* of the quantum curve.

The Eynard-Orantin's *topological recursion* introduced in [**EO07**] is closely related to both of the quantum curves and Painlevé equations (and many other topics). Topological recursion is a recursive algorithm to compute the $1/N$-expansion of the correlation functions and the partition function of matrix models from its *spectral curve*, and it is generalized to any algebraic curve which may not come from a matrix model. In this context, quantum curves were first discussed in [**BE09**] for the Airy spectral curve, and generalized to spectral curves with various backgrounds (see [**DM14c**, **DM**, **DBMN⁺13**, **?**, **MS12**] and the survey article [**Nor15**]). The spectral curves are recovered as the *semi-classical limit* $\hbar \to 0$ of the quantum curves. Moreover, the topological recursion is also closely related to integrability ([**MBE15**,**BE10**,**IM**]) as is the relationship between matrix models and integrable systems ([**PDFZJ95**, **Kon92**]).

The aim of this paper is to relate quantum curves and the *first Painlevé equation* with a formal parameter $\hbar$

$$(P_{\text{I}}) \ : \ \hbar^2 \frac{d^2 q}{dt^2} = 6q^2 + t. \tag{5.1.1}$$

The (semi-classical) spectral curve for the isomonodormy system associated with (P_{I}) is given by

$$y^2 = 4(x - q_0)^2 (x + 2q_0) \tag{5.1.2}$$

where $q_0 = q_0(t)$ is an explicit function of t. This is a family of algebraic curves in (x, y)-space parametrized by t. (The curve (5.1.2) appeared in [**EO07**, §10.6] as the spectral curve of (3,2)-minimal model.) Our main result claims that, starting from the spectral curve (5.1.2), its *quantization* through the Eynard-Orantin's topological recursion (in the sense of [**DM14c**, **DM**]) recovers the *whole* isomonodoromy system for (P_{I}).

The precise statement of our main theorem is as follows. Let $W_{g,n}(z_1, \ldots, z_n)$ be the *Eynard-Orantin differential* of type (g, n) defined from the spectral curve (5.1.2) (see §5.3.1). These are meromorphic multi-differential forms, and z_i's are copies of a coordinate on the spectral curve (5.1.2). $W_{g,n}$'s also depend on t since the spectral curve depends on t. Then, our main result states the following.

Theorem 5.1.1 (Theorem 5.3.1). *The following WKB-type formal series $\psi(x, t, \hbar)$ defined by*

$$\psi(x(z), t, \hbar) := \exp\left(\sum_{g \geq 0, n \geq 1} \hbar^{2g-2+n} \frac{1}{n!} \frac{1}{2^n} \int_{\bar{z}}^{z} \cdots \int_{\bar{z}}^{z} W_{g,n}(z_1, \cdots, z_n) \right) \tag{5.1.3}$$

satisfies the isomonodromy system associated with (P_{I}). Here $x(z)$ is an explicit rational function of z which appears in the parametrization of the spectral curve (5.1.2), and $\bar{z} = -z$.

The above theorem tells us that the isomonodoromy system associated with $(P_{\rm I})$ is a quantum curve, and its particular WKB solution is constructed by the topological recursion as (5.1.3). The main differences between our theorem and previous results on quantum curves are the following:

- Our quantum curve is a restriction of a certain *partial differential equation* (a holonomic system).
- There are *infinitely many $\hbar$-correction terms* in the quantum curve, and these correction terms are essentially given by the *asymptotic expansion* of the solution of $(P_{\rm I})$ for $\hbar \to 0$.

This chapter is organized as follows. In sec. 2, we briefly review some known facts about $(P_{\rm I})$ together with an important result on the WKB analysis of isomonodromic systems developed by Kawai-Takei [**KT96**]. Our main theorem will be formulated in sec. 3 after recalling the notion of topological recursion. We will give a proof of the main results in sec. 4.

Remark 5.1.1. *After writing the draft version of this chapter, K. Iwaki was informed that B. Eynard also has the same result which has not been published yet, but presented in [**Eyn**]. See also [**Eyn16**, Chapter 5].*

5.2. The first Painlevé equation and isomonodromy system

Let us consider the *first Painlevé equation* with a formal parameter $\hbar$:

(5.2.1)
$$(P_{\rm I}) \ : \ \hbar^2 \frac{d^2 q}{dt^2} = 6q^2 + t.$$

The equation $(P_{\rm I})$ is obtained from

$$\frac{d^2 \tilde{q}}{d\tilde{t}^2} = 6\tilde{q}^2 + \tilde{t}$$

via the rescaling $\tilde{t} = \hbar^{-4/5} t$, $\tilde{q} = \hbar^{-2/5} q$. We will regard $\hbar$ as a small parameter (i.e. Planck's constant), and investigate a particular formal solution of $(P_{\rm I})$ which has an $\hbar$-expansion.

5.2.1. Formal solution of $(P_{\rm I})$.

$(P_{\rm I})$ has the following formal power series solution:

(5.2.2)
$$q(t,\hbar) = \sum_{n=0}^{\infty} \hbar^{2n} q_{2n}(t) = q_0(t) + \hbar^2 q_2(t) + \hbar^4 q_4(t) + \cdots$$

It contains only even order terms of $\hbar$ since (P_{I}) is invariant under $\hbar \mapsto -\hbar$. The leading term $q_0 = q_0(t)$ satisfies

$$6q_0^2 + t = 0 \quad (\text{hence } q_0(t) = \sqrt{-t/6}), \tag{5.2.3}$$

and the subleading terms are recursively determined by

$$q_{2(k+1)}(t) = \frac{1}{12q_0(t)} \left(\frac{d^2 q_{2k}}{dt^2}(t) - 6 \sum_{k_1+k_2=k+1, k_i>0} q_{2k_1}(t) q_{2k_2}(t) \right) \ (k \geq 1). \tag{5.2.4}$$

As we will see, the coefficients of the formal series appearing in this paper are multivalued functions of t and are defined on the Riemann surface of q_0. Thus, in what follows, we may use q_0 instead of t when we express coefficients.

The relation (5.2.4) implies

$$q_{2k} = c_{2k} q_0^{1-5k} \ (c_{2k} \in \mathbb{C}).$$

It is obvious that the coefficients $q_{2k}(t)$ have a singularity at $q_0 = 0$ (i.e., $t = 0$). This special point is called a *turning point* of (P_{I}) ([**KT96**, Definition 2.1]; see also [**KT05**, §4]). Throughout the paper, we assume the following:

ASSUMPTION 5.2.1. *The independent variable t of (P_{I}) lies on a domain that doesn't contain the origin.*

REMARK 5.2.1. *The formal solution* (5.2.2) *is called a* 0-parameter solution *of (P_{I}) since it doesn't contain free parameters. More general formal solutions having one or two free parameters (called 1- or 2-parameter solutions) are constructed in [**TAT98**] for all Painlevé equations of second order. See also [**TAU13**] for a construction of general formal solutions of* higher *order Painlevé equations.*

REMARK 5.2.2. *The formal solution* (5.2.2) *is in fact a* divergent *series. However, [**KK12**, Theorem 1.1] proved that the formal solution is* Borel summable *when q_0 satisfies $q_0 \neq 0$ and* $\arg q_0 \notin \{\frac{2\ell}{5}\pi \mid \ell \in \mathbb{Z}\}$. *The exceptional set is called the* Stokes curve *of (P_{I}). (See [**KT96**, Definition 2.1] for the notion of Stokes curves of Painlevé equations with a small parameter $\hbar$.) That is, there exists a function which is analytic in $\hbar$ on a sectorial domain with the center at the origin (which* *is also analytic in* t*) such that* (5.2.2) *is the asymptotic expansion of the function for* $\hbar \to 0$ *in the sector. The analytic function is called the* Borel sum *of the formal series* (5.2.2)*, and it gives an analytic solution of* (P_{I}) *(see [**Cos08**] for Borel summation method). This particular asymptotic solution obtained by the Borel summation method is called the* tri-tronquée solution *of* (P_{I}) *(see [**JK01**]), and the* non-linear Stokes phenomena *on Stokes curves are analyzed by [**AFN06**, **Kap88**, **Tak99**].*

5.2.2. Isomonodromy system and the τ-function. It is known that (P_{I}) describes the compatibility condition for the following system of linear PDEs (cf., [**JM81**, Appendix C]):

$$\hbar \frac{\partial \Psi}{\partial x} = A\Psi, \quad \hbar \frac{\partial \Psi}{\partial t} = B\Psi, \tag{5.2.5}$$

where

$$\begin{aligned} A &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} := \begin{pmatrix} p & 4(x-q) \\ x^2 + qx + q^2 + \dfrac{t}{2} & -p \end{pmatrix} \\ B &= \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} := \begin{pmatrix} 0 & 2 \\ \dfrac{x}{2} + q & 0 \end{pmatrix}. \end{aligned}$$

The compatibility condition

$$\hbar \frac{\partial A}{\partial t} - \hbar \frac{\partial B}{\partial x} + [A, B] = 0 \tag{5.2.6}$$

is equivalent to the following Hamiltonian system

$$\hbar \frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \hbar \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \tag{5.2.7}$$

where the (time-dependent) Hamiltonian is given by

$$H = H(q, p, t) := \frac{1}{2}p^2 - 2q^3 - tq. \tag{5.2.8}$$

We can easily check that (5.2.7) and (P_{I}) are equivalent. The above system of linear ODEs is called the *isomonodromy system* associated with (P_{I}) (see [**JM81**, **JMU81**]).

Let $(q,p) = (q(t,\hbar), p(t,\hbar))$ be a formal power series solution of the Hamiltonian system (5.2.7); that is, $q(t,\hbar)$ is the formal solution (5.2.2) of $(P_{\rm I})$, and

$$p(t,\hbar) = \hbar\frac{dq(t,\hbar)}{dt} = \sum_{n=0}^{\infty} \hbar^{2n+1} p_{2n+1}(t). \tag{5.2.9}$$

The corresponding Hamiltonian function is denoted by

$$\sigma(t,\hbar) := H(q(t,\hbar), p(t,\hbar), t). \tag{5.2.10}$$

We can check that (5.2.10) is invariant under $\hbar \mapsto -\hbar$, and hence it has the following expansion:

$$\sigma(t,\hbar) = \sum_{n=0}^{\infty} \hbar^{2n} \sigma_{2n}(t). \tag{5.2.11}$$

DEFINITION 5.2.1 ([**JMU81**, **Oka80**]). *The* τ-function *(corresponding to the formal solution* (5.2.2)*) of* $(P_{\rm I})$ *is defined by*

$$\hbar^2 \frac{d}{dt} \log \tau(t,\hbar) = \sigma(t,\hbar) \tag{5.2.12}$$

up to constant.

The τ-function can also be defined in terms of a solution of (5.2.5) ([**JMU81**]; see also subsection 5.5). The expansion (5.2.11) implies that the τ-function (5.2.12) has an expansion of the form

$$\log \tau(t,\hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} \tau_{2g}(t).$$

5.2.3. Spectral curve. In what follows, we assume that the formal solution $(q(t,\hbar), p(t,\hbar))$ of (5.2.7) constructed above is substituted into the coefficients of the isomonodromy system (5.2.5). Then, the coefficients of the isomonodromy system has the following $\hbar$-expansions:

$$\begin{aligned} A &= A_0(x,t) + \hbar A_1(x,t) + \hbar^2 A_2(x,t) + \cdots, \\ B &= B_0(x,t) + \hbar B_1(x,t) + \hbar^2 B_2(x,t) + \cdots, \end{aligned}$$

whose top terms are given by

$$A_0(x,t) = \begin{pmatrix} 0 & 4(x-q_0) \\ x^2+q_0x+q_0^2+\frac{t}{2} & 0 \end{pmatrix}, \tag{5.2.13}$$

$$B_0(x,t) = \begin{pmatrix} 0 & 2 \\ \frac{x}{2}+q_0 & 0 \end{pmatrix}. \tag{5.2.14}$$

Observe that, since q_0 satisfies (5.2.3), the algebraic curve defined by

$$\det\left(y - A_0(x,t)\right) = y^2 - 4\left(x-q_0\right)^2\left(x+2q_0\right) = 0 \tag{5.2.15}$$

has *genus 0*. Actually, this gives a family of algebraic curves in $\mathbb{C}^2_{(x,y)}$ parametrized by t. Since we have assumed that $t \neq 0$, $x = q_0$ and $x = -2q_0$ are distinct.

Definition 5.2.2. *We call the algebraic curve* (5.2.15) the semi-classical spectral curve*, or* the spectral curve *of (the first equation of) the isomonodromy system* (5.2.5)*.*

Remark 5.2.3. *It is shown in [**KT96**, Proposition 1.3] that, for all (second order) Painlevé equations with a similar formal parameter ħ, the semi-classical spectral curves corresponding to the same type of formal power series solution as* (5.2.2) *have* genus 0*.*

Remark 5.2.4. *Since we are taking the semi-classical limit (i.e., top term in ħ-expansion), our spectral curve* (5.2.15) *is different from usual spectral curves for isomonodromic deformation equations discussed in e.g., [**Ols99**, **Tak98**]. The spectral curves in the above papers have* higher genus*. Recently, Nakamura [**Nak15**] investigates the geometry of genus* 2 *spectral curves which appear in an* autonomous limit *of the* 4th order Painlevé equations*, and use them to classify the Painlevé equations. See [**KNS12**] for the list of 4th order Painlevé equations.*

5.2.4. WKB analysis of isomonodromy system in scalar form. Denote the unknown vector function of (5.2.5) by $\Psi = {}^t(\psi_1, \psi_2)$. Then, $\psi = \psi_1$ satisfies the following scalar version of isomonodromy system:

$$\begin{cases} \left(\left(\hbar\frac{\partial}{\partial x}\right)^2 + f\left(\hbar\frac{\partial}{\partial x}\right) + g\right)\psi = 0, \\ \hbar\frac{\partial\psi}{\partial t} = \frac{1}{2(x-q)}\left(\hbar\frac{\partial\psi}{\partial x} - p\psi\right), \end{cases} \tag{5.2.16}$$

where

$$\begin{aligned} f = f(x,t,\hbar) &:= -\mathrm{tr}A - \hbar\frac{\partial}{\partial x}\log A_{12} = -\hbar\frac{1}{x-q}, \\ g = g(x,t,\hbar) &:= \det A - \hbar\frac{\partial A_{11}}{\partial x} + \hbar A_{11}\frac{\partial}{\partial x}\log A_{12} \\ &= -(4x^3 + 2tx + p^2 - 4q^3 - 2tq) + \hbar\frac{p}{x-q}. \end{aligned}$$

The coefficients of f and g have an $\hbar$-expansion since q and p are contained in them:

$$f = -\hbar\frac{1}{x-q_0} + \hbar^3\frac{1}{1728q_0^4(x-q_0)^2} + \hbar^5\frac{49x-51q_0}{5971968q_0^9(x-q_0)^3} + \cdots \tag{5.2.17}$$

$$g = -4(x-q_0)^2(x+2q_0) - \hbar^2\frac{x+11q_0}{144q_0^2(x-q_0)} - \hbar^4\frac{7x^2+34q_0x-53q_0^2}{248832q_0^7(x-q_0)^2} + \cdots \tag{5.2.18}$$

The top term of g appears in the defining equation of the spectral curve (5.2.15), and its zeros are called *turning points* of the first equation of (5.2.16) in the WKB analysis. In particular, under the assumption $t \neq 0$, there is

- a *simple* turning point at $x = -2q_0$ which is a branch point of the spectral curve (5.2.15), and
- a *double* turning point at $x = q_0$ which is a singular point of the spectral curve (5.2.15).

Consider the *Riccati equation*

$$\hbar^2\left(P^2 + \frac{\partial P}{\partial x}\right) + f\hbar P + g = 0. \tag{5.2.19}$$

This is equivalent to the first equation in (5.2.16) by

$$\psi = \exp\left(\int^x P dx\right) \quad \left(\text{i.e., } P = \frac{1}{\psi}\frac{\partial \psi}{\partial x}\right). \tag{5.2.20}$$

Let

$$P^{(\pm)}(x,t,\hbar) = \sum_{m=0}^{\infty} \hbar^{m-1} P_m^{(\pm)}(x,t) \tag{5.2.21}$$

be the formal solutions of (5.2.19) with the top term

$$P_0^{(\pm)}(x,t) = \pm 2(x-q_0)\sqrt{x+2q_0}. \tag{5.2.22}$$

The coefficients $P_m^{(\pm)}(x,t)$ are recursively determined by

$$2P_0^{(\pm)}P_{m+1}^{(\pm)} + \sum_{\substack{a+b=m+1\\ a,b\geq 1}} P_a^{(\pm)}(P_b^{(\pm)} + f_b) + \frac{\partial P_m^{(\pm)}}{\partial x} + g_{m+1} = 0 \text{ for } m \geq 0, \tag{5.2.23}$$

where f_a and g_a are the coefficient of $\hbar^a$ in f and g, respectively. Explicit forms of the first few terms are given by

$$\begin{aligned}
P_1^{(\pm)} &= -\frac{1}{4(x+2q_0)}, \\
P_2^{(\pm)} &= \pm\frac{x+17q_0}{576q_0^2(x+2q_0)^{5/2}}, \\
P_3^{(\pm)} &= -\frac{2x^2+20q_0x+77q_0^2}{6912q_0^4(x+2q_0)^4}, \\
P_4^{(\pm)} &= \pm\frac{28x^4+500q_0x^3+3684q_0^2x^2+14273q_0^3x+27307q_0^4}{3981312q_0^7(x+2q_0)^{11/2}}.
\end{aligned}$$

It is obvious from (5.2.23) that $P_m^{(\pm)}(x,t)$ are holomorphic except at the turning points and $x=\infty$ (and multivalued for even m). It also follows from the recursion relation (5.2.23) that

$$P^{(\pm)}(x,t,\hbar) = \pm\left(\frac{2}{\hbar}x^{3/2} + \frac{t}{2\hbar}x^{-1/2} \mp \frac{1}{4}x^{-1} + \frac{\sigma(t,\hbar)}{2\hbar}x^{-3/2} + O(x^{-2})\right) \tag{5.2.24}$$

holds when $x \to \infty$.

REMARK 5.2.5. *We can check that* $P_m^{(\pm)}(x,t)$*'s have the follwoing asymptotic expansion for* $x \to \infty$*:*

$$P_0^{(\pm)}(x,t) = \pm\left(2x^{3/2} + \frac{t}{2}x^{-1/2} + O(x^{-3/2})\right), \tag{5.2.25}$$

$$P_1^{(\pm)}(x,t) = -\frac{1}{4}x^{-1} + O(x^{-3/2}), \tag{5.2.26}$$

$$P_m^{(\pm)}(x,t) = O(x^{-3/2}) \text{ for } m \geq 2, \tag{5.2.27}$$

and we have (5.2.24) *after summing up* $\hbar^{m-1}P_m^{(\pm)}(x,t)$*. Once you know that* $P^{(\pm)}(x,t,\hbar)$ *has an asymptotic expansion in this sense, subleading terms in* (5.2.24) *can be computed from the Riccati equation* (5.2.19)*.*

Define

$$P_{\rm odd}(x,t,\hbar) := \frac{1}{2}\left(P^{(+)}(x,t,\hbar) - P^{(-)}(x,t,\hbar)\right), \tag{5.2.28}$$

$$P_{\rm even}(x,t,\hbar) := \frac{1}{2}\left(P^{(+)}(x,t,\hbar) + P^{(-)}(x,t,\hbar)\right). \tag{5.2.29}$$

It is easy to check that (c.f., [**KT05**, §2])

$$P_{\rm even}(x,t,\hbar) = -\frac{1}{2}\frac{\partial}{\partial x}\log\frac{\hbar P_{\rm odd}(x,t,\hbar)}{2(x-q(t,\hbar))} \tag{5.2.30}$$

and

$$P_{\rm odd}(\sigma(x),t,\hbar) = -P_{\rm odd}(x,t,\hbar) \tag{5.2.31}$$

hold. Here x is regarded as a coordinate on the spectral curve, and σ is the covering involution for the spectral curve: $P^{(+)}(\sigma(x),t) = -P^{(-)}(x,t)$.

Since

$$\frac{\hbar P_{\rm odd}(x,t,\hbar)}{2(x-q(t,\hbar))} = \sqrt{x+2q_0}\,(1+O(\hbar))\,,$$

the right hand-side of (5.2.30) is the derivative of the formal power series

$$-\frac{1}{2}\log\frac{\hbar P_{\rm odd}(x,t,\hbar)}{2(x-q(t,\hbar))} = -\frac{1}{4}\log(x+2q_0) + O(\hbar).$$

Thus the ambiguity of the branch of the logarithm only appears in the top term, but we care about the ambiguity since it doesn't matter in our computation.

The following theorem was applied in the *transformation theory of Painlevé equations* in [**KT96**]. We will use the fact in the proof of our main theorem.

THEOREM 5.2.2 (c.f., [**KT96**, Proposition 1.2 and Theorem 1.1]).

(i) *The formal series* $P^{(\pm)}(x,t,\hbar)$ *satisfies*

$$\hbar\frac{\partial}{\partial t}P^{(\pm)}(x,t,\hbar) = \frac{\partial}{\partial x}\left(\frac{\hbar P^{(\pm)}(x,t,\hbar) - p(t,\hbar)}{2(x-q(t,\hbar))}\right). \tag{5.2.32}$$

In particular, $P_{\rm odd}(x,t,\hbar)$ *satisfies*

$$\frac{\partial}{\partial t}P_{\rm odd}(x,t,\hbar) = \frac{\partial}{\partial x}\left(\frac{P_{\rm odd}(x,t,\hbar)}{2(x-q(t,\hbar))}\right). \tag{5.2.33}$$

(ii) *All coefficients of* $P^{(\pm)}(x,t,\hbar)$ *are holomorphic except at the simple turning point* $x=-2q_0$ *and* $x=\infty$*. In particular, they are holomorphic at the double turning point* $x=q_0$*.*

(iii) *The formal series*

$$\begin{aligned}\psi_\pm(x,t,\hbar) &:= \exp\left(\pm\int_v^x P_{\rm odd}(x',t,\hbar)dx' - \frac{1}{2}\log\frac{\hbar P_{\rm odd}(x,t,\hbar)}{2(x-q(t,\hbar))}\right)\\ &= \left(\frac{2(x-q(t,\hbar))}{\hbar P_{\rm odd}(x,t,\hbar)}\right)^{1/2}\exp\left(\pm\int_v^x P_{\rm odd}(x',t,\hbar)dx'\right)\end{aligned} \tag{5.2.34}$$

satisfies the isomonodoromy system (5.2.16)*. Here* v *is the simple turning point* $-2q_0$*. The integral from* v *is defined by*

$$\int_v^x P_{\rm odd}(x',t,\hbar)dx' = \frac{1}{2}\int_{\gamma_x} P_{\rm odd}(x',t,\hbar)dx', \tag{5.2.35}$$

where the path γ_x *is depicted in Figure 5.1 (c.f., [**KT05**, §2]).*

PROOF. Although the scalar version of isomonodromy system (5.2.16) is different from that used in [**KT96**], they are related by a gauge transformation $\psi \mapsto (x-q)^{1/2}\psi$. Therefore, the equalities (5.2.32) and (5.2.33) in (i) together with the holomorphicity of each coefficient of $P_{\rm odd}(x,t,\hbar)$ at $x=q_0$ follows from [**KT96**, Proposition 1.2 and Theorem 1.1]. Then, it turns out that the

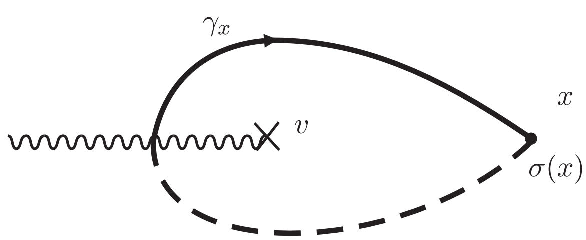


FIGURE 5.1. For a given x, the path γ_x starts from the point $\sigma(x)$ and ends at x. The wiggly lines designate a branch cut, and the solid (resp., dotted) part represents a part of path on the first (resp., the second) sheet of the spectral curve.

coefficients of

$$\frac{P_{\rm odd}(x,t,\hbar)}{x-q(t,\hbar)}$$

are also holomorphic due to (5.2.33). Then, (5.2.30) implies that each coefficient of $P_{\rm even}(x,t,\hbar)$ is also holomorphic at $x = q_0$. Thus we have proved (ii).

The claim (iii) follows from a straightforward computations:

$$\begin{aligned}
\frac{1}{\psi_\pm}\frac{\partial\psi_\pm}{\partial x} &= \pm P_{\rm odd} - \frac{1}{2}\frac{\partial}{\partial x}\log\left(\frac{\hbar P_{\rm odd}}{2(x-q)}\right) = \pm P_{\rm odd} + P_{\rm even} = P^{(\pm)}. \\
\hbar\frac{1}{\psi_\pm}\frac{\partial\psi_\pm}{\partial t} &= \frac{1}{2}\left(\frac{-\hbar(dq/dt)}{x-q} - \frac{\hbar}{P_{\rm odd}}\frac{\partial P_{\rm odd}}{\partial t}\right) \pm \int_v^x \hbar\frac{\partial P_{\rm odd}}{\partial t}dx \\
&= -\frac{p}{2(x-q)} - \frac{\hbar}{P_{\rm odd}}\left(\frac{1}{2(x-q)}\frac{\partial P_{\rm odd}}{\partial x} - \frac{P_{\rm odd}}{2(x-q)^2}\right) \pm \hbar\frac{P_{\rm odd}}{2(x-q)} \\
&= -\frac{p}{2(x-q)} + \frac{\hbar P^{(\pm)}}{2(x-q)} = \frac{1}{2(x-q)}\left(\frac{\hbar}{\psi_\pm}\frac{\partial\psi_\pm}{\partial x} - p\right).
\end{aligned}$$

□

As we will see below, an *isomonodromic WKB solution* such as (5.2.34) is constructed from just a family of algebraic curves (5.2.15) by *the topological recursion* ([**EO07**]). In particular, the first equation in (5.2.16) gives a *quantization* of the spectral curve (5.2.15) in the sense of [**DM14c**,**DM**].

REMARK 5.2.6. *In the above computation the normalization* (5.2.34) *is essential. Since* $P_{\rm odd}$ *is anti-invariant under the covering involution* σ *as* (5.2.31) *and the integral in* (5.2.34) *is defined as* *a contour integral* (5.2.35)*, we don't need to take care of the branch point v in the computation:*

$$\int_v^x \frac{\partial P_{\text{odd}}}{\partial t} dx = \frac{1}{2} \int_{\sigma(x)}^x \frac{\partial}{\partial x}\left(\frac{P_{\text{odd}}}{2(x-q)}\right) dx = \frac{P_{\text{odd}}}{2(x-q)}.$$

Remark 5.2.7. *We can also construct a WKB-type formal solution of matrix isomonodromy system* (5.2.5)*. Define*

$$\begin{aligned} \tilde{\psi}_\pm(x,t,\hbar) &= \frac{\hbar \frac{d\psi_\pm}{dx}(x,t,\hbar) - A_{11}(x,t,\hbar)\psi_\pm(x,t,\hbar)}{A_{12}(x,t,\hbar)} \\ &= \frac{\hbar P^{(\pm)}(x,t,\hbar) - A_{11}(x,t,\hbar)}{A_{12}(x,t,\hbar)} \psi_\pm(x,t,\hbar). \end{aligned} \tag{5.2.36}$$

Then, the matrix valued formal series

$$\Psi(x,t,\hbar) = \begin{pmatrix} \psi_+(x,t,\hbar) & \psi_-(x,t,\hbar) \\ \tilde{\psi}_+(x,t,\hbar) & \tilde{\psi}_-(x,t,\hbar) \end{pmatrix} \tag{5.2.37}$$

gives a fundamental formal solution of the isomonodoromy system (5.2.5).

5.3. Topological recursion and quantum curve theorem

In this section we review the Eynard-Oranatin's topological recursion ([**EO07**]) for our spectral curve (5.2.15), and formulate our main theorem.

5.3.1. Topological recursion. The topological recursion is an algorithm associating some differential forms $W_{g,n}$ and numbers F_g given the following source data:

- A plane curve $(\mathcal{C}, x, y)$: $\mathcal{C}$ is a compact Riemann surface, $x, y : \mathcal{C} \to \mathbb{P}^1$ are meromorphic functions.
- The Bergman kernel B: It is a symmetric differential form on $\mathcal{C} \times \mathcal{C}$ with poles of order 2 along the diagonal, and satisfying some normalization conditions.

In our case, $\mathcal{C} = \mathbb{P}^1$ and x, y are rational functions which parametrize the spectral curve (5.2.15):

$$\begin{cases} x(z) = z^2 - 2q_0 \\ y(z) = 2z(z^2 - 3q_0). \end{cases} \tag{5.3.1}$$

Here z is a coordinate on $\mathbb{P}^1$. The Bergman kernel is given by

$$B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} \tag{5.3.2}$$

since the spectral curve is of genus 0. Zeros of dx are called *ramification points* of the spectral curve (5.3.1). Our spectral curve has only one ramification point at $z = 0$.

The topological recursion for our spectral curve (5.3.1) is formulated as follows (see [**EO07**] for general case):

DEFINITION 5.3.1 ([**EO07**, Definition 4.2] (see also [**DM14c**, §3])). *The* Eynard-Orantin differential $W_{g,n}(z_1, \ldots, z_n)$ *of type* (g, n) *is a meromorphic* n*-differential on the* n*-times product of the spectral curve* (5.3.1) *defined by the following* topological recursion relation*:*

- *For* $2g - 2 + n \le 0$*:*

$$W_{0,1}(z_1) := y(z_1)dx(z_1) \left(= 4z_1^2(z_1^2 - 3q_0)dz_1\right), \tag{5.3.3}$$

$$W_{0,2}(z_1, z_2) := B(z_1, z_2) \left(= \frac{dz_1 dz_2}{(z_1 - z_2)^2}\right), \tag{5.3.4}$$

- *For* $2g - 2 + n = 1$*:*

$$W_{0,3}(z_1, z_2, z_3) := \frac{1}{2\pi i} \oint_{\gamma_0} K(z, z_1)\Big[W_{0,2}(z, z_2)W_{0,2}(\bar{z}, z_3) + W_{0,2}(z, z_3)W_{0,2}(\bar{z}, z_2)\Big], \tag{5.3.5}$$

$$W_{1,1}(z_1) := \frac{1}{2\pi i} \oint_{\gamma_0} K(z, z_1)W_{0,2}(z, \bar{z}), \tag{5.3.6}$$

- *For* $2g - 2 + n \ge 2$*:*

$$\begin{aligned} W_{g,n}(z_1, \ldots, z_n) &:= \frac{1}{2\pi i} \oint_{\gamma_0} K(z, z_1) \\ &\times\Bigg[\sum_{j=2}^{n} \Big(W_{0,2}(z, z_j)W_{g,n-1}(\bar{z}, z_{[\hat{1},\hat{j}]}) + W_{0,2}(\bar{z}, z_j)W_{g,n-1}(z, z_{[\hat{1},\hat{j}]})\Big) \\ &+W_{g-1,n+1}(z, \bar{z}, z_{[\hat{1}]}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J=[\hat{1}]}}^{stable} W_{g_1,|I|+1}(z, z_I)W_{g_2,|J|+1}(\bar{z}, z_J)\Bigg]. \end{aligned} \tag{5.3.7}$$

Here γ_0 *is a small cycle (in* z*-plane) which encircles the ramification point* $z = 0$ *in the counter-clockwise direction,* $\bar{z} = -z$ *is the conjugate of* z *near the ramification point, and the* recursion kernel $K(z, z_1)$ *is given by*

$$K(z, z_1) = -\frac{\omega^{\bar{z}-z}(z_1)}{2(y(z) - y(\bar{z}))dx(z)}, \quad \omega^{\bar{z}-z}(z_1) = \int_z^{\bar{z}} W_{0,2}(\cdot, z_1). \tag{5.3.8}$$

Also, we use the index convention $[\hat{j}] = \{1, \dots, n\} \smallsetminus \{j\}$ *and so on. Lastly, the sum in the third line of* (5.3.7) *is taken for indices in the stable range (i.e. only* $W_{g,n}$*'s with* $2g - 2 + n \ge 1$ *appear).*

The explicit form of some of Eynard-Orantin differentials are given as follows:

$$\begin{aligned}
W_{0,3} &= \frac{1}{12q_0 z_1^2 z_2^2 z_3^2} dz_1 dz_2 dz_3, \\
W_{0,4} &= \frac{z_1^2 z_2^2 z_3^2 z_4^2 + 3q_0(z_1^2 z_2^2 z_3^2 + z_2^2 z_3^2 z_4^2 + z_3^2 z_4^2 z_1^2 + z_4^2 z_1^2 z_2^2)}{144 q_0^3 z_1^4 z_2^4 z_3^4 z_4^4} dz_1 dz_2 dz_3 dz_4, \\
W_{1,1} &= \frac{z_1^2 + 3q_0}{288 q_0^2 z_1^4} dz_1, \\
W_{1,2} &= \frac{2z_1^4 z_2^4 + 6q_0(z_1^4 z_2^2 + z_1^2 z_2^4) + 3q_0^2(5z_1^4 + 3z_1^2 z_2^2 + 5z_2^4)}{3456 q_0^4 z_1^6 z_2^6} dz_1 dz_2, \\
W_{2,1} &= \frac{28 z_1^8 + 84 q_0 z_1^6 + 252 q_0^2 z_1^4 + 609 q_0^3 z_1^2 + 945 q_0^4}{1990656 q_0^7 z_1^{10}} dz_1.
\end{aligned}$$

Eynard-Orantin differentials have the following properties (see [**EO07**]):

- As a differential form on each variable z_i, $W_{g,n}$, for $2g - 2 + n \ge 1$, is *holomorphic* except for the ramification point 0 and may have a pole at 0.
- $W_{g,n}$ is *symmetric*; that is, they are invariant under any permutation of variables.
- For $2g - 2 + n \ge 1$, $W_{g,n}$ is *anti-invariant* under the involution $z_i \mapsto \bar{z}_i$ for each variable:

$$W_{g,n}(z_1, \dots, \bar{z}_j, \dots, z_n) = -W_{g,n}(z_1, \dots, z_j, \dots, z_n) \text{ for } j = 1, \dots, n. \tag{5.3.9}$$

- $W_{g,n}$ is also *holomorphic* in t except for $t = 0$ (i.e., $q_0 = 0$). There is a formula for the derivative of $W_{g,n}$ with respect to t; see §5.3.5.

5.3.2. Quantum curve theorem. In this section we describe our main result which claims that the scalar isomonodromy system (5.2.16) gives a *quantum curve*.

Definition 5.3.2. *For $g \geq 0, n \geq 1$ satisfying $2g-2+n \geq 1$, define* open free energy *of type (g,n) by*

$$F_{g,n}(z_1,\dots,z_n) := \frac{1}{2^n}\int_{\bar{z}_1}^{z_1}\cdots\int_{\bar{z}_n}^{z_n} W_{g,n}(z_1,\dots,z_n). \tag{5.3.10}$$

It follows from the definition that open free energies satisfy

$$d_{z_1}\dots d_{z_n}F_{g,n}(z_1,\dots,z_n) = W_{g,n}(z_1,\dots,z_n), \tag{5.3.11}$$

$$F_{g,n}(z_1,\dots,\bar{z}_j,\dots,z_n) = -F_{g,n}(z_1,\dots,z_j,\dots,z_n) \text{ for } j=1,\dots,n. \tag{5.3.12}$$

Explicit computation shows that

$$\begin{aligned}
F_{0,3}(z_1,z_2,z_3) &= -\frac{1}{12q_0z_1z_2z_3},\\
F_{0,4}(z_1,z_2,z_3,z_4) &= \frac{z_1^2z_2^2z_3^2z_4^2 + q_0\left(z_1^2z_2^2z_3^2 + z_2^2z_3^2z_4^2 + z_3^2z_4^2z_1^2 + z_4^2z_1^2z_2^2\right)}{144q_0^3z_1^3z_2^3z_3^3z_4^3},\\
F_{1,1}(z_1) &= -\frac{z_1^2+q_0}{288q_0^2z_1^3},\\
F_{1,2}(z_1,z_2) &= \frac{2z_1^4z_2^4 + 2q_0\left(z_1^4z_2^2+z_1^2z_2^4\right) + q_0^2\left(3z_1^4+z_1^2z_2^2+3z_2^4\right)}{3456q_0^4z_1^5z_2^5},\\
F_{2,1}(z_1) &= -\frac{140z_1^8+140q_0z_1^6+252q_0^2z_1^4+435q_0^3z_1^2+525q_0^4}{9953280q_0^7z_1^9}.
\end{aligned}$$

We also introduce functions $\{S_m(x,t)\}_{m\geq 0}$ by

$$S_0(x,t) := \int_v^x y(z(x'))dx', \quad S_1(x,t) := -\frac{1}{2}\log\left(\frac{y(z(x))}{2(x-q_0)}\right), \tag{5.3.13}$$

and for $m \geq 2$,

$$S_m(x,t) := \sum_{\substack{2g-2+n=m-1\\ g\geq 0, n\geq 1}} \left.\frac{F_{g,n}(z,\dots,z)}{n!}\right|_{z=z(x)} \tag{5.3.14}$$

where $z(x) = \sqrt{x + 2q_0}$ is the inverse function of $x(z)$. After computations we have

$$\begin{aligned}
S_0(x,t) &= \frac{4}{5}(x - 3q_0)(x + 2q_0)^{3/2}, \\
S_1(x,t) &= -\frac{1}{4}\log(x + 2q_0), \\
S_2(x,t) &= -\frac{x + 7q_0}{288q_0^2(x + 2q_0)^{3/2}}, \\
S_3(x,t) &= \frac{2x^2 + 14q_0x + 35q_0^2}{6912q_0^4(x + 2q_0)^3}, \\
S_4(x,t) &= -\frac{140x^4 + 1580q_0x^3 + 7476q_0^2x^2 + 18739q_0^3x + 23499q_0^4}{9953280q_0^7(x + 2q_0)^{9/2}}.
\end{aligned}$$

THEOREM 5.3.1. *The formal series* $\psi(x,t,\hbar)$ *given by*

$$\psi(x,t,\hbar) := \exp(S(x,t,\hbar)), \tag{5.3.15}$$

$$S(x,t,\hbar) := \sum_{m=0}^{\infty} \hbar^{m-1} S_m(x,t) \tag{5.3.16}$$

satisfies both of the differential equations in scalar-version of the isomonodromy system (5.2.16)*. That is,* (5.3.16) *satisfies the following differential equations which are equivalent to* (5.2.16)*:*

$$\hbar^2\left(\left(\frac{\partial S}{\partial x}\right)^2 + \frac{\partial^2 S}{\partial x^2}\right) = \frac{\hbar}{x - q}\left(\hbar\frac{\partial S}{\partial x} - p\right) + (4x^3 + 2tx + p^2 - 4q^3 - 2tq), \tag{5.3.17}$$

$$\hbar\frac{\partial S}{\partial t} = \frac{1}{2(x - q)}\left(\hbar\frac{\partial S}{\partial x} - p\right). \tag{5.3.18}$$

Thus, the *principal specialization* (i.e., setting $z_i = z$ for all $i = 1,\dots,n$) of the open free energies gives an isomonodromic WKB solution. Theorem 5.3.1 implies

$$\frac{\partial S}{\partial x}(x,t,\hbar) = P^{(+)}(x,t,\hbar) \tag{5.3.19}$$

holds (under a suitable choice of the branch of $\sqrt{x + 2q_0}$). The computational results in §5.2.4 show that (5.3.19) holds up to $\hbar^4$. A full proof of Theorem 5.3.1 will be given in §5.4 together with that of Theorem 5.3.2 below.

REMARK 5.3.1. *In the topological recursion* (5.3.7)*, we take residues only at the ramification point* $z = 0$*. Thus* $W_{g,n}$*'s defined here are different from those in [**DM**]; in particular, our quantum curve* (5.2.16) *has* infinitely many $\hbar$-corrections *as in* (5.2.17) *and* (5.2.18) *(but recovers the same spectral curve in the semi-classical limit).*

REMARK 5.3.2. *In Theorem 5.3.1, the choice of the lower end points of the integral in* (5.3.10) *is important. Different choice also give a WKB solution of the first equation in* (5.2.16)*, but it may not satisfy the second equation in general.*

5.3.3. Closed Free energies and the τ-function. The other main result of this paper is giving another proof of the known fact about the relationship between the closed free energies and the τ-function of $(P_{\rm I})$ (c.f., [**PDFZJ95**], [**EO07**]).

DEFINITION 5.3.3 ([**EO07**, Definition 4.3]). *Define the* closed free energy $F_g = F_g(t)$ *for* $g \geq 2$ *by*

$$F_g(t) = \frac{1}{2\pi i(2-2g)} \oint_{\gamma_0} \Phi(z) W_{g,1}(z), \tag{5.3.20}$$

where

$$\Phi(z) = \int_{z_o}^{z} y(z)dx(z) \quad \left(= \frac{4}{5}z^5 - 4q_0 z^3 + constant \right) \tag{5.3.21}$$

and z_o *is a generic point. Free energies* F_0 *and* F_1 *for* $g = 0, 1$ *are also defined but in a different manner (see [**EO07**, §4.2.2 and §4.2.3] for the definition).*

Note that F_g defined here is different from $F_{g,n}$ defined in the previous subsection. F_g's are also called *symplectic invariants* since they are invariant under symplectic transformations of the spectral curve (see [**EO07**]). Explicit computation shows that

$$\begin{aligned} F_0(t) &= -\frac{48q_0^5}{5}, \\ F_1(t) &= -\frac{1}{24}\log(-3q_0), \\ F_2(t) &= \frac{7}{207360q_0^5}, \\ F_3(t) &= \frac{245}{429981696q_0^{10}}. \end{aligned}$$

Theorem 5.3.2 ([**PDFZJ95**], [**EO07**, §10.6]). *The generating function of the free energy $F_g(t)$ gives a τ-function of (P_{I}):*

$$\log\tau(t,\hbar) = \sum_{g=0}^{\infty}\hbar^{2g-2}F_g(t). \tag{5.3.22}$$

Namely,

$$\frac{dF_g(t)}{dt} = \sigma_{2g}(t). \tag{5.3.23}$$

The proof will be given in §5.4.

It is worth mentioning that the closed free energies specify one particular τ-function although there is an ambiguity in Definition 5.2.1.

Proposition 5.3.1. *For $g \geq 2$, we have*

$$F_g(t) = \int_{\infty}^{t}\sigma_{2g}(t')dt'. \tag{5.3.24}$$

Proof. Let us describe the behavior of the $W_{g,n}$'s when $q_0 \to \infty$ (i.e., $t \to \infty$). When q_0 tends to ∞, no singular point of the integrand in the right hand-side of (5.3.7) on the z-plane hits the integration cycle γ_0. Thus, we can show that

$$W_{g,n}(z_1,\ldots,z_n) = O(q_0^{-(2g-2+n)}) \tag{5.3.25}$$

for $2g-2+n \geq 0$. This implies that

$$F_g(t) = O(q_0^{-(2g-2)}) \tag{5.3.26}$$

holds since $\Phi(z) \sim q_0$ as $q_0 \to \infty$ (but we can verify that F_g for $g \geq 2$ has a stronger decay in the above explicit computations). This completes the proof of (5.3.24). □

5.3.4. Asymptotics of Eynard-Orantin differnetials. The rest of this section will be devoted to show some important properties of $W_{g,n}$ and $F_{g,n}$. Firstly, we will describe the asymptotic behavior of them near $z_i = \infty$.

LEMMA 5.3.1. (i) *For $2g - 2 + n \geq 0$, we have*

$$W_{g,n}(z_1, \ldots, z_n) = \left(\frac{c_{g,n}}{z_1^2 \cdots z_n^2} + O(z_1^{-4} \cdots z_n^{-4}) \right) dz_1 \cdots dz_n \tag{5.3.27}$$

as $z_i \to \infty$ for all $i = 1, \ldots, n$. Here $c_{g,n} \in \mathbb{C}$ is a constant.

(ii) *For $2g - 2 + n \geq 0$, we have*

$$F_{g,n}(z_1, \ldots, z_n) = \frac{c'_{g,n}}{z_1 \cdots z_n} + O(z_1^{-3} \cdots z_n^{-3}) \quad (c'_{g,n} \in \mathbb{C}) \tag{5.3.28}$$

as $z_i \to \infty$ for all $i = 1, \ldots, n$.

PROOF. The first property (5.3.27) follows from the analyticity of $W_{g,n}$ at $z_i = \infty$. The second property (5.3.28) follows from (5.3.27) immediately because $F_{g,n}(z_1, \ldots, z_n)$ doesn't have a constant term due to the definition (5.3.10). □

As a corollary, the principal specialization of open free energies satisfies

$$F_{g,n}(z, \ldots, z) = O(z^{-n}) \tag{5.3.29}$$

when $z \to \infty$.

5.3.5. Variation of spectral curve. There is a formula (for "variation of spectral curves") that allows us to compute derivatives of $W_{g,n}$ etc. with respect to the parameter t.

THEOREM 5.3.3 (c.f., [**EO07**, Theorem 5.1]).

(i) *For $2g - 2 + n \geq 0$, we have*

$$\frac{\partial}{\partial t} W_{g,n}(z(x_1), \ldots, z(x_n)) = -2 \operatorname*{Res}_{x_{n+1}=\infty} z(x_{n+1}) W_{g,n+1}\big(z(x_1), \ldots, z(x_n), z(x_{n+1})\big). \tag{5.3.30}$$

(ii) *For $g \geq 1$, we have*

$$\frac{dF_g}{dt}(t) = -2 \operatorname*{Res}_{x=\infty} z(x) W_{g,1}(z(x)) = -\operatorname*{Res}_{z=\infty} z W_{g,1}(z). \tag{5.3.31}$$

(iii) *For $2g-2+n \geq 1$, we have*

$$\frac{\partial}{\partial t} F_{g,n}(z(x_1), \dots, z(x_n)) = -2 \operatorname*{Res}_{x_{n+1}=\infty} z(x_{n+1}) d_{x_{n+1}} F_{g,n+1}(z(x_1), \dots, z(x_n), z(x_{n+1})), \tag{5.3.32}$$

or equivalently,

$$\begin{aligned} &\frac{\partial}{\partial t} F_{g,n}(z(x_1), \dots, z(x_n)) \\ &\qquad = \lim_{z_{n+1} \to \infty} \left(z_{n+1}^2 \frac{\partial}{\partial z_{n+1}} F_{g,n+1}(z_1, \dots, z_n, z_{n+1}) \right) \Big|_{(z_1, \dots, z_n) = (z(x_1), \dots, z(x_n))}. \end{aligned} \tag{5.3.33}$$

Proof. Set $\Lambda(z) := z$. Then, we can check $\Lambda(z)$ satisfies the required condition

$$\operatorname*{Res}_{z=\infty} (\Lambda(z) W_{0,2}(z, z_1)) = -dz_1 = -\left(\frac{\partial y}{\partial t}(z_1) dx(z_1) - \frac{\partial x}{\partial t}(z_1) dy(z_1) \right)$$

to apply [**EO07**, Theorem 5.1]. Thus the claim (i) and (ii) are proved. Integrating both hand-sides of (5.3.30), we have (iii). □

5.3.6. Differential recursion for open free energies. Here we give a key theorem in the proof of our main results. We have the following *differential recursion* which is a modification of the one obtained in [**DM14c**, **DM**].

THEOREM 5.3.4. *The open free energies for $2g-2+n \geq 2$ satisfy the following equatinos:*

$$\frac{\partial F_{g,n}}{\partial z_1}(z_1,\dots,z_n) = \sum_{j=2}^{n} \frac{-2z_j}{z_1^2 - z_j^2}\left(\frac{1}{2y(z_1)\frac{dx}{dz}(z_1)}\frac{\partial F_{g,n-1}}{\partial z_1}(z_{[\hat{j}]}) - \frac{1}{2y(z_j)\frac{dx}{dz}(z_j)}\frac{\partial F_{g,n-1}}{\partial z_j}(z_{[\hat{1}]})\right)$$
$$- \frac{1}{2y(z_1)\frac{dx}{dz}(z_1)}\frac{\partial^2}{\partial u_1 \partial u_2}\left(F_{g-1,n+1}(u_1,u_2,z_{[\hat{1}]}) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=[\hat{1}]}}^{\text{stable}} F_{g_1,|I|+1}(u_1,z_I)F_{g_2,|J|+1}(u_2,z_J)\right)\Bigg|_{u_1=u_2=z_1}$$
$$+ \frac{s}{\frac{dy}{dz}(s)\frac{dx}{dz}(s)(z_1^2 - s^2)}\Bigg[\sum_{j=2}^{n}\frac{-2z_j}{z_j^2 - s^2}\frac{\partial F_{g,n-1}}{\partial z_1}(s, z_{[\hat{1},\hat{j}]})$$
$$+ \frac{\partial^2}{\partial u_1 \partial u_2}\left(F_{g-1,n+1}(u_1,u_2,z_{[\hat{1}]}) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=[\hat{1}]}}^{\text{stable}} F_{g_1,|I|+1}(u_1,z_I)F_{g_2,|J|+1}(u_2,z_J)\right)\Bigg|_{u_1=u_2=s}\Bigg]. \tag{5.3.34}$$

Here

$$s = (3q_0)^{1/2} \tag{5.3.35}$$

is a zero of $y(z)$.

PROOF. This can be proved by a similar technique used in [**DM14c**, Theorem 4.7], as follows. Integrating the topological recursion relation (5.3.7) with respect to $z_2,\dots,z_n$, we have

$$\begin{aligned}\frac{\partial}{\partial z_1}F_{g,n}(z_1,\dots,z_n) &= \frac{1}{2^{n-1}}\int_{\bar{z}_2}^{z_2}\cdots\int_{\bar{z}_n}^{z_n} W_{g,n}(z_1,\dots,z_n) \\ &= \frac{1}{2\pi i}\frac{1}{2^{n-1}}\oint_{\gamma_0} K(z,z_1)R_{g,n}(z,z_2,\dots,z_n),\end{aligned} \tag{5.3.36}$$

where

$$R_{g,n}(z,z_2,\dots,z_n) = \sum_{j=2}^{n}\Bigg[\left(\int_{\bar{z}_j}^{z_j} W_{0,2}(z,z_j)\right)\left(\int_{\bar{z}_{[\hat{1},\hat{j}]}}^{z_{[\hat{1},\hat{j}]}} W_{g,n-1}(\bar{z}, z_{[\hat{1},\hat{j}]})\right)$$
$$- \left(\int_{\bar{z}_j}^{z_j} W_{0,2}(\bar{z},z_j)\right)\left(\int_{\bar{z}_{[\hat{1},\hat{j}]}}^{z_{[\hat{1},\hat{j}]}} W_{g,n-1}(z, z_{[\hat{1},\hat{j}]})\right)\Bigg]$$
$$+ \int_{\bar{z}_{[\hat{1}]}}^{z_{[\hat{1}]}} W_{g-1,n+1}(z,\bar{z},z_{[\hat{1}]}) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=[\hat{1}]}}^{\text{stable}}\left(\int_{\bar{z}_I}^{z_I} W_{g_1,|I|+1}(z,z_I)\right)\left(\int_{\bar{z}_J}^{z_J} W_{g_2,|J|+1}(\bar{z},z_J)\right). \tag{5.3.37}$$

Here, for a set $L = \{\ell_1, \dots, \ell_k\} \subset \{1, \dots, n\}$ of indices, we have used the notation

$$\int_{\bar{z}_L}^{z_L} W_{g,n}(z_1, \dots, z_n) := \int_{\bar{z}_{\ell_1}}^{z_{\ell_1}} \cdots \int_{\bar{z}_{\ell_k}}^{z_{\ell_k}} W_{g,n}(z_1, \dots, z_n).$$

On the z-plane, the integrand $K(z, z_1) R_{g,n}(z, z_1, \dots, z_n)$ in the right hand-side of (5.3.36) has poles at

- at $z = z_1, \bar{z}_1$ which are poles of $K(z, z_1)$,
- at $z = z_2, \dots, z_n, \bar{z}_2, \dots, \bar{z}_n$ which are poles of $W_{0,2}(z, z_j)$ and $W_{0,2}(\bar{z}, z_j)$,
- at $z = s, \bar{s}$ which are poles of $K(z, z_1)$,

and all of them are simple poles. Then, the equalities

$$\begin{aligned} \int_{\bar{z}_j}^{z_j} W_{0,2}(z, z_j) &= \left(\frac{1}{z - z_j} - \frac{1}{z - \bar{z}_j} \right) dz, \\ \frac{1}{2^{n-2}} \int_{\bar{z}_{[\hat{1},\hat{j}]}}^{z_{[\hat{1},\hat{j}]}} W_{g,n}(z, z_{[\hat{1},\hat{j}]}) &= \frac{\partial F_{g,n-1}}{\partial z_1}(z, z_{[\hat{1},\hat{j}]}) \end{aligned}$$

and the residue theorem show (5.3.34). □

REMARK 5.3.3. *Note that the first two blocks in the right hand-side of* (5.3.34) *coincide with that obtained in [**DM14c, DM**]. Unlike the case of [**DM14c, DM**], we need more terms arising from* $z = s$ *corresponding to the singular point* $(x, y) = (q_0, 0)$ *of the spectral curve* (5.2.15) *since it becomes a (simple) pole of the recursion kernel* $K(z, z_1)$. *It also worth mentioning that the right hand-side of* (5.3.34) *doesn't have singularity at* $z_j = s$ *for* $j = 1, \dots, n$.

Using this differential recursion, we can give an alternative expression of (5.3.33) as follows.

THEOREM 5.3.5. *For* $2g - 2 + n \geq 1$, *the following holds:*

$$\frac{\partial}{\partial t} F_{g,n}(z(x_1), \dots, z(x_n)) = E_{g,n}(z(x_1), \dots, z(x_n)), \tag{5.3.38}$$

where

(5.3.39) $E_{g,n}(z_1,\dots,z_n) :=$

$$\sum_{j=1}^{n} \frac{2z_j}{2y(z_j)\frac{dx}{dz}(z_j)} \frac{\partial F_{g,n}}{\partial z_j}(z_1,\dots,z_n) + \frac{s}{\frac{dy}{dz}(s)\frac{dx}{dz}(s)} \sum_{j=1}^{n} \frac{-2z_j}{z_j^2 - s^2} \frac{\partial F_{g,n}}{\partial u_1}(u_1, z_{[\hat{j}]})\Big|_{u_1=s}$$

$$+ \frac{s}{\frac{dy}{dz}(s)\frac{dx}{dz}(s)} \frac{\partial^2}{\partial u_1 \partial u_2}\Bigg(F_{g-1,n+2}(u_1,u_2,z_1,\dots,z_n) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=\{1,\dots,n\}}}^{\text{stable}} F_{g_1,|I|+1}(u_1,z_I)F_{g_2,|J|+1}(u_2,z_J)\Bigg)\Bigg|_{u_1=u_2=s}.$$

Proof. The equality (5.3.33) shows that the left hand-side of (5.3.38) coincides with

$$\lim_{z_{n+1}\to\infty} z_{n+1}^2 \frac{\partial}{\partial z_{n+1}} F_{g,n+1}(z_1,\dots,z_n,z_{n+1})$$

after the substitution $z_i \mapsto z(x_i)$ for $i = 1,\dots,n$. Then, the equality follows from the asymptotic behavior (5.3.28) of $F_{g,n}$'s and the above differential recursion (5.3.34) for $2g-2+(n+1) \geq 2$. □

5.4. Proof of main theorems

5.4.1. Strategy for the proof. What we will show here is that the formal series $S(x,t,\hbar)$ defined in (5.3.16) satisfies the system of equations (5.3.17) and (5.3.18). In addition, we will also prove the equality (5.3.23). These equalities will be proved by an induction as follows.

Theorem 5.4.1. *Let $[\bullet]_{\hbar^m}$ be the coefficient of $\hbar^m$ in a formal series $\bullet$ of $\hbar$. For an even integer $k \geq 2$, assume that*

$$\begin{cases} \dfrac{\partial S_m}{\partial x}(x,t) = P_m(x,t), \quad \dfrac{\partial S_m}{\partial t}(x,t) = \left[\dfrac{1}{2(x-q)}\left(\hbar\dfrac{\partial S}{\partial x} - p\right)\right]_{\hbar^m} & \text{for } m = 0,\dots,k-1, \\ \dfrac{dF_g}{dt}(t) = \sigma_{2g}(t) & \text{for } g = k/2 \end{cases} \tag{5.4.1}$$

holds. Here $P_m(x,t) = P_m^{(+)}(x,t)$ is the coefficient of $\hbar^{m-1}$ in the formal solution $P^{(+)}(x,t,\hbar)$ of the Riccati equation (5.2.19) *constructed in §5.2.4, and σ_{2g} is given in* (5.2.11). *Then, we have*

(A) *The following equality holds for $m = k$ and $k+1$:*

$$\left[\hbar^2\left(\left(\frac{\partial S}{\partial x}\right)^2 + \frac{\partial^2 S}{\partial x^2}\right)\right]_{\hbar^m} = \left[2\hbar^2\frac{\partial S}{\partial t} + (4x^3 + 2tx + p^2 - 4q^3 - 2tq)\right]_{\hbar^m}. \tag{5.4.2}$$

(B) *The following equalities hold:*

$$\frac{\partial S_k}{\partial x}(x,t) = P_k(x,t), \quad \frac{\partial S_k}{\partial t}(x,t) = \left[\frac{1}{2(x-q)}\left(\hbar\frac{\partial S}{\partial x} - p\right)\right]_{\hbar^k} \tag{5.4.3}$$

$$\frac{\partial S_{k+1}}{\partial x}(x,t) = P_{k+1}(x,t), \quad \frac{\partial S_{k+1}}{\partial t}(x,t) = \left[\frac{1}{2(x-q)}\left(\hbar\frac{\partial S}{\partial x} - p\right)\right]_{\hbar^{k+1}} \tag{5.4.4}$$

$$\frac{dF_g}{dt}(t) = \sigma_{2g}(t) \text{ for } g = (k+2)/2. \tag{5.4.5}$$

It is obvious that our main theorems (Theorem 5.3.1 and 5.3.2) follow from the statements in (A) and (B). The rest of this section is devoted to give a proof of (A) and (B).

5.4.2. Proof of (A). We emphasize that the results shown in §5.4.2.1 below are proved without using the assumption (5.4.1). We also note that we only use the second equality in assumption (5.4.1) in §5.4.2.2 to prove (A).

5.4.2.1. *Computation of principal specializations.* Define

$$\begin{aligned} G_{g,n}(z_1,\dots,z_n) &:= \frac{\partial F_{g,n}}{\partial z_1}(z_1,\dots,z_n) \\ &- \sum_{j=2}^{n} \frac{-2z_j}{z_1^2 - z_j^2}\left(\frac{1}{2y(z_1)\frac{dx}{dz}(z_1)}\frac{\partial F_{g,n-1}}{\partial z_1}(z_{[\hat{j}]}) - \frac{1}{2y(z_j)\frac{dx}{dz}(z_j)}\frac{\partial F_{g,n-1}}{\partial z_j}(z_{[\hat{1}]})\right) \\ &+ \frac{1}{2y(z_1)\frac{dx}{dz}(z_1)}\frac{\partial^2}{\partial u_1 \partial u_2}\left(F_{g-1,n+1}(u_1,u_2,z_{[\hat{1}]}) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=[\hat{1}]}}^{\text{stable}} F_{g_1,|I|+1}(u_1,z_I)F_{g_2,|J|+1}(u_2,z_J)\right)\Bigg|_{u_1=u_2=z_1}. \end{aligned} \tag{5.4.6}$$

The technique developed in [**DM14c**, **DM**] enables us to show the following.

LEMMA 5.4.1 (c.f., [**DM14c**, Theorem 6.5]). *For $m \geq 2$, we have*

$$\left(\frac{2y(z)}{\frac{dx}{dz}(z)}\sum_{\substack{2g-2+n=m \\ g\geq 0, n\geq 1}} \frac{G_{g,n}(z,\dots,z)}{(n-1)!}\right)\Bigg|_{z=z(x)} = \sum_{\substack{a+b=m+1 \\ a,b\geq 0}} \frac{\partial S_a}{\partial x}\frac{\partial S_b}{\partial x} + \frac{\partial^2 S_m}{\partial x^2} - \frac{1}{x-q_0}\frac{\partial S_m}{\partial x}. \tag{5.4.7}$$

PROOF. As is shown in [**DM14c**, Theorem 6.5], applying $\sum_{2g-2+n=m} \frac{1}{(n-1)!}$ and the principal specialization to (5.4.6), we have

$$\sum_{\substack{2g-2+n=m \\ g\geq 0, n\geq 1}} \frac{G_{g,n}(z,\ldots,z)}{(n-1)!} = \frac{1}{2y(z)\frac{dx}{dz}(z)} \left(\sum_{\substack{a+b=m+1 \\ a,b\geq 2}} \frac{\partial S_a(x(z))}{\partial z} \frac{\partial S_b(x(z))}{\partial z} + \frac{\partial^2 S_m(x(z))}{\partial z^2} \right) \tag{5.4.8}$$

$$+ \frac{\partial S_{m+1}(x(z))}{\partial z} + \left\{ \frac{\partial}{\partial z} \left(\frac{1}{2y(z)\frac{dx}{dz}(z)} \right) \right\} \frac{\partial S_m(x(z))}{\partial z}$$

in the coordinate z. After the coordinate change $z = z(x)$, the right hand-side becomes

$$\frac{\frac{dx}{dz}(z(x))}{2y(z(x))} \left(\sum_{\substack{a+b=m+1 \\ a,b\geq 2}} \frac{\partial S_a}{\partial x} \frac{\partial S_b}{\partial x} + \frac{\partial^2 S_m}{\partial x^2} + 2y(z(x)) \frac{\partial S_{m+1}}{\partial x} - \frac{1}{y(z(x))} \frac{\partial y(z(x))}{\partial x} \frac{\partial S_m}{\partial x} \right)$$

Then, the desired equality (5.4.7) follows from the above equality and

$$\frac{\partial S_0}{\partial x} = y(z(x)), \quad \frac{\partial S_1}{\partial x} = -\frac{1}{2y(z(x))} \frac{\partial y(z(x))}{\partial x} + \frac{1}{2(x-q_0)}.$$

□

Note that the right hand-side of (5.4.7) coincides with

$$\left[\hbar^2 \left(\left(\frac{\partial S}{\partial x} \right)^2 + \frac{\partial^2 S}{\partial x^2} \right) \right]_{\hbar^{m+1}} - \frac{1}{x-q_0} \frac{\partial S_m}{\partial x}.$$

Thus, Lemma 5.4.1 relates the principal specialization of $G_{g,n}$ to the left hand-side of (5.4.2). Next we also relate them to the right hand-side of (5.4.2) under the assumption (5.4.1).

LEMMA 5.4.2. *Let $E_{g,n}(z_1,\ldots,z_n)$ be the functions defined by (5.3.39). Then, the following equality holds for $m \geq 2$.*

$$\sum_{\substack{2g-2+n=m \\ g\geq 0, n\geq 2}} \left(\frac{2y(z)}{\frac{dx}{dz}(z)} \frac{G_{g,n}(z,\ldots,z)}{(n-1)!} - \frac{2E_{g,n-1}(z,\ldots,z)}{(n-1)!} \right) \Bigg|_{z=z(x)} = -\frac{1}{x-q_0} \frac{\partial S_m}{\partial x} \tag{5.4.9}$$

PROOF. Theorem 5.3.4 shows that (5.4.6) can also be written as

$$G_{g,n}(z_1,\dots,z_n) = \frac{s}{\frac{dy}{dz}(s)\frac{dx}{dz}(s)(z_1^2-s^2)}\sum_{j=2}^{n}\frac{-2z_j}{z_j^2-s^2}\frac{\partial F_{g,n-1}}{\partial z_1}(s,z_{[\hat{1},\hat{j}]}) + \frac{s}{\frac{dy}{dz}(s)\frac{dx}{dz}(s)(z_1^2-s^2)}\frac{\partial^2}{\partial u_1\partial u_2}\Bigg(F_{g-1,n+1}(u_1,u_2,z_{[\hat{1}]})+ \sum_{\substack{g_1+g_2=g\\ I\sqcup J=[\hat{1}]}}^{\text{stable}} F_{g_1,|I|+1}(u_1,z_I)F_{g_2,|J|+1}(u_2,z_J)\Bigg)\Bigg|_{u_1=u_2=s}. \tag{5.4.10}$$

Taking the principal specialization of (5.4.10) and (5.3.39) with $n\mapsto n-1$, we have

$$\frac{2y(z)}{\frac{dx}{dz}(z)}\frac{G_{g,n}(z,\dots,z)}{(n-1)!} - \frac{2E_{g,n-1}(z,\dots,z)}{(n-1)!} = -\frac{4z}{2y(z)\frac{dx}{dz}(z)}\left(\frac{1}{(n-1)!}\frac{\partial}{\partial z}F_{g,n-1}(z,\dots,z)\right) \tag{5.4.11}$$

for any $g\geq 0$ and $n\geq 2$ satisfying $2g-2+n\geq 2$. Then, summing up (5.4.11) for $g\geq 0, n\geq 2$ satisfying $2g-2+n=m$, we obtain (5.4.9) after the coordinate change $z=z(x)$. □

On the other hand, Theorem 5.3.5 implies that

$$\sum_{\substack{2g-2+n=m\\ g\geq 0,n\geq 2}}\frac{2E_{g,n-1}(z,\dots,z)}{(n-1)!}\Bigg|_{z=z(x)} = 2\frac{\partial}{\partial t}S_m = \left[2\hbar^2\frac{\partial S}{\partial t}\right]_{\hbar^{m+1}} \tag{5.4.12}$$

holds for $m\geq 2$. Therefore we have the following.

LEMMA 5.4.3. *The equality*

$$\left[\hbar^2\left(\left(\frac{\partial S}{\partial x}\right)^2+\frac{\partial^2 S}{\partial x^2}\right)\right]_{\hbar^{m+1}} = \left[2\hbar^2\frac{\partial S}{\partial t}\right]_{\hbar^{m+1}} + \sum_{\substack{2g-2+n=m\\ g\geq 0,n\geq 1}}\frac{2y(z)}{\frac{dx}{dz}(z)}\frac{G_{g,n}(z,\dots,z)}{(n-1)!} - \sum_{\substack{2g-2+n=m\\ g\geq 0,n\geq 2}}\frac{2y(z)}{\frac{dx}{dz}(z)}\frac{G_{g,n}(z,\dots,z)}{(n-1)!} \tag{5.4.13}$$

holds for $m\geq 2$.

5.4.2.2. *Completion of the proof of* (A). Lemma 5.4.3 implies

$$\begin{cases} \left[\hbar^2\left(\left(\frac{\partial S}{\partial x}\right)^2+\frac{\partial^2 S}{\partial x^2}\right)\right]_{\hbar^{m+1}} = \left[2\hbar^2\frac{\partial S}{\partial t}\right]_{\hbar^{m+1}} & \text{if } m \text{ is even.} \\ \left[\hbar^2\left(\left(\frac{\partial S}{\partial x}\right)^2+\frac{\partial^2 S}{\partial x^2}\right)\right]_{\hbar^{m+1}} = \left[2\hbar^2\frac{\partial S}{\partial t}\right]_{\hbar^{m+1}} + \frac{2y}{\frac{dx}{dz}}G_{(m+1)/2,1} & \text{if } m \text{ is odd.} \end{cases} \tag{5.4.14}$$

On the other hand, it follows from (5.2.11) that

$$\left[4x^3+2tx+p^2-4q^3-2tq\right]_{\hbar^{m+1}} = \begin{cases} 0 & \text{if } m \text{ is even.} \\ 2\sigma_{m+1} & \text{if } m \text{ is odd.} \end{cases} \tag{5.4.15}$$

Therefore, under the assumption (5.4.1), the desired equality (5.4.2) follows from (5.4.14) and Lemma 5.4.4 below.

Lemma 5.4.4. *For $g \geq 2$, we have*

$$\frac{2y(z)}{\frac{dx}{dz}(z)}G_{g,1}(z) = 2\frac{dF_g}{dt}(t) \tag{5.4.16}$$

Proof. Firstly, we note that

$$\frac{2y(z)}{\frac{dx}{dz}(z)}G_{g,1}(z) = \frac{1}{4s^2}\left(\frac{\partial^2 F_{g-1,2}}{\partial z_1 \partial z_2}(s,s) + \sum_{\substack{g_1+g_2=g \\ g_1,g_2\geq 1}} \frac{\partial F_{g_1,1}}{\partial z_1}(s)\frac{\partial F_{g_2,1}}{\partial z_2}(s)\right) \tag{5.4.17}$$

holds. Using the differnetial recursion (5.3.34) for $n=1$, we have

$$\begin{aligned} \frac{\partial F_{g,1}}{\partial z_1}(z) = &-\frac{1}{2y(z)\frac{dx}{dz}(z)}\left(\frac{\partial^2 F_{g-1,2}}{\partial z_1 \partial z_2}(z,z) + \sum_{\substack{g_1+g_2=g \\ g_1,g_2\geq 1}} \frac{\partial F_{g_1,1}}{\partial z_1}(z)\frac{\partial F_{g_2,1}}{\partial z_1}(z)\right) \\ &+\frac{s}{\frac{dy}{dz}(s)\frac{dx}{dz}(s)(z^2-s^2)}\left(\frac{\partial^2 F_{g-1,2}}{\partial z_1 \partial z_2}(s,s) + \sum_{\substack{g_1+g_2=g \\ g_1,g_2\geq 1}} \frac{\partial F_{g_1,1}}{\partial z_1}(s)\frac{\partial F_{g_2,1}}{\partial z_1}(s)\right). \end{aligned}$$

Then, Lemma 5.3.1 imples that

$$z\frac{\partial F_{g,1}}{\partial z_1}(z)dz = zW_{g,1}(z) = \frac{1}{8s^2}\left(\frac{\partial^2 F_{g-1,2}}{\partial z_1 \partial z_2}(s,s) + \sum_{\substack{g_1+g_2=g \\ g_1,g_2\geq 1}} \frac{\partial F_{g_1,1}}{\partial z_1}(s)\frac{\partial F_{g_2,1}}{\partial z_2}(s)\right)\frac{dz}{z} + O(1)$$

holds when $z \to \infty$. Then the equality (5.4.16) follows from (5.3.31) and (5.4.17). □

5.4.3. Proof of (B)**.** The desired equality (5.4.3) is proved as follows.

LEMMA 5.4.5. *Under the assumption* (5.4.1)*, we have*

$$\frac{\partial S_k}{\partial x}(x,t) = P_k(x,t). \tag{5.4.18}$$

$$S_k(x,t) = \int_\infty^x P_k(x',t)dx'. \tag{5.4.19}$$

$$\frac{\partial S_k}{\partial t}(x,t) = \left[\frac{1}{2(x-q)}\left(\hbar\frac{\partial S}{\partial x} - p\right)\right]_{\hbar^k}. \tag{5.4.20}$$

PROOF. The equality (5.4.2) for $m = k$ and the equalities in the first line of the assumption (5.4.1) imply

$$\left[\hbar^2\left(\left(\frac{\partial S}{\partial x}\right)^2 + \frac{\partial^2 S}{\partial x^2}\right)\right]_{\hbar^k} = \left[\frac{\hbar}{x-q}\left(\hbar\frac{\partial S}{\partial x} - p\right) + (4x^3 + 2tx + p^2 - 4q^3 - 2tq)\right]_{\hbar^k}.$$

Thus $\partial S_k/\partial x$ and P_k satisfy the same equation (5.2.23). Then the uniqueness of the solution of (5.2.23) implies (5.4.18).

Since $S_m(x)$ for $m \geq 2$ decay when $x \to \infty$ (c.f., (5.3.29)), the equality (5.4.19) immediately follows from (5.4.18). Then, the equality (5.2.32) shows

$$\begin{aligned}
\frac{\partial}{\partial t}S_k(x,t) &= \int_\infty^x \left[\hbar\frac{\partial}{\partial t}P\right]_{\hbar^k} dx \\
&= \int_\infty^x \frac{\partial}{\partial x}\left[\frac{\hbar P - p}{2(x-q)}\right]_{\hbar^k} dx \\
&= \left[\frac{1}{2(x-q)}\left(\hbar\frac{\partial S}{\partial x} - p\right)\right]_{\hbar^k}.
\end{aligned} \tag{5.4.21}$$

The last equality follows from the assumption (5.4.1) and the fact that $P_m(x,t)$'s decay when $x \to \infty$ for $m \geq 1$ (see Remark 5.2.5), and

$$\lim_{x\to\infty}\frac{P_0(x,t)}{(x-q_0)^2} = 0.$$

Thus we have proved (5.4.20).

□

Since we have also already proved (5.4.2) for $m = k+1$, we can prove (5.4.4) by the same discussion as the proof of Lemma 5.4.5 above. Then, finally we obtain

LEMMA 5.4.6. *The equality* (5.4.4) *is true; namely, we have*

$$\frac{dF_{(k+2)/2}}{dt} = \sigma_{2k+2}. \tag{5.4.22}$$

PROOF. It follows from the equality (5.4.14) (for the odd number $m = k+1$) that

$$2\frac{\partial S_0}{\partial x}\frac{\partial S_{k+2}}{\partial x} + \sum_{\substack{a+b=k+2 \\ a,b\geq 1}} \frac{\partial S_a}{\partial x}\frac{\partial S_b}{\partial x} + \frac{\partial^2 S_{k+1}}{\partial x^2} - 2\frac{\partial S_{k+1}}{\partial t} = 2\frac{dF_{(k+2)/2}}{dt} \tag{5.4.23}$$

holds. On the other hand, we know that P_{k+2} satisfies

$$2P_0P_{k+2} + \sum_{\substack{a+b=k+2 \\ a,b\geq 1}} P_aP_b + \frac{\partial P_{k+1}}{\partial x} - \left[\frac{\hbar}{x-q}\left(\hbar P - p\right)\right]_{\hbar^{k+2}} = 2\sigma_{k+2} \tag{5.4.24}$$

(c.f., (5.2.23)). Under our assumption, comparing (5.4.23) and (5.4.24), we have

$$\frac{\partial S_0}{\partial x}\left(\frac{\partial S_{k+2}}{\partial x} - P_{k+2}\right) = \frac{dF_{(k+2)/2}}{dt} - \sigma_{k+2}. \tag{5.4.25}$$

Note that the right hand-side doesn't depend on x. Then, thanks to the fact

$$\left.\frac{\partial S_0}{\partial x}\right|_{x=q_0} = 0$$

and the holomorphicity of $S_m(x)$ and $P_m(x)$ (see Theorem 5.2.2), we have the desired equality (5.4.22) by substituting $x = q_0$ into (5.4.25). □

This completes the proof of (B) and Theorem 5.4.1. Thus we have proved Theorem 5.3.1 and 5.3.2.

REMARK 5.4.1. *Since the spectral curve* (5.2.15) *has only one branch point, we have*

$$\int_v^x P_m(x',t)dx' = \int_\infty^x P_m(x',t)dx' \tag{5.4.26}$$

for all even $m \geq 2$. *This implies that the WKB solution* (5.3.15) *defined by the topological recursion coincides with the WKB solution* (5.2.34) *constructed in §5.2.4. However, the above equality* (5.4.26) *may not hold for other Painlevé equations since, their spectral curves have more branch points in general.*

5.5. Alternative definition of the τ-function by Jimbo-Miwa-Ueno

There is another definition of τ-function (5.2.12) in terms of the formal solution (5.2.37) of the isomonodromy system.

PROPOSITION 5.5.1 ([**JMU81**, §5]; see also [**MBE15**, §4.2] and [**BE10**, §1.5]). *The τ-function satisfies*

$$\frac{d}{dt}\log\tau(t,\hbar) = -2\operatorname*{Res}_{x=\infty}\left(\frac{1}{\hbar}\frac{\partial T_\infty}{\partial t}(x,t)\mathcal{W}_1(x,t,\hbar)dx\right), \tag{5.5.1}$$

where

$$T_\infty(x,t) := \frac{4x^{5/2}}{5} + tx^{1/2}$$

(which is the divergent part of $\int^x P_0^{(+)}(x',t)dx'$ as $x \to \infty$), and

$$\mathcal{W}_1(x,t,\hbar) = \frac{\partial\psi_+}{\partial x}(x,t,\hbar)\tilde{\psi}_-(x,t,\hbar) - \frac{\partial\tilde{\psi}_+}{\partial x}(x,t,\hbar)\psi_-(x,t,\hbar). \tag{5.5.2}$$

PROOF. It follows from the definition (5.2.37) of Ψ that

$$\mathcal{W}_1(x,t,\hbar) = P^{(+)}(x,t,\hbar) + \frac{A_{12}(x,t,\hbar)}{2\hbar P_{\mathrm{odd}}(x,t,\hbar)}\frac{\partial}{\partial x}\left(\frac{\hbar P^{(+)}(x,t,\hbar) - A_{11}(x,t,\hbar)}{A_{12}(x,t,\hbar)}\right). \tag{5.5.3}$$

Then, the asymptotics (5.2.24) of $P^{(\pm)}(x,t,\hbar)$ imples that

$$\mathcal{W}_1(x,t,\hbar) = \frac{2}{\hbar}x^{3/2} + \frac{t}{2\hbar}x^{-1/2} + \frac{\sigma(t,\hbar)}{2\hbar}x^{-3/2} + O(x^{-2})$$

holds when $x \to \infty$, and thus we have (5.5.1). □

APPENDIX A

Geometry

A.1. Blow Ups

A.1.1. Properties and introduction. The blow-up is a technique in algebraic geometry that is used to resolve singularities. The simplest example is when you have a curve in a plane that intersects itself at a point. That is, we obtain something that locally looks like a cross on the plane. As a manifold, this is singular because the cross-point is not locally Euclidean. Algebraically, the tangent space at the cross-point is 2 dimensional which is higher than the expected 1 dimension. The resolution, one of many, of this singularity is easy. Introduce a time parameter and parametrize the curve using such parameter. In order to make sure that we parametrize the curve smoothly, one needs to consider the tangent vector of the path at the cross-point and check that it varies smoothly at the cross-point. Thus, we have resolved our singularity by embedding the curve in a 3-dimensional space and separating the cross-strands, but this resolution is not the blow-up resolution.

The reader should now consider this example and note that this resolution is global, meaning that we have changed the ambient space from 2 dimensions to 3 dimensions, and more importantly that the devil lies in the tangent space of the cross-point. In contrast, the blow-up is a local resolution that only uses the topology of the normal space of the cross-point. In the case of the cross, the blow-up cuts out a neighborhood of the cross-point and glues in its place a copy of the compactified normal bundle of the cross-point in the plane. This changes the embedding locally where the ambient space remains 2-dimensional but it is no longer the plane, and it separates the cross-strands according to the tangent space.

Generally, one can blow up any variety X along any closed subvariety Z, and topologically, this corresponds to cutting out a tubular neighborhood of Z and glueing the projective normal tangent bundle in its place. We don't go into the details of this techniques as it doesn't enter our computations and instead we cite the results that show that the blow-up doesn't affect our computations. We reference the reader to [**Har77**] for further details.

First, we note where the blow-up comes in our computations. Choosing specific affine charts, we may write the coordinates of our rational maps before blow-ups as

$$\psi_{(k,l)}(\vec{z},\vec{w}) = \tilde{w}_{(k,l)} = -\frac{F(z_k,z_l)}{F(z_l,z_k)}.$$

For some polynomial in two variables, $F(x,y)$. We remarked that this is not well-defined when $F(z_k,z_l) = 0 = F(z_l,z_k)$ since $0/0$ is not well-defined. To fix this problem, note that these equations define a closed subvariety $Z \subseteq X$ where the ψ function is not well-defined. Also, note that the problem comes from the fact that if you approach Z along distinct paths, you will obtain different limits of $-\frac{F(z_k,z_l)}{F(z_l,z_k)}$. For example, if we approach Z along $\{F(z_k,z_l) = 0\}$, we get the limit of the fraction to be 0. While, if we approach Z along $\{F(z_l,z_k) = 0\}$, we get the limit of the fraction to be ∞. Thus, we need to pull apart all the distinct limits of the function $-\frac{F(z_k,z_l)}{F(z_l,z_k)}$ as we did with the cross-singularity, and note that the limit depends only on the normal bundle of Z (i.e. how we approach Z). Thus, we blow up X along Z to make ψ well-defined and smooth. Again, we will not go through the details explicitly, but we will focus on the properties of the blow-up.

Lemma A.1.1 (p. 602 [**GH78**]). *There exist a smooth map $\pi : \mathrm{Blow}_Z(X) \to X$ such that the following properties hold*

(1) $\pi^{-1}(Z) = \mathbb{P}(T_Z X)$*, the inverse image of Z is diffeomorphic to the projective normal tangent bundle of Z.*
(2) $\pi : \mathrm{Blow}_Z(X) \setminus \pi^{-1}(Z) \to X \setminus Z$ *is an isomorphism.*

This lemma gives us the topology of the blow-up, but we are most interested in the homology of the blow-up space as it is where we do our computations.

Theorem A.1.1 (Prop p. 606 [**GH78**]). *The induced map ring $\pi^* : H^*(X) \to H^*(\mathrm{Blow}_Z(X))$ embeds $H^*(X)$ as a direct summand of $H^*(\mathrm{Blow}_Z(X))$ (i.e. $H^*(\mathrm{Blow}_Z(X)) = H^*(X) \oplus H$)*

This theorem allows us to factorize the maps we are working with and determine that our computations only depend on $H^*(X)$.

A.1.2. Charts. We need to show that a smooth resolution of the ψ map exists. For this, we go into the details and describe the charts of the blow-up of a manifold along a submanifold. This is all standard material that can be found in *Principle of Algebraic Geometry* by Griffiths and Harris. (We copy much of this material straight from the book for reference.)

Let Δ be an n-dimensional disc with holomorphic coordinates $z_1, \ldots, z_n$, and let $V \subseteq \Delta$ be the locus $z_{k+1} = \cdots = z_n = 0$. Let $[l_{k+1} : \cdots : l_n]$ be the homogeneous coordinates of $\mathbb{CP}^{n-k-1}$, and let

$$\tilde{\Delta} \subseteq \Delta \times \mathbb{CP}^{n-k-1}$$

be the smooth variety defined by the relations

$$\tilde{\Delta} = \{(z, l) | z_i l_j = z_j l_i, k + 1 \leq i, j \leq n\}$$

The projection $\pi : \tilde{\Delta} \to \Delta$ on the first factor is clearly an isomorphism away from V, while the inverse image of a point $z \in V$ is a projective space $\mathbb{CP}^{n-k-1}$. The manifold $\tilde{\Delta}$ with the map $\pi : \tilde{\Delta} \to \Delta$ is called the *blow-up of* Δ *along* V; the inverse image $E = \pi^{-1}(V)$ is called the *exceptional divisor* of the blow-up.

$\tilde{\Delta}$ may be covered by coordinate patches

$$U_j = (l_j \neq 0), \qquad j = k + 1, \ldots, n$$

with holomorphic coordinates

$$\begin{aligned} z_i &= z_i, && i = 1, \ldots, k \\ z(j)_i &= \frac{l_i}{l_j} = \frac{z_i}{z_j}, && i = k + 1, \ldots, \hat{j}, \ldots n \\ z_j &= z_j && i = j \end{aligned}$$

on U_j; the coordinates $\{z(j)_i\}$ are Euclidean coordinates on each fiber of $\pi^{-1}(p) \cong \mathbb{CP}^{n-k-1}$ of the exceptional divisior. Moreover, one can show that the blow-up $\pi : \tilde{\Delta} \to \Delta$ is independent of the coordinates chosen. This allows us to globalize the construction. That is, given a manifold X and a submanifold Z of codimension k, we may choose a collection $\{U_\alpha\}$ of discs covering Z such that in each disc the subvariety $Z \cap U_\alpha$ may be given as the locus $(z_{k+1} = \cdots = z_n = 0)$, and then blow-up each disc as above and patch them together.

Now, we want to know how the subvarieties of $Y \subseteq X$ transform as we blow-up X along Z. We define the *proper transform* $\tilde{Y} \subseteq \tilde{X}_Z$ of Y in the blow-up $\tilde{X}_Z$ to be the closure in $\tilde{X}_Z$ of the inverse image

$$\pi^{-1}(Y - Z) = \pi^{-1}(Y) - E$$

of Y away from the exceptional divisor E.

A.1.3. Resolution of ψ_N. In this section, we explicitly construct the resolution of the map $\psi_N : X \dashrightarrow X$. We are heavily influenced by the ideas of Hironaka's Theorem on the resolution of singularities [**Hir64a**, **Hir64b**, **Hau03**].

Remark A.1.1. *In this section, our methods don't apply for the case when the hopping rate is* $p = 1/2$. *This is due to the fact that the subvaritey corresponding to the points where the map* ψ *is not well-defined, for* $p = 1/2$, *has multiplicity greater than one, as it is apparent in* (A.1.1). *This case can be approached with techniques from [**Har77**].*

First, we decompose $X = X_1 \times X_2$ as in Proposition 4.4.1, and ψ_N also decomposes

$$f \times g : X_1 \times X_2 \dashrightarrow X_2 \times X_1$$

where $g : X_2 \to X_1$ is actually a smooth map and $f : X_1 \dashrightarrow X_2$ is the rational part of the ψ_N map. Thus, it suffices to resolve $f : X_1 \dashrightarrow X_2$. We do this by blowing up the domain, X_1, several times and then we show how f extends on the blow-ups.

First, recall that

$$f : \left([z_0^i : z_1^i]\right)_{i=1}^N \mapsto \left([\omega_0^{(k,l)} : \omega_1^{(k,l)}]\right)_{1 \le k < l \le N}$$

where

$$[\omega_0^{(k,l)} : \omega_1^{(k,l)}] = [p z_0^k z_0^l + q z_1^k z_1^l - z_0^k z_1^l : -(p z_0^k z_0^l + q z_1^k z_1^l - z_0^l z_1^k)]$$

where the map is undefined on the subvarieties

$$Z^i_{(k,l)} = \left\{ pt \in X \mid \pi_k(p) = \pi_l(p) = [2q : 1 + (-1)^i \sqrt{1 - 4pq}] \right\} \tag{A.1.1}$$

Then, after a change of variable we may assume, for convenience, that

$$f : \left([z_0^i : z_1^i]\right)_{i=1}^N \mapsto \left([\omega_0^{(k,l)} : \omega_1^{(k,l)}]\right)_{1\leq k<l\leq N}$$

where

$$[\omega_0^{(k,l)} : \omega_1^{(k,l)}] = [a(z_0^k z_1^l + z_0^l z_1^k) + z_0^l z_1^k : a(z_0^k z_1^l + z_0^l z_1^k) + z_0^k z_1^l]$$

and now f is undefined in the subvarieties

$$Z_{(k,l)}^0 = \left\{\left([z_0^i : z_1^i]\right)_{i=1}^N \mid [z_0^k : z_1^k] = [z_0^l : z_1^l] = [0:1]\right\}$$
$$Z_{(k,l)}^1 = \left\{\left([z_0^i : z_1^i]\right)_{i=1}^N \mid [z_0^k : z_1^k] = [z_0^l : z_1^l] = [1:0]\right\}$$

Now, recall, that we have that the blow-up is a local construction. Thus, to make everything explicit we will work with affine chart and describe the blow-up on each of these charts. Of course, as we performed blow-ups, we will need more charts to describe the resulting space, and we will introduce new charts accordingly. Thus, we start by introducing charts of

$$(\mathbb{CP}^1)^N = \left\{\left([z_0^i : z_1^i]\right) | i = 1, \dots, N\right\}$$

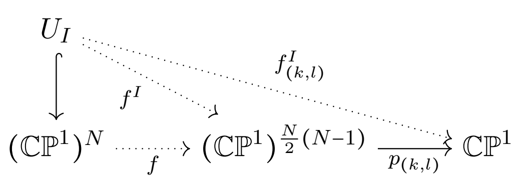
indexed by subsets I of $[N]$ and given by

$$U_I := \left(z_0^i \neq 0\right) \cap \left(z_1^i \neq 0\right)$$

with Euclidean coordinates

$$z_i^I = \begin{cases} z_1^i / z_0^i \text{ for } i \in I \\ z_0^i / z_1^i \text{ for } i \notin I \end{cases}.$$

Then, we have the commutative diagram



where $p_{(k,l)} : (\mathbb{CP}^1)^{\frac{N}{2}(N-1)} \to \mathbb{CP}^1$ is the projection onto the $(k,l)^{th}$ factor of $(\mathbb{CP}^1)^{\frac{N}{2}(N-1)}$, and we can describe $f^I_{(k,l)}$ explicitly

$$(z^I) \mapsto \begin{cases} [a(z^I_k + z^I_l) + z^I_l : a(z^I_k + z^I_l) + z^I_k] & \text{for } k,l \in I \\ [a(z^I_k + z^I_l) + z^I_k : a(z^I_k + z^I_l) + z^I_l] & \text{for } k,l \notin I \\ [a(z^I_k z^I_l + 1) + 1 : a(z^I_k z^I_l + 1) + z^I_k z^I_l] & \text{for } k \in I, l \notin I \\ [a(z^I_k z^I_l + 1) + z^I_k z^I_l : a(z^I_k z^I_l + 1) + 1] & \text{for } l \in I, k \notin I. \end{cases}$$

Note that, in the last two cases, we have that $f^I_{(k,l)}$is smooth. Then, after any blow-ups the proper transform of $f^I_{(k,l)}$ will remain smooth of the last two cases. Now, we wish to define the blow-ups recursively for $0 \le n \le N$ by

$$C_n := \mathrm{Blow}_{Z_{n-1}} C_{n-1}$$

where we explain our notation now and we start by defining

$$C_0 := (\mathbb{CP}^1)^N$$
$$Z_0 := \bigcup_{I \in [N]} \left(\overline{W^I_0} \cup \overline{Y^I_0} \right)$$

with

(A.1.2)
$$W^I_0 = \begin{cases} \{ z^I_i = 0 | i = 1, \dots, N \} & \text{if } [N] \subseteq I \\ \varnothing & \text{otherwise} \end{cases}$$

(A.1.3)
$$Y^I_0 = \begin{cases} \{ z^I_i = 0 | i = 1, \dots, N \} & \text{if } [N] \subseteq I^c \\ \varnothing & \text{otherwise.} \end{cases}$$

Observe that, for some (i.e. $I \neq [N], \varnothing$), we'll have $W^I_0 = Y^I_0 = \varnothing$. Regardless, we have

$$\overline{W^I_0}, \overline{Y^I_0} \subseteq U^I.$$

Also, we have that at least one of the subvarieties W^I_0 or Y^I_0 is empty, if not both, and we define

$$Z_0^I := W_0^I \cup Y_0^I. \tag{A.1.4}$$

Thus, we have the decomposition of the blow-up C_1 by

$$\begin{aligned} C_1 &= \mathrm{Blow}_{Z_0} C_0 \\ &= \bigcup_{I \subseteq [N]} \mathrm{Blow}_{Z_0^I} U^I. \end{aligned}$$

Moreover, we have that

$$\mathrm{Blow}_{Z_0^I} U^I = \begin{cases} \left\{ \left(z^I, l^I \right) \in U^I \times \mathbb{CP}^{N-1} | z_i^I l_j^I = z_j^I l_i^I \text{ for } 1 \leq i, j \leq N \right\} & \text{if } Z_0^I \neq \varnothing \\ U^I & \text{if } Z_0^I = \varnothing. \end{cases}$$

Next, we have affine charts $V_i^I \subseteq \mathrm{Blow}_{Z_0^I} U^I$ for $i = 1, \ldots, N$ given by

$$V_i^I = \begin{cases} \left(l_i^I \neq 0 \right) & \text{if } Z_0^I \neq \varnothing \\ U^I & \text{if } Z_0^I = \varnothing \end{cases}$$

and if if $Z_0^I \neq \varnothing$ we have the Euclidean coordinates for the chart V_i^I

$$z^I(i)_j = \begin{cases} l_j^I / l_i^I & \text{for } j \neq i \\ z_i^I & \text{for } j = i \end{cases}$$

and if if $Z_0^I = \varnothing$ we have the Euclidean coordinates for the chart V_i^I

$$z^I(i)_j = z_i^I \text{ for all } j.$$

Then, we have that the V_i^I for $I \subseteq [N]$ and $i = 1, \ldots, N$ are the charts of C_1, with some redundancies. Now, we can define the blow-up recursively as follows. Given C_n with affine charts $V_{\vec{i}}^I$ where $I \subseteq N$ and $\vec{i} = (i_1, \ldots, i_n)$ with each $i_t \in [N]$ and pair-wise distinct, we define

$$W_n^{I;i_1,\ldots,i_n} = \begin{cases} \{z(i_1,\ldots,i_n)_j = 0 \mid j \in [N] - \{i_1,\ldots,i_n\}\} & \text{if } [N] - \{i_1,\ldots,i_n\} \subseteq I \\ \varnothing & \text{otherwise} \end{cases} \tag{A.1.5}$$

$$Y_n^{I;i_1,\ldots,i_n} = \begin{cases} \{z(i_1,\ldots,i_n)_j = 0 \mid j \in [N] - \{i_1,\ldots,i_n\}\} & \text{if } [N] - \{i_1,\ldots,i_n\} \subseteq I^c \\ \varnothing & \text{otherwise.} \end{cases} \tag{A.1.6}$$

Note that either $W_n^{I;\vec{i}}$ or $Y_n^{I;\vec{i}}$ is empty, if not both. Then, define

$$Z_n^{I;\vec{i}} := W_n^{I;\vec{i}} \cup Y_n^{I;\vec{i}} \tag{A.1.7}$$

and

$$Z_n := \bigcup_{I \subseteq [N]} \bigcup_{\vec{i}} Z_n^{I;\vec{i}}$$

Then, we have the inductive definition

$$C_{n+1} := \mathrm{Blow}_{Z_n} C_n$$

Actually, we still have to specify the order in which we perform the blow-up along the subvarieties $Z_n^{I;\vec{i}}$, but we claim;

Lemma A.1.2.

$$I \neq J \text{ or } \{i_1,\ldots,i_n\} \neq \{j_1,\ldots,j_n\} \Rightarrow Z_n^{I;\vec{i}} \cap Z_n^{J;\vec{j}} = \varnothing \tag{A.1.8}$$

$$I = J \text{ and } \{i_1,\ldots,i_n\} = \{j_1,\ldots,j_n\} \Rightarrow Z_n^{I;\vec{i}} = Z_n^{J;\vec{j}} \tag{A.1.9}$$

which says that the submanifolds are disjoint or the same, meaning that the order in which we perform the blow-ups does not make a difference.

Proof. The (A.1.9) statement is clear by blowing down the subvarieties and seeing that they agree in the blow-down.

Now, assume that

$$Z_n^{I;\vec{i}} \neq \varnothing \neq Z_n^{J;\vec{j}}.$$

That is, either

$$[N] - \vec{i} \subseteq I \qquad \text{or} \qquad [N] - \vec{i} \subseteq I^c$$

and either

$$[N] - \vec{j} \subseteq J \qquad \text{or} \qquad [N] - \vec{j} \subseteq J^c.$$

In any case, recall that

$$Z_n^{I;\vec{i}} = \{z(i_1, \ldots, i_n)_j = 0 \mid j \in [N] - \{i_1, \ldots, i_n\}\} \subseteq V_{\vec{i}}^I = \left(l^I(\vec{i} - i_n)_{i_n} \neq 0\right).$$

Assume, for simplicity, that $I = J$. Also, by (A.1.9), we may assume that the order of the indices of $\vec{i}$ and $\vec{j}$ is such that

$$i_t = j_t \qquad \text{for} \qquad t \leq k$$

$$\text{and} \qquad \{i_{k+1}, \ldots, i_n\} \cap \{j_{k+1}, \ldots, j_n\} = \varnothing.$$

Now, define for any $1 \leq t \leq n$

$$Z_t^{I;\vec{i}} = \{z(i_1, \ldots, i_t)_j = 0 | j \in [N] - \{i_1, \ldots, i_n\}\} \subseteq V_{i_1, \ldots, i_t}^I$$

and note that $Z_n^{I;\vec{i}}$ is the proper transform of $Z_t^{I;\vec{i}}$ under the blow-up

$$\pi_t^n : C_n \to C_t.$$

Also, we have that

$$\pi_t^n\left(Z_n^{I;\vec{i}}\right) \cap \pi_t^n\left(Z_n^{J;\vec{j}}\right) = Z_t^{I;i_1, \ldots i_k}.$$

Then, we have

$$\begin{aligned}\pi_{k+1}^n\left(Z_n^{I;\vec{i}}\right) \cap \pi_{k+1}^n\left(Z_n^{J;\vec{j}}\right) &= Z_{k+1}^{I;\vec{i}} \cap Z_{k+1}^{J;\vec{j}} \\ &= Z_{k+1}^{I;i_1, \ldots, i_{k+1}} \cap Z_{k+1}^{J;j_1, \ldots, j_{k+1}}.\end{aligned}$$

If $k + 1 < n$, then the argument follows by induction . Thus, we may assume that,

$$i_1 = j_1, \ldots, i_{n-1} = j_{n-1}, \text{ and } i_n \neq j_n$$

Then, for any $pt \in \overline{Z_n^{I;\vec{i}}}$ we have

$$\begin{aligned} z^I(\vec{i})_{j_n}(pt) = 0 &\Rightarrow l^I(\vec{i})_{j_n}(pt) = 0 \\ &\Rightarrow pt \notin V_{\vec{j}}^I \\ &\Rightarrow pt \notin \overline{Z_n^{J;\vec{j}}}. \end{aligned}$$

Therefore, we must have that

$$\overline{Z_n^{I;\vec{i}}} \cap \overline{Z_n^{J;\vec{j}}} = \varnothing.$$

Now, we need to consider the case when $I \neq J$. First, note that if $i \in I \cap J^c$, then we have

$$\begin{aligned} pt \in \pi_0^n\left(\overline{Z_n^{I;\vec{i}}}\right) &\Rightarrow z_{i_t}^I(pt) = 0 \\ &\Rightarrow z_1^{i_t}(pt) = 0 \\ &\Rightarrow pt \notin U_I \\ &\Rightarrow pt \notin \pi_0^n\left(\overline{Z_n^{J;\vec{j}}}\right) \\ &\Rightarrow \pi_0^n\left(\overline{Z_n^{I;\vec{i}}}\right) \cap \pi_0^n\left(\overline{Z_n^{J;\vec{j}}}\right) = \varnothing \\ &\Rightarrow \overline{Z_n^{I;\vec{i}}} \cap \overline{Z_n^{J;\vec{j}}} = \varnothing. \end{aligned}$$

Thus, if we want a non-empty intersection, we must have that $I \cap J^c = \varnothing$, and by symmetry of the argument, we also have $I^c \cap J = \varnothing$. That is, we must have $I = J$ if we want a non-empty intersection, which is as we assumed in the earlier argument. □

From this, we can write down the blow-up explicitly

$$\text{Blow}_{Z_n} C_n = \bigcup_{I \subseteq [N]} \bigcup_{\vec{i}} \text{Blow}_{Z_n^{I;\vec{i}}} V_{\vec{i}}^I$$

with

$$\text{Blow}_{Z_n^{I;\vec{i}}} V_{\vec{i}}^I = \left\{(z^I(\vec{i}), l^I(\vec{i})) \in \mathbb{C}^N \times \mathbb{CP}^{N-n-1} | z^I(\vec{i})_i l^I(\vec{i})_j = z^I(\vec{i})_j l^I(\vec{i})_i \text{ with } i, j \in [N] - \vec{i}\right\}$$

if $Z_n^{I;\vec{i}}$ is non-empty and otherwise we have

$$\mathrm{Blow}_{Z_n^{I;\vec{i}}} V^I_{\vec{i}} = V^I_{\vec{i}}.$$

Then, we can write affine charts $V^I_{\vec{i}+i_{n+1}}$ of $\mathrm{Blow}_{Z_n^{I;\vec{i}}} V^I_{\vec{i}}$ with $i_{n+1} \in [N] - \vec{i}$ and

$$V^I_{\vec{i}+i_{n+1}} = \left(l^I(\vec{i})_{i_{n+1}} \neq 0\right) \subseteq V^I_{\vec{i}}$$

with Euclidean coordinates

$$\begin{aligned} z^I(\vec{i}+i_{n+1})_j &= l^I(\vec{i})_j / l^I(\vec{i})_{i_{n+1}} && \text{for } j \in [N] - \{i_1, \dots, i_{n+1}\} \\ z^I(\vec{i}+i_{n+1})_j &= z^I(\vec{i})_j && \text{for } j \in \{i_1, \dots, i_{n+1}\} \end{aligned}$$

$Z_n^{I;\vec{i}}$ is non-empty and otherwise we have

$$V^I_{\vec{i}+i_{n+1}} = V^I_{\vec{i}}$$

with Euclidean coordinates

$$z^I(\vec{i}+i_{n+1})_j = z^I(\vec{i})_j \text{ for all } j \in \{i_1, \dots, i_{n+1}\}.$$

Thus, from

$$\left(C_n, \left\{V^I_{i_1,\dots,i_n}\right\}_{I;i_1,\dots,i_n}\right)$$

we construct

$$\left(C_{n+1}, \left\{V^I_{i_1,\dots,i_{n+1}}\right\}_{I;i_1,\dots,i_{n+1}}\right)$$

where C_{n+1} is the blow-up of C_n, and this concludes our explicit construction of the blow-up for $0 \leq n \leq N$. Now, we wish to show that these blow-ups resolve the map $f : (\mathbb{CP}^1)^N \dashrightarrow (\mathbb{CP}^1)^{\frac{N}{2}(N-1)}$. Thus, we consider

$$\begin{array}{ccccc} C_n & \hookrightarrow & V^I_{\vec{i}} & & \\ \downarrow & \overset{f^n}{\searrow} & & \overset{f^{I;\vec{i}}_{(k,l)}}{\searrow} & \\ C_0 & \overset{f}{\dashrightarrow} & (\mathbb{CP}^1)^{\frac{N}{2}(N-1)} & \xrightarrow{p_{(k,l)}} & \mathbb{CP}^1 \end{array}$$

where we write

$$f_{(k,l)}^{I;\vec{i}} := f^n|_{V_{\vec{i}}^I} \circ p_{(k,l)}.$$

Now, note that if $k, l \notin \vec{i}$, we have

$$f_{(k,l)}^{I;\vec{i}} : \left(z(\vec{i})_j\right)_{j=1}^N \mapsto$$

$$\begin{cases} \left[a\left(z^I(\vec{i})_k + z^I(\vec{i})_l\right) + z^I(\vec{i})_k : a\left(z^I(\vec{i})_k + z^I(\vec{i})_l\right) + z^I(\vec{i})_l\right] & \text{if } k, l \in I^c \\ \left[a\left(z^I(\vec{i})_k + z^I(\vec{i})_l\right) + z^I(\vec{i})_l : a\left(z^I(\vec{i})_k + z^I(\vec{i})_l\right) + z^I(\vec{i})_k\right] & \text{if } k, l \in I \\ \left[a\left(z^I(\vec{i})_k z^I(\vec{i})_l + 1\right) + 1 : a\left(z^I(\vec{i})_k z^I(\vec{i})_l + 1\right) + z^I(\vec{i})_l z^I(\vec{i})_k\right] & \text{if } l \in I^c, k \in I \\ \left[a\left(z^I(\vec{i})_k z^I(\vec{i})_l + 1\right) + z^I(\vec{i})_k z^I(\vec{i})_l : a\left(z^I(\vec{i})_k z^I(\vec{i})_l + 1\right) + 1\right] & \text{if } l \in I, k \in I^c \end{cases} \tag{A.1.10}$$

where the first two cases are singular and the other two are smooth, and we note that blow-ups along $Z_{n+1}^{I;\vec{i}+i_k}$ and $Z_{n+1}^{I;\vec{i}+i_l}$ resolve the map (A.1.10). Indeed, when we blow-up $V_{\vec{i}}^I$ along $Z_{n+1}^{I;\vec{i}+i_k}$ we get that the map, in the singular cases, is given in the local coordinates by

$$f_{(k,l)}^{I;\vec{i}+i_k} : \left(z(\vec{i}+i_k)_j\right)_{j=1}^N \mapsto$$

$$\begin{cases} \left[a\left(z^I(\vec{i}+i_k)_k + z^I(\vec{i}+i_k)_l z^I(\vec{i}+i_k)_k\right) + z^I(\vec{i}+i_k)_k \right. \\ \left. : a\left(z^I(\vec{i}+i_k)_k + z^I(\vec{i}+i_k)_l z^I(\vec{i}+i_k)_k\right) + z^I(\vec{i}+i_k)_l z^I(\vec{i}+i_k)_k\right] \text{ if } k, l \in I^c \\ \left[a\left(z^I(\vec{i}+i_k)_k + z^I(\vec{i}+i_k)_l z^I(\vec{i}+i_k)_k\right) + z^I(\vec{i}+i_k)_l z^I(\vec{i}+i_k)_k \right. \\ \left. : a\left(z^I(\vec{i}+i_k)_k + z^I(\vec{i}+i_k)_l z^I(\vec{i}+i_k)_k\right) + z^I(\vec{i}+i_k)_k\right] \text{ if } k, l \in I. \end{cases}$$

One can check that in both cases this map only has removable singularities at the points where we have the equality $\left(z^I(\vec{i}+i_k)_l = z^I(\vec{i}+i_k)_k = 0\right)$, which can be resolved by factoring out a $z^I(\vec{i}+i_k)_k$ term from both entries. Now, if we blow up along $Z_{n+1}^{I;\vec{i}+i_l}$, we get a similar resolution where the indices i_k and i_l are switched. One should note that we didn't describe what happened to the maps under the blow-up in the cases where the maps are smooth. This is because by properties of the blow-up the map will remain smooth, which is what we need. Actually, once we have a smooth resolution of a map, it will stay smooth under further blow-ups. Moreover, note that for

$n = N$, $\{i_1, \ldots, i_n\} = \{1, \ldots, N\}$. Thus, $f^n \circ p_{(k,l)}$ for every pair (k,l) and we have that

$$f^N : C_N \to (\mathbb{P}^1)^{\frac{N(N-1)}{2}}$$

is smooth. This completes the proof.

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